Knight Independence on Triangular Hexagon Boards

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ABSTRACT. The independence number β_n for knights on equilateral triangular boards T_n of regular hexagons is determined for all n.

1. Introduction

Triangular hexagon boards T_n with n consecutive hexagons at each side are parts of the regular hexagonal tessellation of the plane (see T_{13} in Figure 1). Corresponding to knights on chess boards we consider knights on T_n which can move as indicated in Figure 1 (from the center to the hexagons marked with stars). The independence number β_n is the maximum number of pairwise nonattacking knights on T_n.

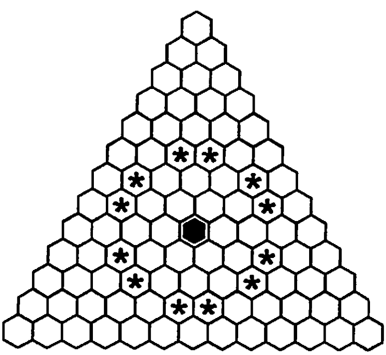

Figure 1. Knight's moves on T_{13}.

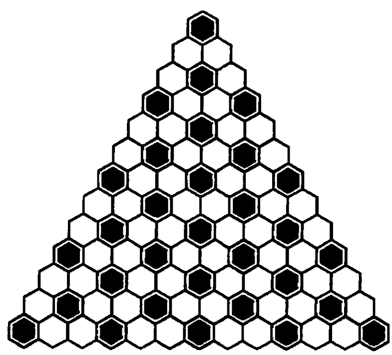

Figure 2. General lower bound.

In [1] partial results and exact values of β_n for $n \leq 14$ are given. Here we will present β_n for all n.

Theorem 1. *For knights the independence number β_n on triangular hexagon boards is*

$$\beta_n = \left\lceil \frac{n(n+1)}{6} \right\rceil \qquad \text{for } n \geq 30,$$

and all smaller values are as in Table 1.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
β_n	1	3	6	7	7	9	15	18	21	21	27	31	36	42	42

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29
β_n	47	54	62	70	70	77	85	95	105	109	117	126	136	147

Table 1.

It can be remarked that $\beta_n = \lceil n(n+1)/6 \rceil$ also holds for $n=1, 20, 21, 22,$ 25, 26, 27, and 28. For the remaining small values of n the independence numbers exceed one third of the number of hexagons of T_n at least by one.

The independence numbers for rooks and for two types of kings on T_n have been settled in [4] and [2], respectively. Independence numbers and similar invariants for different chess pieces on square boards are surveyed in [3].

2. Proof of Theorem 1

For $n \leq 14$ the values β_n have been determined in [1].

As a general lower bound we obtain $\beta_n \geq \lceil n(n+1)/6 \rceil = b_n$ if in rows $\equiv 1$, 2, and 0 (mod 3) knights are placed in positions $\equiv 1, 0, 2 \pmod 3$, respectively (see Figure 2). Figures 3 and 4 show exceptional lower bounds for $n=16$ and 17. For the remaining values $n=15, 18, 19, 23, 24,$ and 29 lower bounds $> b_n$ are obtained if in rows $\equiv 2$, 3, and 4 (mod 5) knights are placed in positions $\equiv 1$ and 2, in positions 1, 2, and 3, and in positions 2 and 3 (mod 5), respectively (see Figure 5 for $n=15$).

To prove the upper bounds for small values of n we use a computer together with the following method of [1]. We choose a partition of the hexagons into a number B of triples, pairs and singles such that every pair and every triple can contain at most one independent knight. By computer we try to find this upper bound B rather close to the asserted value of β.

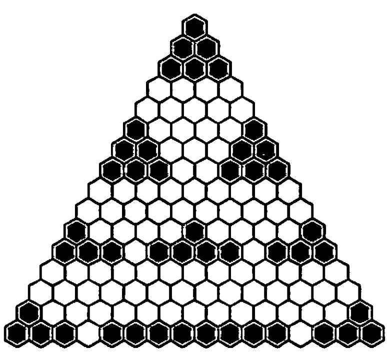

Figure 3. Lower bound for T_{16}.

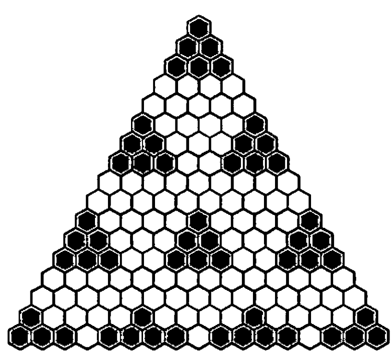

Figure 4. Lower bound for T_{17}.

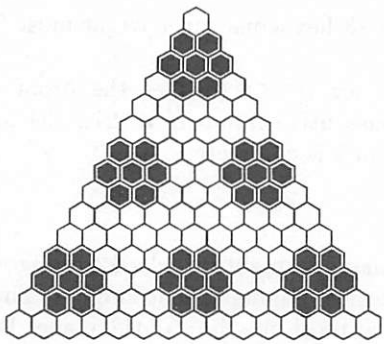

Figure 5. Lower bound for T_{15}.

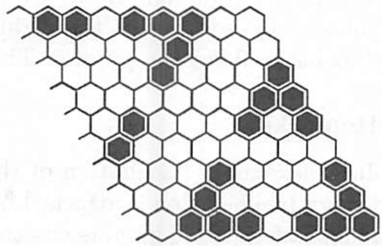

Figure 6. Last 8 positions of TR_n.

Then we try by a backtracking algorithm to place independent knights on as many as possible of the chosen tuples. If in all end situations the hexagons of at least $B-\beta$ tuples are attacked by knights from other tuples then β is proved as an upper bound. In case of T_n this procedure was successful for

$$n = 15, 16, 17, 18, 19, 20, 23, 24, 25, \text{ and } 29,$$

and as induction basis for

$$n = 21, 22, 26, 27, 28, 32, 33, 34, \text{ and } 38.$$

Thus we have obtained all values β_n of Table 1.

For $n \geq 30$ we use induction on n modulo 9. Let TR_{n+1} denote the trapezium consisting of the last 9 rows of T_{n+9} and $\bar{\beta}_{n+1}$ its independence number. If $\beta_n = b_n$ and $\bar{\beta}_{n+1} \leq 3n+15$ is assumed then

$$\beta_{n+9} \leq \beta_n + \bar{\beta}_{n+1} \leq \left\lceil \frac{n(n+1)}{6} \right\rceil + 3n + 15 = \left\lceil \frac{(n+9)(n+10)}{6} \right\rceil = b_{n+9}.$$

With the basis of above the induction is complete if

$$\bar{\beta}_n \leq 3n + 12 \quad \text{for } n = 22, 23 \text{ and } n \geq 27$$

which remains to be proved.

For the asserted values of $n \leq 34$ we have proved $\bar{\beta}_n \leq 3n+12$ by using the same algorithm as before. By computer we have checked that for every independent set of knights in TR_n an i with $1 \leq i \leq 8$ exists such that at most $3i$ knights are contained in the last i hexagons of all 9 rows. This may be done by a backtracking algorithm starting with all possible sets of knights in the last positions of the 9 rows. Then in the preceding positions, simultaneously for all rows, all independent sets of knights are considered. Whenever the number of placed knights does not exceed one third of the so far considered hexagons then one can go back. In Figure 6 the unique possibility is shown that more then $3i$ knights are in the last i hexagons of all 9 rows for i=1,...,7, however, the last 8 hexagons can have at most 24 knights.

For $n \geq 35$ induction from n-i to n for $1 \leq i \leq 8$ finishes the proof of $\bar{\beta}_n \leq 3n+12$ by addition if the eight consecutive numbers n=27,...,34 are used as basis. Thus the proof of Theorem 1 is complete.

3. Remarks

In a hexagonal tessellation of the plane a knight attacks 12 hexagons and every free hexagon is attacked by at most 6 independent knights. Thus an independent set of knights can cover at most one third of the plane. For $n \geq 30$ the ratio of independent knights against all hexagons of T_n cannot exceed one third. For smaller values of n more knights near the border allow a large ratio.

For TR_n (trapezium with 9 rows) we can use knights as in Figure 2 to prove $\bar{\beta}_n \geq 3n+12$ in general, that is one third of all hexagons. Only for $n \leq 12$ and n=14, 15, 16, 20, 21, 26 the independence number $\bar{\beta}_n$ exceeds $3n+12$.

In general we may ask to characterize all parts of the regular hexagonal tessellation for which the border allows an independence number β exceeding one third of all hexagons at least by one.

References

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