On the Existence of Balanced Arrays of Strength Five

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Abstract

Summary. In this paper we present some inequalities on balanced arrays (B-arrays) of strength five with two symbols.

1. Introduction.

A balanced array (B-array) T of strength t with two symbols (say, 0 and 1), m constraints (rows), N columns (treatment-combinations) is merely a matrix T $(m \times N)$ with elements 0 and 1 such that in every $(t \times N)$ submatrix T^* of T, every vector $\underline{\alpha}$ $(t \times 1)$ of weight i $(0 \leq i \leq t$; the weight of a vector $\underline{\alpha}$ is the number of 1's in it) appears as a fixed number (say) μ_i, $(0 \leq i \leq t)$ times. The vector $\underline{\mu}'=(\mu_0, \mu_1, \mu_2, \ldots, \mu_t)$ is called the index set of the B-array T, and T is sometimes denoted by $T\ (m, N, 2, t)$. It is quite clear that

$$N = \sum_{i=0}^{t} \binom{t}{i} \mu_i.$$

If $\mu_i = \mu$ for each i, then the B-array is called an orthogonal array (O-array), and clearly for an O-array we have $N = \mu 2^t$. Thus O-arrays form a subset of B-arrays. B-arrays have been found to be quite useful in constructing symmetrical and asymmetrical factorial designs. Furthermore, the incidence matrix of a balanced incomplete block design (BIBD) is a certain kind of B-array. In this paper, we restrict ourselves to B-arrays of strength t = 5. The problem of constructing B-arrays for a given $m (m > t)$ and $\underline{\mu}'$ is non-trivial. In this paper we present some inequalities which are necessary conditions for the existence of B-arrays, and consequently we obtain an upper bound on m for a given $\underline{\mu}'$ (an important problem in combinatorics and statistical design of experiments).

The existence problems and the ones concerned with obtaining an upper bound on m for O-arrays and B-arrays have been studied, among others, by

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Bose and Bush [1], Bush [2], Chopra [3],[4], Chopra and Dios [5], Rafter and Seiden [7], Saha et al [8], Seiden and Zemach [9], Yamamoto et. al. [10] etc. Readers interested to gain further insight into the importance and usefulness of these arrays to combinatorics and factorial designs may consult the list of references provided at the end of this paper.

2. Preliminaries.

The following results are easy to establish.

Lemma 2.1. A B-array T with $t = m = 5$ always exists.

Lemma 2.2. A B-array T $(m, N, 2, t = 5;\ \underline{\mu}'=(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5))$ is also of strength $t'$where $0 \leq t' \leq 5$ The index set $(\underline{\mu}'^{*} = \mu_0^{*}, \mu_1^{*}, \ldots, \mu_{t'}^{*})$,of T (considered as an array of strength t') is given by $\mu_j^{*} = \sum_{i=0}^{5-t'} \binom{5-t'}{i} \mu_{i+j}$. $j = 0, 1, 2, \ldots, t'$,with the convention that $\binom{5-t'}{i} = 1$ when $i = 5 - t' = 0$.

Remark: Using Lemma 2.2, one can easily obtain the index sets $\underline{\mu}^{*\prime}$ for $t' = 4, 3, 2$ and 1 which are respectively $(A_i;\ i = 0, 1, 2, 3, 4$ with $A_i = \mu_i + \mu_{i+1}), (B_i;\ B_i = A_i + A_{i+1}$with $i = 0, 1, 2, 3), .(C_i;\ C_i = B_i + B_{i+1}$with $i = 0, 1, 2)$, and $(D_i;\ D_i = C_i + C_{i+1}$ with $i = 0, 1)$. It is not difficult to see that the successive index sets $\underline{\mu}^{*\prime}$ for various values of t' are merely linear functions of μ_i's.

Lemma 2.3. Consider a B-array T $(m, N, \underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_5))$. Let $x_j (0 \leq j \leq m)$ be the number of columns of weight j in T. Then the following results hold:

$$\sum_{j=0}^{m} x_j = N \tag{2.1a}$$

$$\sum j x_j = m_1 D_1 = M_1 \tag{2.1b}$$

$$\sum j^2 x_j = m_2 C_2 + m_1 D_1 = M_2 \tag{2.1c}$$

$$\sum j^3 x_j = m_3 B_3 + 3 m_2 C_2 + m_1 D_1 = M_3 \tag{2.1d}$$

$$\sum j^4 x_j = m_4 A_4 + 7 m_3 B_3 + 6 m_2 C_2 + m_1 D_1 = M_4 \tag{2.1e}$$

$$\sum j^5 x_j = m_5 \mu_5 + 15 m_4 A_4 + 25 m_3 B_3 + 10 m_2 C_2 + m_1 D_1 = M_5 \tag{2.1f}$$

where $m_r = m(m-1)(m-2)\ldots(m-r+1)$.

Remark: Lemma 2.3 expresses various moments of the weights of the columns of T in terms of its parameters m and μ_i's.
Proof outline: The above results can be easily derived by considering T as an array of strength $t'(t' = 1, 2, 3, 4, 5)$ and counting in two ways (through columns and rows) the total number of column vectors of weight t'.

3. Main Results

In order to derive the main results, in this section we present some results on the existence of B-arrays of strength $t = 5$ by using some classical inequalities and results from the previous section.

Theorem 3.1. Consider a B-array T with m rows and with $\underline{\mu}' = (\mu_0, \mu_1, ..., \mu_5)$. Then the following must hold:

$$(M_4 + 1)\left[\ln(M_5 + 1) - \ln(M_4 + 1)\right] > M_4\left[\ln M_5 - \ln M_4\right] \tag{3.1}$$

Proof: The following result is true (Mitrinovic, [6]):
$\left(\frac{a+1}{b+1}\right)^{b+1} > \left(\frac{a}{b}\right)^{b}, a \neq b; a, b > 0.$
Setting here $a = \sum j^5 x_j = M_5$ and $b = \sum j^4 x_j = M_4$ we obtain the result after some simplification.

Theorem 3.2. For a B-array T $(m \times N)$ with $\underline{\mu}' = (\mu_0, \mu_1, ..., \mu_5)$ to exist, we must have

$$\begin{aligned}&(N + 5m_1 D_1 + 20m_2 C_2 + 33m_3 B_3 + +16m_4 A_4 + m_5\mu_5)^6 \geq \\ &\geq 46,656 N M_1 M_2 M_3 M_4 M_5\end{aligned} \tag{3.2}$$

where M_i's are defined in Lemma 2.3 and are functions of m, and μ_i's.
Proof: It is well-known that the arithmetic mean amongst $\sum_{j=0}^{m} j^k x_j (k = 0, 1, 2, 3, 4, \text{and } 5)$ is always greater than or equal to their geometric mean. It means $N + \sum_{i=1}^{5} M_i \; / \; 6 \geq \left[N \prod_{i=1}^{5} M_i\right]^{1/6}$. Next, we substitute the values of M_i's from (2.1a-2.1f) and simplify to obtain the desired result.

Result: In order to prove the next result, we make use of the following classical inequalities:

$$\left(\sum a_k b_k\right)^2 \leq \sum a_k^2 \sum b_k^2 \tag{A1}$$

$$\left[\sum (a_k b_k c_k d_k)\right]^2 \leq \sum a_k^2 \sum b_k^2 \sum c_k^2 \sum d_k^2 \tag{A2}$$

Theorem 3.3. Consider a B-array T of strength $t = 5$ with m rows, and having $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$. Then the following must hold:

$$\text{(a)}\quad M_5^2 \leq \frac{m^2(m+1)^2(2m+1)}{12} M_3 M_4 \qquad (3.3a)$$

$$\text{(b)}\quad M_4^2 \leq \left[\frac{m(m+1)}{2}\right]^3 M_1 M_3 \qquad (3.3b)$$

$$\text{(c)}\quad M_3^2 \leq N \left[\frac{m(m+1)}{2}\right]^2 M_4 \qquad (3.3c)$$

$$\text{(d)}\quad M_2^2 \leq M_1 M_3 \qquad (3.3d)$$

Proof: In order to derive (3.3a), substitute in result (A2), $a_k = k^2\sqrt{x_k}, b_k = k^{3/2}\sqrt{x_k}, c_k = k, d_k = \sqrt{k}$ and we obtain the result. Similarly to obtain (3.3b, 3.3c), we make the following substitutions: to obtain (3.3b), set $a_k = k^{3/2}, b_k = k^{1/2}, c_k = k^{1/2}\sqrt{x_k}, d_k = k^{3/2}\sqrt{x_k}$ in (A2), for (3.3c), we take $a_k = k^{1/2}, b_k = \sqrt{x_k}, c_k = k^{1/2}$, and $d_k = k^2\sqrt{x_k}$. For (3.3d) we take $a_k = \sqrt{kx_k}$ and $b_k = \sqrt{k^3 x_k}$ in (A1).

Remark: Every inequality is a polynomial in terms of m. We may point out the results derived here are necessary conditions for the existence of B-arrays of strength $t = 5$. If all the conditions are satisfied for a given m and $\underline{\mu}'$, it does not imply the corresponding B-array will exist. However, if some condition is contradicted for a given $\underline{\mu}'$ with $m = m^* + 1$, then we can say that such an array does not exist for the number of constraints $m \geq m^* + 1$. Incidentally it also tells us that m^* is an upper bound on the number of constraints m for the B-array with index set $\underline{\mu}'$. In order to check the existence conditions given in this paper, one can easily prepare a computer program. For a given $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, ..., \mu_5)$ every inequality is only in terms of m. One can start with $m = 6$, and see at what value of m the contradiction takes place. It is quite obvious that the B-array T will not exist for $m \geq (m^* + 1)$ if it does not exist for $m = m^*$(say).

Example. Consider the array (2,0,0,0,1,2). Using the inequality in (3.3d), we obtain a contradiction for $m = 8$ for which $LHS = 107584$ and $RHS = 107136$. Hence, for this array, $m \leq 7$.

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