Equitable Labelings of Corona Graphs

Christian Barrientos

Departament de Matemàtica Aplicada i Telemàtica
Universitat Politècnica de Catalunya
c/ Jordi Girona 1-3, Mòdul C3 Campus Nord
08034 Barcelona, Spain
cbarrien@mat.upc.es

Abstract

A labeling f of the vertices of a graph G is said *k-equitable* if each weight induced by f on the edges of G, appears exactly k times. A graph G is said *equitable* if for every proper divisor k of its size, the graph G has a k-equitable labeling.

A graph G is a corona graph if G is obtained from two graphs, G_1 and G_2, taking one copy of G_1, which is supposed to have order p, and p copies of G_2, and then joining by an edge the k^{th} vertex of G_1 to every vertex in the k^{th} copy of G_2. We denote G by $G_1 \otimes G_2$.

In this paper we proved that the corona graph $C_n \otimes K_1$ is equitable. Moreover, we show k-equitable labelings of the corona graph $C_m \otimes nK_1$, for some values of the parameters k, m, and n.

1 Introduction

A *labeling* f a graph G is a one-to-one mapping from the vertex set of G into a set of integers. A *difference vertex labeling* is a labeling f such that for each edge $e = uv \in E(G)$, the weight induced by f on e is the number $|f(u) - f(v)|$. A labeling f of G is called *proper* if $f(v) \geq 0$ for every vertex $v \in V(G)$. Let f be a labeling of a graph G, we say that the labeling f is *complete* if the weights induced by f are the integers $1, 2, \ldots, w$, where w is the number of different weights indeced by f. We say that f is *optimal* if the labels assigned by f to the vertices of G are consecutive integers. Graph theory terminology and notation are taken from [6].

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A labeling f of a graph G is *k-equitable* if for each weight h induced by f there are exactly k edges of G having weight equal to h. If a graph G has a k-equitable labeling then, G is said to be *k-equitable.* Equitable labelings of graphs were introduced by Bloom and Ruiz in [5].

Let f be a k-equitable labeling of a (p, q)-graph G. Clearly k divides the size of G, so the number of different weights induced by f is $w = q/k$. If for every k proper divisor of q, G has a k-equitable labeling, G is said *equitable.* Not every graph is equitable, for instance, the complete graph is not 3-equitable. Until now, few families of graphs are known to be equitable. It was proved, independently by Barrientos [1] and Wojciechowski [9], that the cycle C_n is equitable. In [2], Barrientos, Dejter, and Hevia proved that linear forest are equitables. We also know that other families of graphs, related with cycles and linear forest are equitable: dragons and the union of a cycle and a linear forest (see [4]).

In the next section we present some conditions that a graph must satisfy to have a k-equitable labeling and two preliminary results.

2 Initial results

In [2], was established that if a graph G is k-equitable then the maximum degree $\Delta(G)$ of G is bounded from above. In [3], other necessary condition to have a k-equitable labeling of a graph G was presented, but now bounding from below the minimum degree $\delta(G)$ of G. We state these conditions in the following theorems.

Theorem 1 *If G is a k-equitable graph of size $q = kw$, then $\Delta(G) \leq 2w$.*

Theorem 2 *If G is a k-equitable graph of size $q = kw$, then $\delta(G) \leq w$.*

As we mentioned before, some families of equitable graphs are known: cycles, disjoint union of a cycle and a linear forest, and dragons. For a dragon we means a cycle with a path attached to a vertex.

Theorem 3 *The cycle C_n is k-equitable optimal for every proper divisor k of its size n.*

Let G be the disjoint union of a cycle C_n and a linear forest F of size m. Then we may state the following.

Theorem 4 *The graph G is k-equitable for every proper divisor k of its size $n + m$.*

Let $R(n, m)$ denote the dragon formed by the cycle C_n and the path P_m. As a corollary of the last theorem we can establish the following theorem.

Theorem 5 *The dragon $R(n, m)$ is k-equitable for every proper divisor k of its size $n + m$.*

In the next section, we will use Theorem 3 to obtain a new family of equitable graphs.

3 Corona graphs

The *corona* $G_1 \otimes G_2$ of two graphs, G_1 and G_2, is the graph obtained by taking one copy of G_1, which is supposed to have order p, and p copies of G_2, and then joining by an edge the kth vertex of G_1 to every vertex in the kth copy of G_2. In [7], Frucht introduced graceful numberings of this kind of graphs, in particular, for the case $C_n \otimes K_1$. Let $G = C_n \otimes K_1$, since $\Delta(G) = 3$, $\delta(G) = 1$, and the size is $2n$,then for every k proper divisor of $2n$, both degree conditions, given in theorems 1 and 2, are satisfied. In the next theorem we present k-equitable labelings for every k proper divisor of $2n$.

Theorem 6 *The corona $C_n \otimes K_1$ is k-equitable for every proper divisor k of the size $2n$.*

Proof Let the cycle C_n be described by a circuit $v_1, v_2, \ldots, v_n, v_1$. Denote by $w_1, w_2, \ldots, w_n$ the n vertices of degree 1. We broke the proof in three cases.

Case 1: If k is a proper divisor of n.

Suppose that C_n has been labeled with a k-equitable labeling (see [9] or [4]). Let $v_i w_i \in E(C_n \otimes K_1)$, for every $1 \leq i \leq n$, where v_i has label equal to i. Let f be a labeling of the vertices w defined, for every $i = 1, 2, \ldots, n/k$ and $j = 1, 2, \ldots, k$ as follow:

$$f(w_{ki-j+1}) = n - j + (k+1)i$$

The weights induced on these edges are the integers $n, n+1, \dots, n+\frac{n}{k}-1$, each of them with frecuency k. So, we have a k-equitable labeling of the graph $C_n \otimes K_1$. In Figure 1 (a), we show a 3-equitable labeling of . $C_{12} \otimes K_1$

Case 2: If $k = n$.

Assume that $v_i w_i \in E(C_n \otimes K_1)$, for every $1 \le i \le n$. Let f be a labeling defined on the vertices of $C_n \otimes K_1$ by:

$f(v_i) = i$, $1 \le i \le n$. (This labeling induces the weight 1 with frecuency $n-1$ and the weight $n-1$ with frecuency 1.)

$f(w_1) = 0$. (The weight induced here is 1, with frecuency 1.)

$f(w_i) = n - 1 + i$, $2 \le i \le n$. (The weight induced here is $n-1$ with frecuency $n-1$.)

So, we have a n-equitable labeling of $C_n \otimes K_1$. In Figure 1 (b),we present an example for $C_7 \otimes K_1$.

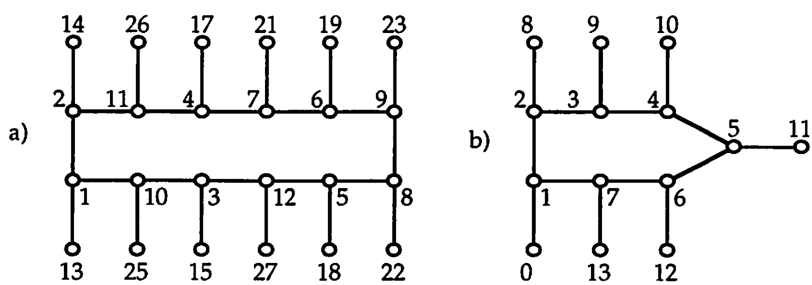

Figure 1

Case 3: If k is not a divisor of n.

For the Division Algorithm, given n and k, there exist q and r such that $n = kq + r$. Let f be a labeling defined by:

$$f(v_1) = 1 \text{ and } f(v_{i-1}) + j, \text{ where } k(j-1) + 2 \le i \le kj + 1, \text{ and } 1 \le j \le q.$$

Note that if $k = 2$, we have labeled all the vertices of C_n, which is not true otherwise. We will distinguish three subcases:

Subcase 3.1: If $k = 2$.

Until now, the weights indeced by f are $1, 2, \dots, q$ each of them with frecuency 2, and the weight $m^2 + m$ with frecuency 1. For convenience, let α denote the number $m^2 + m + 1$, that correspond to the maximum label used. Now, we define f over the vertices of degree 1.

$f(w_1) = 2 - \alpha$, and $f(w_i) = f(v_i) + \alpha - s - 2 + j$, where $kj \le i \le kj + 1$, and $1 \le j \le q$.

Clearly the weights induced are $m^2 + m$ with frecuency 1 and $m^2 + m - q, m^2 + m - q + 1, \ldots, m^2 + m - 1$, each of them with frecuency 2. Thus, we have the desired labeling of $C_n \otimes K_1$. See an example in Figure 2 for $C_7 \otimes K_1$.

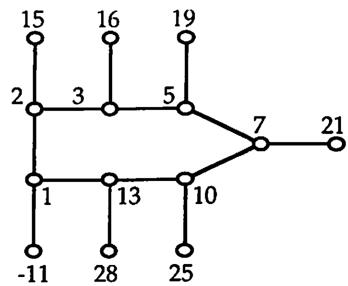

Figure 2

Recall that if $k \neq 2$ not all the vertices of C_n have been yet labeled. Therefore, we proceed as follows.

Subcase 3.2: If $r \ge q$.

To label the remaining vertices of C_n we define by: $f(v_i) = f(v_{i-1}) + r + 1$, $kq \le i \le n$. Let $f(v_n) = \alpha$. Now, the weights induced are $r + 1$ with frecuency $r - 1$ and $\alpha - 1$ with frecuency 1. Suppose that $v_i w_i \in E(C_n \otimes K_1)$, for every $1 \le i \le n$. Define f over the vertices w by:

$$f(w_i) = i - (r + 1) \text{ where } 1 \le i \le r + 1.$$

We obtain the weight $r + 1$ with frecuency $r + 1$; actually its frecuency is k. And,

$$f(w_i) = f(v_i) + \alpha - 1 \text{ where } n - k + 2 \le i \le n.$$

Noe we have obtained the weight $\alpha - 1$ with frecuency $k - 1$; so its frecuency is k. Finally,

$$f(w_i) = f(v_i) + \alpha - s - 2 + j, \text{ where } k(j - 1) - 1 \le i \le kj - 2, \text{ and } 2 \le j \le q.$$

The weights induced here by f are $\alpha - q, \alpha - q + 1, \ldots, \alpha - 2$, each of them with frecuency k. Therefore, f so defined is a k-equitable labeling of $C_n \otimes K_1$. The next figure show as example a 4-equitable labeling of $C_{10} \otimes K_1$.

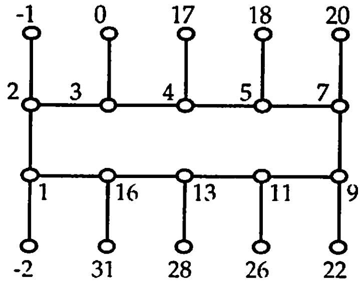

Figure 3

Subcase 3.3: If $r < q$.

To complete the labeling of C_n we define by: $f(v_i) = f(v_{i-1}) + q + 1$ where $kq + 2 \leq i \leq n$. Again, let $f(v_n) = \alpha$. The weights induced are $q + 1$ with frecuency $r - 1$ and $\alpha - 1$ with frecuency 1. If $v_i w_i \in E(C_n \otimes K_1)$, for every $1 \leq i \leq n$, we define f over the vertices w by:

$$f(w_i) = i - (q + 1) \text{ where } 1 \leq i \leq r + 1.$$

This induces the weight $q + 1$ with frecuency $r + 1$; so its final frecuency is k. And,

$$f(w_i) = f(v_i) + \alpha - 1 \text{ where } n - k + 2 \leq i \leq n.$$

The weight obtained now is $\alpha - 1$ with frecuency $k - 1$; so its final frecuency is k. Finally,

$$f(w_i) = f(v_i) + \alpha - s - 2 + j, \text{ where } k(j - 1) - 1 \leq i \leq kj - 1, \text{ and } 2 \leq j \leq q.$$

In this step the weights induced are $\alpha - q, \alpha - q + 1, \ldots, \alpha - 2$, each of them with frecuency k. Hence, f so defined is a k-equitable labeling of $C_n \otimes K_1$. The Figure 4 show an example of a 4-equitable labeling of $C_{14} \otimes K_1$. This complete the proof. ■

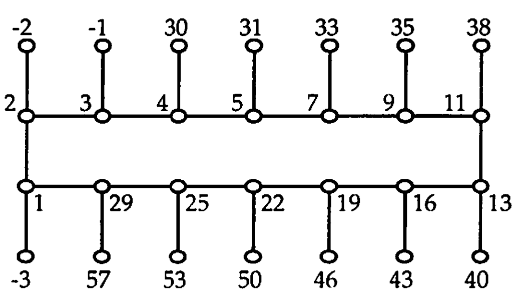

Figure 4

To transform the last three labelings into proper ones, is enough add to each label a suitable positive constant.

Recall that the graph presented above, belongs to the family of corona graphs $C_m \otimes nK_1$, for which we are interested in to find a k-equitable labeling. Three parameters are involved: the length m of the cycle, the number n of "pendant" edges, and the frecuency k of the weights. The degree condition (Theorem 1) establish a relation between them: $k(n+2) \leq 2m(n+1)$. The last theorem fixed the number of pendant edges as $n = 1$, however the amount of labels used is too much bigger than the order of the graph, and the set of weights has no restrictions. In the following theorems we have imposed some conditions on these sets and fixed the values of m and k.

As we mentioned before, a labeling is optimal if the labels used are consecutive integers and using the fact that if m is odd, the size of $C_m \otimes nK_1$ is not always divisible by 2, we find an optimal 2-equitable labeling of $C_m \otimes nK_1$ when m is even. Moreover, a labeling is complete if the weights induced are the first consecutive integers until the number of weights, In this case, we find a 2-equitable (when n is odd) and a 3-equitable labeling of $C_3 \otimes nK_1$, being both of them complete labelings.

Before that, as a particular case of Theorem 3, we have that every cycle of even size is optimal 2-equitable. We proved this fact in the following lemma.

Lemma 7 *The cycle C_m is optimal 2-equitable for every m even.*
Proof Let P_1 and P_2 be two paths of order $m/2$. Denote by $v_1, v_2, \ldots, v_{m/2}$ and $u_1, u_2, \ldots, u_{m/2}$ the consecutive vertices of P_1 and P_2, respectively. If we join, by an edge, the vertex v_1 to u_1, and the vertex $v_{m/2}$ to $u_{m/2}$, we obtain the cycle C_m. Consider the following labeling of its vertices:

$$f(v_i) = \begin{cases} i, & i \text{ is odd} \\ m+1-i, & i \text{ is even} \end{cases} \quad \text{and} \quad f(u_i) = \begin{cases} i+1, & i \text{ is odd} \\ m+2-i, & i \text{ is even} \end{cases}$$

The weights induced by f inside each path are the integers $2, 4, \ldots, m-2$ (because f over P_1 or P_2 is a dilation by 2 of the graceful labeling of $P_{(m-2)/2}$ gaven by Rosa in [8]). We obtain the weight 1 on the edges $v_{m/2}u_{m/2}$ and v_1u_1. Therefore, f is a 2-equitable labeling of C_m. ■

Theorem 8 *The corona $C_m \otimes nK_1$ is optimal 2-equitable, for every m even.*

Proof Use Lemma 7 to obtain an optimal 2-equitable labeling of a cycle of even size C_m. If we multiply by $n+1$ the vertex labels of C_m, we still have a 2-equitable labeling of C_m, with labels $n+1, 2n+2, \ldots, mn+m$, and weights $n+1, 2(n+1), 4(n+1), \ldots, (m-2)(n+1)$. Denote by v_i the vertex of C_m that has label equal to $i(n+1)$, $1 \le i \le m$. And let $v_{i,j}$ denote the vertices of degree 1 adjacent to v_i, where $1 \le j \le n$. Let f be a labeling defined on the vertices of degree 1 by:

$$f(v_{i,j}) = \begin{cases} (m-i-1)(n+1)+j, & i \text{ odd } 1 \le j \le n \\ (m-i+1)(n+1)+j, & i \text{ even } 1 \le j \le n \end{cases}$$

Is possible to see, that for i odd, the edges $v_{i,j}v_i$ and $v_{i+1,j}v_{i+1}$ has the same weights. So each weight induced appears twice. Recall that the weights over C_m are $n+1, 2(n+1), 4(n+1), \ldots, (m-2)(n+1)$. If $m \equiv 0(mod\ 4)$, the weights over these edges are $2(n+1) \pm j$, $6(n+1) \pm j, \ldots, (m-2)(n+1) \pm j$. And $m \equiv 2(mod\ 4)$, they are j, $4(n+1) \pm j$, $8(n+1) \pm j, \ldots, (m-2)(n+1) \pm j$. Therefore, we have the corona $C_m \otimes nK_1$ with an optimal 2-equitable labeling. ■

See an example in the next figure for $C_6 \otimes 3K_1$.

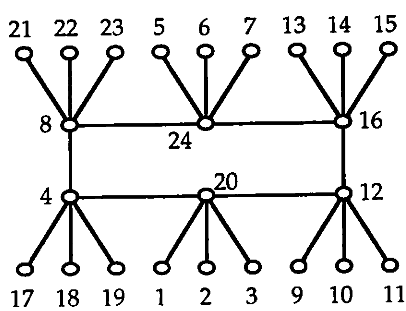

Figure 5

Theorem 9 *If n is odd, the corona $C_3 \otimes nK_1$ is a complete 2-equitable graph.*

Proof Let v_1, v_2, and v_3 be the vertices of C_3 and denote by $v_{i,j}$, where $1 \le j \le n$ and $i = 1, 2, 3$, the vertices of degree 1 adjacent to v_i. Define a labeling f of the vertices of $C_3 \otimes nK_1$ by:

$$f(v_1) = 3(n+1)/2, \quad f(v_{1,j}) = \begin{cases} j-1, & 1 \le j \le (n+1)/2 \\ j+n+1, & (n+3)/2 \le j \le n \end{cases}$$

$$f(v_2) = 2(n+1), \quad f(v_{2,j}) = j + (n+1)/2, \ 1 \le j \le n$$
$$f(v_3) = 3(n+1), \quad f(v_{3,j}) = j + 2(n+1), \ 1 \le j \le n.$$

The weights on the triangle are: $(n+1)/2$, $n+1$, and $3(n+1)/2$. On the edges $v_1v_{1,j}$ the weights are $1, 2, \ldots, (n-1)/2$ and $n+2, n+3, \ldots, 3(n+1)/2$. On the edges $v_2v_{2,j}$ are: $(n+3)/2, (n+5)/2, \ldots, (3n+1)/2$. And on the edges $v_3v_{3,j}$ are: $1, 2, \ldots, n$. So, each weight appears exactly twice and they are consecutive integers. Therefore, f is a complete 2-equitable labeling of $C_3 \otimes nK_1$, when n is odd. ■

In Figure 6 we show an example for $C_3 \otimes 5K_1$.

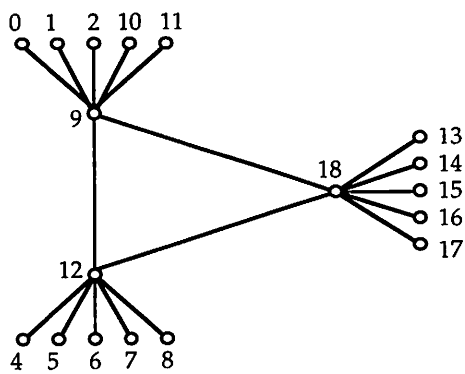

Figure 6

Theorem 10 *The corona $C_3 \otimes nK_1$ is a complete 3-equitable graph, for every positive value of n.*

Proof Let v_1, v_2, and v_3 be the vertices of C_3 and denote by $v_{i,j}$, where $1 \le j \le n$ and $i = 1, 2, 3$, the vertices of degree 1 adjacent to v_i. Define a labeling f of the vertices of $C_3 \otimes nK_1$ by:

$f(v_1) = n+1, \quad f(v_{1,j}) = j-1, \quad 1 \le j \le n$

$f(v_2) = n+2, \quad f(v_{2,1}) = n, \quad f(v_{2,j}) = j+n+1, \ 2 \le j \le n$

$f(v_3) = 2n+2, \quad f(v_{3,1}) = 2n+3, \quad f(v_{3,j}) = j+2n+3, \ 2 \le j \le n$

The weights on the triangle are: 1, n, $n+1$. On the edges $v_1v_{1,j}$ the weights are 2, 3, . . . , $n+1$. On the edges $v_2v_{2,j}$ are: 2, 1, 2, 3, ... $n-1$. And on the edges $v_3v_{3,j}$ are: 1, 3, 4, . . . , $n+1$. So, each weight appears exactly three times and they are the $n+1$ first integers. Hence, f so defined is a complete 3-equitable labeling of $C_3 \otimes nK_1$. ■

In Figure 7 we show an example for $C_3 \otimes 5K_1$.

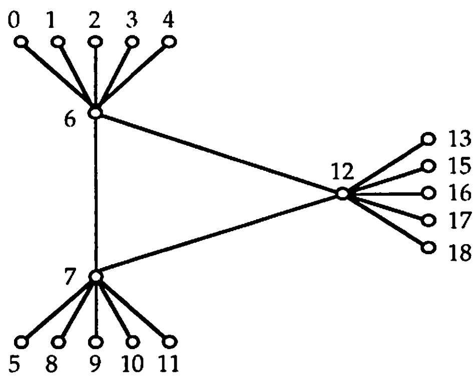

Figure 7

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