Some Concepts in List Coloring

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Abstract

In this paper uniquely list colorable graphs are studied. A graph G is said to be uniquely k-list colorable if it admits a k-list assignment from which G has a unique list coloring. The minimum k for which G is not uniquely k-list colorable is called the m-number of G. We show that every triangle-free uniquely colorable graph with chromatic number $k+1$ is uniquely k-list colorable. A bound for the m-number of graphs is given, and using this bound it is shown that every planar graph has m-number at most 4. Also we introduce list criticality in graphs and characterize all 3-list critical graphs. It is conjectured that every χ'_ℓ-critical graph is χ'-critical and the equivalence of this conjecture to the well known list coloring conjecture is shown.

1 Introduction

We consider finite, undirected simple graphs. For necessary definitions and notations we refer the reader to standard texts such as [11].

By a k–list assignment L to a graph G we mean a map which assigns to each vertex v of G a set $L(v)$ of size k. A list coloring for G from L, or an L–coloring for short, is a proper coloring c, in which for each vertex v, $c(v)$ is chosen from $L(v)$. A graph G is called k–choosable if it has a list coloring from any k–list assignment to it. The minimum number k for which G is k-choosable is called the list chromatic number of G and is denoted by

$\chi_\ell(G)$. In the following theorem all 2–choosable graphs are characterized. Before we state the theorem it should be noted that the core of a graph is a subgraph which is obtained by repeatedly deleting a vertex of degree 1, until no vertex of degree 1 remains. By $\theta_{r,s,t}$ we mean a graph consisting of three internally disjoint paths of lengths r, s, and t which have the same start points and the same end points. Similarly one can define the graphs $\theta_{r_1,\ldots,r_p}$.

Theorem A. [4] *A connected graph is* 2*-choosable, if and only if its core is either a single vertex, an even cycle, or* $\theta_{2,2,2r}$, *for some* $r \geqslant 1$.

A graph G is called uniquely k–list colorable, or UkLC for short, if it admits a k–list assignment L such that G has a unique L–coloring. This concept was introduced by Dinitz and Martin [3] and independently by Mahdian and Mahmoodian ([7] and [8]). A characterization of uniquely 2–list colorable graphs follows.

Theorem B. [7] *A graph G is not U2LC if and only if each of its blocks is either a cycle, a complete graph, or a complete bipartite graph.*

It is easy to see that for each graph G there exists a number k such that G is not UkLC. The minimum k with this property is called the m–number of G and is denoted by m(G). It is shown in [8] that every planar graph has m–number at most 5, and it is asked about the existence of planar graphs with m–number equal to 5. We study uniquely list colorable graphs in Section 2, where we prove that every triangle–free uniquely $(k+1)$–colorable graph is uniquely k–list colorable. We also show that every planar graph has m–number at most 4, so the answer to that question in [8] is negative.

In Section 3 we introduce list critical graphs and characterize 3–list critical graphs. Finally we pose a conjecture about list critical graphs which is shown to be equivalent to the list coloring conjecture.

2 Uniquely list colorable graphs

For every positive integer k several examples of UkLC graphs are introduced In [6]. In the following lemma we introduce another class of UkLC graphs. We mention that a uniquely k-colorable graph is a graph with chromatic number k for which every k-coloring induces the same k-partition on its vertex set.

Lemma 1. *Let G be a uniquely colorable graph with chromatic number $k+1$, and c be its unique $(k+1)$-coloring with color classes $C_1, \ldots, C_{k+1}$. If for each $i \leqslant k+1$, $|C_i| \geqslant i-1$, then G is a uniquely k-list colorable graph.*

Proof. We proceed by induction on k and prove that there exists a k-list assignment to such a graph G using exactly $k+1$ colors, which induces a unique list coloring. For $k=1$ the result obviously holds. Let G be a uniquely $(k+1)$-colorable graph as in the statement and $k \geqslant 2$. By induction $G - C_{k+1}$ admits a $(k-1)$-list assignment L' which induces a unique list coloring and uses colors $1, \ldots, k$. For each $v \in V(G) - C_{k+1}$, assign the list $L(v) = L'(v) \cup \{k+1\}$ to v, and since $|C_{k+1}| \geqslant k$, it is possible to assign some lists to C_{k+1} such that $\bigcap_{v \in C_{k+1}} L(v) = \{k+1\}$. Now it is easy to see that L is the desired list assignment. ■

It is shown in [10] that for every $k \geqslant 3$, in a triangle-free uniquely k-colorable graph, each color class has at least $k+1$ vertices. Using this result, we obtain the following theorem.

Theorem 1. *Every triangle-free uniquely $(k+1)$-colorable graph is uniquely k-list colorable.*

On the other hand in [2] it is shown that for each $k \geqslant 2$, there exists a uniquely k-colorable graph with arbitrarily large girth. So by theorem above, for each k, there exists a UkLC graph with arbitrarily large girth.

We need here a definition which is a generalization of the concept of a UkLC graph.

Definition 1. *Let G be a graph and f a be function from $V(G)$ to $\mathbb{N}$. An f–*list assignment* L to G is a list assignment in which $|L(v)| = f(v)$ for each vertex v. The graph G is called to be* uniquely f–list colorable, *or* UfLC *for short, if there exists an f-list assignment L for it such that G has a unique L-coloring.*

By definition above, if G is a UfLC graph, where $f(v) = k$ for each vertex v of G, then G in fact is a UkLC graph. To prove the next theorem, we need a relation which is proved in Truszczyński [9] and states that if G is a uniquely k-colorable graph, then $e(G) \geqslant (k-1)n(G) - \binom{k}{2}$.

Theorem 2. *If G is a UfLC graph, then*

$$\sum_{v \in V(G)} f(v) \leqslant n(G) + e(G).$$

Proof. Suppose that L is an f-list assignment to G using colors $1, 2, \ldots, t$, such that G has a unique L-coloring. We construct a uniquely t-colorable graph G^* as follows. Let $V(G) = \{v_1, \ldots, v_n\}$ and let K_t be a complete graph on the vertex set $\{w_1, \ldots, w_t\}$. Now for G^* consider the union of G and K_t and add edges $v_i w_j$ where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant t$, and $j \notin L(v_i)$.

Consider a t-coloring c of G^*. Without loss of generality we can assume that $c(w_i) = i$ for each $1 \leqslant i \leqslant t$. Since G has a unique L-coloring, by construction of G^*, c is the only t-coloring of G^*. So G^* is a uniquely t-colorable graph. On the other hand G^* has $n(G) + t$ vertices, and $e(G) + \binom{t}{2} + \sum_{v \in V(G)}(t - f(v))$ edges. Therefore as mentioned above, we have

$$e(G) + \binom{t}{2} + \sum_{v \in V(G)} (t - f(v)) \geqslant (n(G) + t)(t - 1) - \binom{t}{2}$$

and after simplification we obtain the result. ■

A natural question which arises here is whether or not equality can hold in Theorem 2? In the following proposition we give a positive answer to this question.

Proposition 1. *For every graph G, there exists $f : V(G) \to \mathbb{N}$ such that G is $UfLC$ and $\sum_{v \in V(G)} f(v) = n(G) + e(G)$.*

Proof. We proceed by induction on the number of vertices of G. For $n(G) = 1$ the statement is obvious. Consider a graph G with $n(G) \geqslant 2$ and a vertex v of G. By induction there exists $f' : V(G - v) \to \mathbb{N}$ and an f'-list assignment L' to $G - v$ such that $G - v$ has a unique L'-coloring, and $\sum_{w \in V(G-v)} f'(w) = n(G - v) + e(G - v)$. Consider a color a which is not used by L', and define a list assignment L to G as follows.

$$L(w) = \begin{cases} a & \text{for } w = v \\ L'(w) \cup \{a\} & \text{for } w \in N(v) \\ L'(w) & \text{otherwise.} \end{cases}$$

It is easy to verify that G has a unique L-coloring and that we have $\sum_{v \in V(G)} |L(v)| = n(G) + e(G)$. ■

Although the proposition above shows that in Theorem 2 equality may hold, but it seems that if $f(v) = k$ for each vertex v, equality does not hold and we have $e(G) > (k - 1)n(G)$.

By definition every graph G for $k = \mathrm{m}(G) - 1$ is UkLC. So by Theorem 2 we have the following.

Theorem 3. *For a graph G let $\bar{d}(G)$ denote the average degree of G, i.e. $\bar{d}(G) = 2e(G)/n(G)$. Then*

$$\mathrm{m}(G) \leqslant \lfloor \frac{\bar{d}(G)}{2} \rfloor + 2.$$

For example suppose that G is a bipartite graph. We have $\bar{d}(G) \leqslant n(G)/2$ so Theorem 3 implies $\mathrm{m}(G) \leqslant \lfloor n(G)/4 + 2 \rfloor$. This bound can be improved to a logarithmic bound as we will show in Theorem 4, but first we need a lemma.

Let L be a k–list assignment to a graph G such that G has a unique L–coloring c. For each vertex v of G, all the elements of $L(v) - \{c(v)\}$ must appear in $N(v)$, so if we denote by $c_N(v)$ the set of colors appearing in $N(v)$, then $|c_N(v)| \geqslant k - 1$. In the following lemma we state a stronger result.

Lemma 2. *Suppose that G is a UkLC graph, and L is a k-list assignment to G such that G has a unique L-coloring c with color classes $C_1, \ldots, C_t$. There exist at least $k - 1$ classes containing a vertex v with $|c_N(v)| \geqslant k$.*

Proof. Let $c(C_i) = \{i\}$ for each $i \in \{1, \ldots, t\}$. Suppose to the contrary that there are at most $k-2$ color classes containing a vertex v with $|c_N(v)| \geqslant k$. So without loss of generality we can assume that for each $\ell \geqslant k - 1$, C_ℓ contains no vertex v with $|c_N(v)| \geqslant k$. Let $v_0 \in C_{k-1}$, $i = c(v_0) = k-1$, and $j \in L(v_0) - \{1, \ldots, k-1\}$. Suppose that G_{ij} is the subgraph of G induced by $C_i \cup C_j$. Since for each vertex v of the component of G_{ij} containing v_0 we have $|c_N(v)| = k - 1$, it is implied that $i, j \in L(v)$. So we can interchange the colors i and j in this component to obtain a new L–coloring for G. This contradiction completes the proof. ■

It is shown in [4] that every non-k-choosable bipartite graph has more than 2^{k-1} vertices. So by applying Lemma 2, we deduce the following theorem.

Theorem 4. *Let G be a bipartite graph. Then* $\mathrm{m}(G) \leqslant 2 + \log_2 n(G)$.

Proof. Suppose that L is a k–list assignment to G such that G has a unique L–coloring c. By Lemma 2, G has a vertex v_0, such that there are at least k colors appearing at $N(v_0)$ in c. Let G' be the graph obtained from G, by duplicating v_0, i.e. adding a new vertex w to G and joining it to $N(v_0)$. Now assign to w a list containing k of the colors appearing at $N(v_0)$ in c, and the list $L(v)$ to each other vertex v of G'. It is clear that

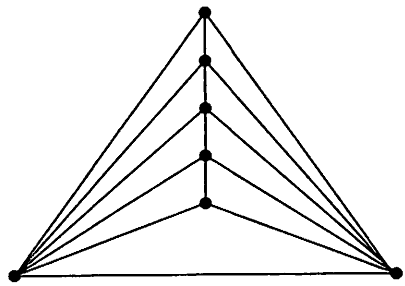

Figure 1: A uniquely 3–list colorable planar graph

G' is a bipartite graph and it has no coloring from these lists, so it is not k-choosable. Hence $n(G') > 2^{k-1}$ vertices. This implies that $n(G) \geqslant 2^{k-1}$, and so $k \leqslant 1 + \log_2 n(G)$. Now we obtain the desired relation by setting $k = \mathrm{m}(G) - 1$. ■

In the remainder of this section we state some consequences of Theorems 2 and 3.

It is well known that a planar graph with $n \geqslant 3$ vertices has at most $3n - 6$ edges. So the following theorem is an immediate consequence of Theorem 3.

Theorem 5. *For every planar graph G we have* $\mathrm{m}(G) \leqslant 4$.

By Lemma 1 the planar graph shown in Figure 1 is a U3LC graph for which the inequalities in Theorem 3 and Theorem 5 turn to be equalities.

Furthermore we know that a triangle–free planar graph G with at least 3 vertices has at most $2n(G) - 4$ edges. So Theorem 3 implies that each triangle–free planar graph G has m–number at most 3. In the following proposition a stronger result is obtained.

Proposition 2. *If a planar graph has at most 7 triangular faces, then* $\mathrm{m}(G) \leqslant 3$.

Proof. Consider a U3LC plane graph G with n vertices, e edges, f faces, and t triangular faces. We have $2e \geqslant 4(f - t) + 3t = 4f - t$, and by Euler formula $f = 2 - n + e$, so $t \geqslant 8 - 4n + 2e$. On the other hand Theorem 2 implies that $e \geqslant 2n$. So $t \geqslant 8$, as desired. ■

The following conjecture is about the structure of U3LC planar graphs which is motivated by the proposition above.

Conjecture 1. *Every U3LC planar graph has K_4 as a subgraph.*

For another application of Theorem 2, we study the edge version of unique list coloring.

A graph G is called to be uniquely k-list edge colorable, if $L(G)$ is a uniquely k-list colorable graph. The edge m-number of G is defined to be $\mathrm{m}(L(G))$, and is denoted by $\mathrm{m}'(G)$. It is straightforward to see that for each graph G, $\overline{d}(L(G)) \leqslant \Delta(L(G)) \leqslant 2\Delta(G) - 2$. So using Theorem 3 we deduce the following.

Theorem 6. *For every graph G, we have $\mathrm{m}'(G) \leqslant \Delta(G) + 1$ and if $\mathrm{m}'(G) = \Delta(G) + 1$ then G is a regular graph.*

Note that in Theorem 6 it is shown that if G is not a regular graph, then $\mathrm{m}'(G) \leqslant \Delta(G)$. So in this case $\mathrm{m}'(G) \leqslant \chi(G)$.

3 List critical graphs

In this section we introduce a concept of list critical graphs and we state some results concerning it.

Definition 2. *A graph G is called χ_ℓ-critical if for each proper subgraph H of it we have $\chi_\ell(H) < \chi_\ell(G)$.*

We sometimes refer to a χ_ℓ-critical graph G as a k-list critical graph, where $k = \chi_\ell(G)$. It can easily be verified that the only 2-list critical graph is K_2, odd cycles are 3-list critical, and the complete graph K_k is k-list critical.

Obviously every graph G contains a χ_ℓ-critical subgraph H such that $\chi_\ell(H) = \chi_\ell(G)$, and by an argument similar to critical graphs, $\delta(G) \geqslant \chi_\ell(G) - 1$ for each χ_ℓ-critical graph G. On the other hand there exists some differences between critical graphs and list critical graphs. For example it is well known that every critical graph is 2-connected. In Figure 2 we have given an example of a 3-list critical graph which is not 2-connected.

In the next theorem 3-list critical graphs are characterized.

Theorem 7. *A graph is 3-list critical if and only if it is either an odd cycle, two even cycles with a path joining them, $\theta_{r,s,t}$ where r, s, t have the same parity, and at most one of them is 2, or $\theta_{2,2,2,2r}$ where $r \geqslant 1$.*

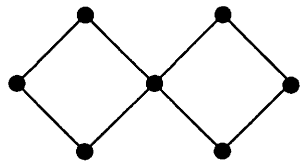

Figure 2: A non-2-connected 3-list critical graph

Proof. By use of Theorem A, it is easy to see that all the graphs listed in the statement are 3-list critical.

For the converse suppose that G is a 3-list critical graph. If G is 2-connected, by a theorem of Whitney [12] G has an ear decomposition $K_2 \cup P^1 \cup \ldots \cup P^q$. If $q \geqslant 4$, deleting an edge of P^q yields a non-2-choosable graph, which contradicts the 3-list criticality of G. So $q \leqslant 3$ and we consider the following three cases.

- If $q = 1$, G is a cycle, and so it is an odd cycle.
- If $q = 2$, $G = \theta_{r,s,t}$. In this case by deleting any edge of G, we obtain a graph whose core is a cycle, and since this cycle must be even, the numbers r, s, and t have the same parity. Now if at least two of r, s, and t are equal to 2, we have $\chi_\ell(G) = 2$, a contradiction.
- The last case is $q = 3$. By deleting any edge of G we obtain a graph whose core is a $\theta_{r,s,t}$, and since this graph must be 2-choosable, we have $r = s = 2$ and t is an even number. Now, by case analysis, it is easy to see that $G = \theta_{2,2,2,2\ell}$.

On the other hand if G is not 2-connected, we consider two end-blocks B_1 and B_2 of G. Since $\delta(G) \geqslant 2$ each of B_1 and B_2 has a cycle. So G has a subgraph H which is composed of two edge-disjoint cycles joined to each other by a path (possibly of length zero). We know that $\chi_\ell(H) = 3$, and so by χ_ℓ-criticality of G, G has no edge outside H, i.e. $G = H$. Hence G satisfies the statement. ∎

Suppose that G is a k-list critical graph, and L is a k-list assignment to G. Consider a vertex v in G and a color $a \in L(v)$. Assign to each vertex u in $G - v$ the list $L(u) - \{a\}$. Since $G - v$ is $(k-1)$-choosable, it has a coloring from the assigned lists, and one can extend this coloring to an L-coloring of G by assigning the color a to v. So there exists an L-coloring for G in which v takes a.

Therefore, every k–list critical graph has at least k colorings from each k–list assignment so every k–list critical graph has m–number at most k.

A graph G is said to be edge k–choosable if the graph $L(G)$ is k–choosable, and the list chromatic index of G written $\chi'_\ell(G)$ is defined to be $\chi_\ell(L(G))$. As in the case of defining χ'–critical graphs, one can define a χ'_ℓ–critical graph G to be a graph in which for each proper subgraph H, $\chi'_\ell(H) < \chi'_\ell(G)$. We recall here the well known List Coloring Conjecture (LCC), which first appeared in print in [1].

Conjecture. [1] *Every graph G satisfies $\chi'_\ell(G) = \chi'(G)$.*

Suppose that G is a counterexample to the LCC with minimum number of vertices and edges. So for each edge uv of G we have $\chi'_\ell(G-uv) = \chi'(G-uv)$, and since $\chi'(G-uv) \leqslant \chi'(G) < \chi'_\ell(G)$, we conclude that $\chi'_\ell(G-uv) = \chi'_\ell(G)-1$. This means that G is a χ'_ℓ–critical graph and therefore χ'_ℓ–critical graphs may be useful to attack the LCC.

In the study of χ'_ℓ–critical graphs we have lead to the following conjecture.

Conjecture 2. *Every χ'_ℓ–critical graph is χ'–critical.*

Proposition 3. *The conjecture above is equivalent with the LCC, while its converse is implied by the LCC.*

Proof. It is straightforward to check that the list coloring conjecture implies Conjecture 2 and its converse. On the other hand suppose that Conjecture 2 is true, and G is a counterexample to the list coloring conjecture with minimum number of edges. As mentioned above G is χ'_ℓ–critical, and by Conjecture 2, it is χ'–critical. By removing an arbitrary edge uv from G we obtain a graph for which the list coloring conjecture holds. So $\chi'_\ell(G-uv) = \chi'(G-uv)$, and this means that $\chi'_\ell(G)-1 = \chi'(G)-1$, a contradiction. ■

In [5] it is proved that every bipartite multigraph fulfills the LCC. On the other hand we know that the only bipartite χ'–critical graphs are stars. So it is implied by Galvin's theorem [5] that the only bipartite χ'_ℓ–critical graphs are stars.

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