Chromatic Index Critical Graphs of Odd Order with Five Major Vertices

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ABSTRACT. In an earlier paper [11], we proved that there does not exist any Δ-critical graph of even order with five major vertices. In this paper, we prove that if G is a Δ-critical graph of odd order $2n+1$ with five major vertices, then $e(G) = n\Delta+1$. This extends an earlier result of Chetwynd and Hilton, and also completes our characterization of graphs with five major vertices. In [9], we shall apply this result to establish some results on class 2 graphs whose core has maximum degree two.

1. Introduction

Throughout this article, all graphs we deal with are finite, simple, and undirected. We use $V(G)$, $|G|$, $E(G)$, $e(G)$, $\Delta(G)$, and $\delta(G)$ to denote respectively the vertex set, order, edge set, size, maximum degree, and minimum degree of a graph G. We also use K_n, O_n, C_n, $G \cup H$, and rG to denote respectively the complete graph of order n, null graph of order n, cycle of order n, union of two vertex-disjoint graphs G and H, and vertex-disjoint union of r copies of a graph G. The *join* $G+H$ of two vertex-disjoint graphs G and H is the graph with the vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. Vertices of maximum degree in G are called *major vertices* and others are called *minor vertices*. We write $G \cong i_1^{n_1} i_2^{n_2} \cdots i_\Delta^{n_\Delta}$ if G has n_j vertices of degree i_j, where $j = 1, \cdots, \Delta = \Delta(G)$. If $x, y \in V(G)$, we use $N_G(x)$ (or simply $N(x)$), $N[x] = N(x) \cup \{x\}$, $d_G(x)$ (or simple $d(x)$), and $d_G(x,y)$ to denote respectively the neighborhood of x, close neighborhood of x, degree of x, and distance between x and y in G. If $A \subseteq V(G)$, we use $G - A$ (or simply $G - x$ if $A = \{x\}$) to denote the

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graph obtained by deleting the set of vertices A from G, and use $G[A]$ (or simply $G[x_1, x_2, \cdots, x_k]$ if $A = \{x_1, x_2, \cdots, x_k\}$) to denote the subgraph of G induced by A. If A and B are disjoint subsets of $V(G)$, we use $e_G(A, B)$ (or simply $e_G(x, B)$ if $A = \{x\}$) to denote the number of edges joining A with B. If $F \subseteq E(G)$, we use $G - F$ to denote the graph obtained by deleting F from G.

An *edge-colouring* of a graph G is a map $\pi : E(G) \to C$, where C is a set of colours, such that no two adjacent edges receive the same colour. If π is an edge-colouring of G with $|C| = k$, then π is called a k-colouring of G. The *chromatic index* $\chi'(G)$ of G is the least value of $|C|$ for which an edge-colouring $\pi : E(G) \to C$ exists. A well-known theorem of Vizing [12] states that, for any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. A graph G is said to be of *class* i, where $i = 1$, 2, if $\chi'(G) = \Delta(G) + i - 1$. A graph G is *overfull* if $e(G) \geq \Delta\lfloor \frac{|G|}{2} \rfloor + 1$. It is easy to see that if G is overfull, then G is of class 2.

The *core* G_Δ of a graph G is the subgraph of G induced by the major vertices of G. We use $d_\Delta(v)$ to denote the number of major vertices of G adjacent to v. If G is a connected class 2 graph with $\Delta(G) = \Delta$ and $\chi'(G - e) < \chi'(G)$ for each edge $e \in E(G)$, then G is said to be Δ-*critical*. From Vizing's Adjacency Lemma (abbreviated as VAL) (see Lemma 2.2 below) we know that if G is Δ-critical, then $|G_\Delta| \geq 3$.

In an earlier paper [11], we proved the following result.

Theorem 1.1. *There does not exist any Δ-critical graph of even order with five major vertices.*

In this paper, we shall apply Theorem 1.1 (together with many other results) to prove Theorem 1.2, which is an extension of a result of Chetwynd and Hilton (see [4] and [5]).

Theorem 1.2. *Let G be a graph of odd order $2n + 1 \geq 7$ with maximum degree $\Delta \geq 3$. Suppose G is Δ-critical and $|G_\Delta| = 5$. Then $e(G) = n\Delta + 1$.*

kThe graph G of Fig.1 is obtained from the Petersen graph by removing one vertex. Observe that G is 3-critical with six major vertices, and $e(G) = 12 < 4 \times 3 + 1$, which indicates that Theorem 1.2 cannot be further extended in general.

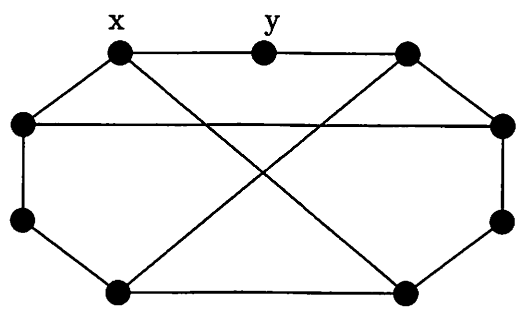

Fig. 1. G

2. Some useful results

In this section we give a list of results which we shall apply later. Proofs of Lemmas 2.1, 2.2, 2.3, 2.4, 2.5 can be found, for instance, in [14]. Lemma 2.6 to Lemma 2.8 were due to Chetwynd and Hilton ([4],[5]). Alternative/shorter proofs of these results and a proof of Lemma 2.10 can be found in Yap and Song [15].

Lemma 2.1 [12]. *For any graph G, $\chi'(G) \leq \Delta(G) + 1$.*

Lemma 2.2 [13]. *Suppose G is a Δ-critical graph and $vw \in E(G)$, where $d(v) = k$. Then*

(i) $d_\Delta(w) \geq \Delta - k + 1$ if $k < \Delta$;

(ii) $d_\Delta(w) \geq 2$ if $k = \Delta$;

(iii) $|G_\Delta| \geq max\{3, \Delta - \delta(G) + 2\}$.

Lemma 2.3 [12]. *Let G be a class 2 graph. Then G contains a k-critical subgraph for each k satisfying $2 \leq k \leq \Delta(G)$.*

Lemma 2.4 [8]. *There are no regular Δ-critical graphs for any $\Delta \geq 3$.*

Lemma 2.5 [5]. *Let $e = vw$ be an edge of a graph G. Suppose $d_\Delta(w) = 1$. Then $\Delta(G - w) = \Delta(G)$ implies that $\chi'(G - w) = \chi'(G)$.*

Lemma 2.6 [5]. *Let G be a connected graph of order n with $\Delta = \Delta(G) \geq 3$. Suppose $|G_\Delta| = 3$. Then G is of class 2 if and only if $G \cong (n-2)^{n-3}(n-1)^3$ (and thus n is odd).*

Lemma 2.7 [4]. *There does not exist any Δ-critical graph of even order with four major vertices.*

Lemma 2.8 [4]. *Let G be a Δ-critical graph of order $2n+1$ with $|G_\Delta| = 4$. Then either*
(i) $G \cong (2n-2)^{2n-3}(2n-1)^4$ or (ii) $G \cong (2n-2)(2n-1)^{2n-4}(2n)^4$. In particular, $e(G) = n\Delta + 1$.

Lemma 2.9 [7]. *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then G has a Hamilton cycle.*

Let J_s be a graph of order s and $G_0 = J_s + O_{s+2}$, and let G_0' be a spanning subgraph of G_0 such that each vertex of O_{s+2} is joined to at least $s-1$ vertices of J_s and at least one vertex of O_{s+2} is joined to exactly $s-1$ vertices of J_s.

Lemma 2.10 [15]. *A graph G of order $2n$ has a 1-factor if*

(i) $\delta(G) \geq n-1$ except when $G = G_0$.

(ii) $\delta(G) = n-2$ except when $G = G_0'$ or $G = 3K_3 + K_1$.

Lemma 2.11 [10]. *Let G be a graph of order $2n$. If $\delta(G) \geq n + |G_\Delta| - 2$, then G is of class 1.*

We observe that by following the proof of Lemma 2.11 given in [10] and choosing two nonadjacent major vertices z_1 and z_2 of G, we obtain the following result.

Lemma 2.12. *Suppose G is a graph of order $2n$ and G_Δ has a maximal matching $M = \{z_1z_r, z_2z_{r-1}, \cdots, z_kz_{r+1-k}\}$, where $r = |G_\Delta|$, $k \geq 2$, and $z_1z_2 \notin E(G)$. If $\delta(G) \geq n + |G_\Delta| - 3$, then G is of class 1.*

Lemma 2.13. *Suppose G is a Δ-critical graph of order $2n+1 \geq 11$ with $|G_\Delta| = 5$ and $\delta(G) \leq \Delta - 2$. Let $x \in V(G)$ be such that $d(x) = \delta(G)$. Then $G - x$ contains a 1-factor.*

Proof. Let $\delta = \delta(G)$, and let A be the set of major vertices of G. By VAL, $5 = |G_\Delta| \geq \Delta - \delta + 2$, and so $\Delta - 3 \leq \delta \leq \Delta - 2$. By VAL again, for any $v \in N(x)$, $d_\Delta(v) \geq \Delta - \delta + 1$. Hence by counting the number of edges incident with A in two different ways, we have

$$(\Delta - \delta + 1)\delta + 2((2n+1) - \delta) \leq 5\Delta. \tag{1}$$

Furthermore, if there is a vertex $u \in N(x)$ such that $d(u) = \delta$, then for any $v \in N[x]$, $d_\Delta(v) \geq \Delta - \delta + 1$, and we have the following better inequality:

$$(\Delta - \delta + 1)(\delta + 1) + 2((2n+1) - (\delta + 1)) \leq 5\Delta. \tag{2}$$

We next show that $\delta(G - x) \geq n - 2$.

Suppose $\delta = \Delta - 3$. Then (1) implies that $\Delta \geq n + 1$. Hence, if $d(v) \geq \Delta - 2$ for any $v \in N(x)$, then $\delta(G - x) \geq \Delta - 3 \geq n - 2$. On the other hand, if $d(v) = \delta$ for some $v \in N(x)$, then we need only to consider the case $\Delta = n + 1$. However, when $\Delta = n + 1$, (2) implies that $n \leq 5$ and thus $\Delta = n + 1 \leq 6$, which contradicts the fact that $\Delta - 3 = \delta = d(x) \geq d_\Delta(x) \geq \Delta - \delta + 1 = 4$.

Suppose $\delta = \Delta - 2$. Then (1) implies that $\Delta \geq n$. Hence, if $d(v) \geq \Delta - 1$ for any $v \in N(x)$, then $\delta(G - x) \geq \Delta - 2 \geq n - 2$. On the other hand, if $d(v) = \delta$ for some $v \in N(x)$, then by (2), $\Delta \geq n + 1$, and again $\delta(G - x) \geq \Delta - 3 \geq n - 2$.

From the above discussion, we have the following observation:

$\Delta \geq n$ and $\Delta = n$ <u>only if</u> $\delta = \Delta - 2$ <u>and the equality in</u> (1) <u>holds.</u>

By Lemma 2.10, $G - x$ has a 1-factor except when $G - x \in \{G_0, G_0', 3K_3 + K_1\}$. However, since $\Delta(G - x) - \delta(G - x) \leq \Delta - (\delta - 1) \leq 4 < 6 = \Delta(3K_3 + K_1) - \delta(3K_3 + K_1)$, we have $G - x \neq 3K_3 + K_1$. Suppose $G - x = G_0$ or $G - x = G_0'$. Then $s = n - 1$. We next show that $A \subseteq V(J_s)$.

Suppose otherwise. Let $a \in A \cap V(O_{s+2})$. Since $d_G(a) = \Delta \geq n = s + 1$, we have $ax \in E(G)$ and a is adjacent to every vertex $v \in V(J_s)$. Now $\Delta - 1 = d_{G-x}(a) \leq s$, together with the inequality $\Delta \geq n = s + 1$, implies that $\Delta = n = s + 1$. Thus from the above observation, it follows that $\delta = \Delta - 2$ and the equality in (1) holds. Since $ax \in E(G)$ and the equality in (1) holds, we have $|A \cap V(J_s)| = d_\Delta(a) = \Delta - \delta + 1 = 3$ and G has exactly $\delta = \Delta - 2 = n - 2$ vertices v with $d_\Delta(v) = \Delta - \delta + 1 = 3$. Moreover, these $n - 2$ vertices are all in $N(x)$ and G has no vertex v with $d_\Delta(v) \geq 4$. Let $B = V(O_{s+2}) \backslash N(x)$. Since $|N(x)| = \Delta - 2 = n - 2$, we have $|B| \geq (s + 2) - (n - 2) = 3$. If $v \in B$ is adjacent to every vertex of J_s in G, then $d_\Delta(v) = |A \cap V(J_s)| = 3$, which contradicts the fact that all the vertices $v \in V(G)$ with $d_\Delta(v) = 3$ are in $N(x)$. On the other hand, if $v \in B$ is adjacent to $s - 1$ vertices of J_s in G, then $d_G(v) = s - 1 = \Delta - 2$. However, since $a \in N(x)$ and $a \notin N(v)$, we have $N(x) \neq N(v)$, and thus there exists $w \in N(v) \backslash N(x)$ such that $d_\Delta(w) \geq \Delta - (\Delta - 2) + 1 = 3$, again contradicting the fact that all the vertices $v \in V(G)$ with $d_\Delta(v) = 3$ are all in $N(x)$ and G has no vertex v with $d_\Delta(v) \geq 4$. Hence $A \subseteq V(J_s)$ as required.

Finally, let $Y = G - V(O_{s+2})$. Since $A \subseteq V(J_s)$, we have $A \subseteq V(Y)$. As for any $u \in N(x) \cap A$, $d_Y(u) \geq d_\Delta(u) \geq 3$, we have $\Delta(Y) \geq 3$. This, together with the fact that $d_Y(v) \geq d_\Delta(v) \geq 2$ for any $v \in V(Y)$, implies that $e(Y) \geq s + 2$. Thus $(s + 2)\delta \leq e(V(O_{s+2}), V(Y)) = \sum_{v \in V(Y)} d_G(v) - 2e(Y) \leq 5\Delta + (s - 5)(\Delta - 1) + \delta - 2(s + 2) = s\Delta - 3s + \delta + 1 \leq s(\delta + 3) - 3s + \delta + 1$, and it follows that $\delta \leq 1$, which is false. Hence $G - x \neq G_0$, G_0', and thus $G - x$ has a 1-factor. ■

3. Proof of Theorem 1.2

Proof. Let $A = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of major vertices of G, let $B = V(G)\backslash A$, and let $\delta = \delta(G)$. As G is Δ-critical, we have

$$e(G) \leq n\Delta + 1. \tag{3}$$

Since each $v \in V(G)$ is adjacent to at least two major vertices of G, we have $2(2n+1) \leq 5\Delta$. Hence

$$\Delta \geq \frac{4n+2}{5}. \tag{4}$$

The results of Beineke and Fiorini [1] as well as Chetwynd and Yap [6] on Δ-critical graphs of order 7 and order 9 confirm that this theorem is true for $n = 3, 4$. Hence, from now on, we assume that

$$n \geq 5. \tag{5}$$

We shall now prove this theorem by induction on Δ. For $\Delta \leq 4$, by (4), we have $n \leq 4$, and so this theorem is true. Hence, from now on, we assume that $\Delta \geq 5$. By VAL, $5 = |A| \geq \Delta - \delta + 2$, and it follows that

$$\delta \geq \Delta - 3. \tag{6}$$

By Lemma 2.4, $\delta \neq \Delta$. Hence, from (6), we have two cases to consider.

Case 1. $\delta \leq \Delta - 2$.

Let $x \in V(G)$ be of degree δ. By Lemma 2.13, $G - x$ has a 1-factor F. Clearly, $G^* = G - F$ is of class 2, $\Delta(G^*) = \Delta - 1$ and $N_{G^*}(A) = V(G^*)$. By Lemma 2.3, G^* contains a $(\Delta - 1)$-critical subgraph H with at most five major vertices.

Suppose H has five major vertices. Then $N_{G^*}(A) = V(G^*)$ implies that $V(H) = V(G^*)$. Thus by the induction hypothesis on Δ, $e(H) = n(\Delta - 1) + 1$, and so $e(G) \geq e(H) + n = n\Delta + 1$. This, together with (3), yields $e(G) = n\Delta + 1$.

Suppose H has four major vertices, say w_2, w_3, w_4, w_5. Since for any $v \in V(G)$, $d_\Delta(v) \geq 2$, we have $x \in V(H)$ and $w_1 \in V(H)$. We claim that $V(H) = V(G^*)$. Suppose otherwise. By Lemma 2.7 and Lemma 2.8, $|G^*| - |H| \geq 2$ and $\delta(H) \geq \Delta(H) - 2 = \Delta - 3$. Since for any $v \in V(G^*)\backslash V(H)$, $d_\Delta(v) \geq 2$, we have $vw_1 \in E(G^*)$. As $d_H(w_1) \geq \delta(H) \geq \Delta - 3$, it follows that $2 \leq |V(G^*)\backslash V(H)| \leq d_{G^*}(w_1) - d_H(w_1) \leq (\Delta - 1) - (\Delta - 3) = 2$. Thus, $|V(G^*)\backslash V(H)| = 2$ and $d_H(w_1) = \Delta - 3$. By Lemma 2.8 again, $\Delta = \Delta(H) + 1 = |H| = (2n+1) - 2 = 2n - 1 \geq 9$ (by (5)). Now let

$y \in V(G^*) \backslash V(H)$. Then $d_\Delta(y) = 2$ (otherwise H has at most three major vertices) and thus $d_G(y) \geq \delta \geq \Delta - 3 \geq 6$. This, however, implies that H contains another vertex $v' \neq w$ such that $d_H(v') \leq \Delta - 3 = \Delta(H) - 2$, which contradicts Lemma 2.8. Hence $V(H) = V(G^*)$ as claimed. By Lemma 2.8, $e(H) = n(\Delta - 1) + 1$, and so $e(G) \geq e(H) + n \geq n\Delta + 1$. This, together with (3), yields $e(G) = n\Delta + 1$.

Suppose H has three major vertices, say w_1, w_2 and w_3. By Lemma 2.6, $\delta(H) = \Delta(H) - 1$, $|H|$ is odd, and $|H| = \Delta(H) + 1 = \Delta < |G|$. Let $U = V(G^*) \backslash V(H)$. Then $|U|$ (≥ 2) is even. Since w_i ($i = 1, 2, 3$) is adjacent to every other vertex of H in G^*, we have $\{w_1u_1, w_2u_2, w_3u_3\} \subset F$, where u_1 ,u_2, and u_3 are distinct vertices in U. Hence $|U| \geq 4$. As each minor vertex $v(\neq x, w_4, w_5)$ of H is of degree $\Delta - 1$ in G, we have

$$e_{G^*}(\{w_1, w_2, w_3\}, U) = 0 \quad \text{and} \quad e_{G^*}(V(H), U) \leq 2. \tag{7}$$

Suppose $A \subset V(H)$. Then the fact that $d_\Delta(u) \geq 2$ for any $u \in U$ implies that $e_{G^*}(U, A) \geq 4$, contradicting (7). Hence $A \cap U \neq \phi$. Let $w_5 \in A \cap U$. Then $d_\Delta(w_5) = 2$ (otherwise H cannot have exactly three major vertices), $w_5 \in \{u_1, u_2, u_3\}$, and $w_5w_4 \in E(G^*)$. Since $d_\Delta(w_5) = 2$ and $d(x) = \delta \leq \Delta - 2$, by VAL, $w_5x \notin E(G)$. Thus $\{w_1, w_2, w_3\} \cap N(x) \neq \phi$, which in turn implies that $x \in V(H)$ (otherwise $e_{G^*}(\{w_1, w_2, w_3\}, U) \neq 0$, which contradicts (7)). Hence, by VAL, $d_\Delta(w_i) \geq \Delta - d(x) + 1 \geq 3$, from which (by (7)) it follows that $w_4 \in V(H)$.

Finally, by (7), we have $|U| \ngeq 6$. Hence $|U| = 4$ and $\Delta = |H| = (2n + 1) - |U| = 2n - 3 \geq 7$ (by (5)). However, as $w_5 \in U$, $|U| = 4$ and $e_{G^*}(V(H), U) \leq 2$, we have $\Delta = d_G(w_5) \leq 5$, a contradiction.

Case 2. $\delta = \Delta - 1$.

Suppose $e(G) \leq n\Delta$. Then $((2n+1) - 5)(\Delta - 1) + 5\Delta = 2e(G) \leq 2n\Delta$, and it follows that

$$\Delta \quad \text{is even and} \quad \Delta \leq 2n - 4. \tag{8}$$

Since $\Delta \geq 5$, by (8),

$$\Delta \geq 6. \tag{9}$$

We shall use the following claims to settle Case 2.

Claim 1. *Suppose* $min\{d_\Delta(v) : v \in B\} \geq 3$. *Then* $G_\Delta \cong K_5$.

Proof. We prove this claim by contradiction. Suppose $\delta(G_\Delta) \leq 3$. Then, by VAL, $2 \leq \delta(G_\Delta) \leq 3$.

Suppose $\delta(G_\Delta) = 2$. Let w_1 be such that $d_\Delta(w_1) = 2$ and let w_1w_2, $w_1w_3 \in E(G)$, where $d_\Delta(w_2) \geq d_\Delta(w_3)$. By (8), G has a minor vertex, say u, such that $w_1u \notin E(G)$. We next show that $G' = G - \{w_1, w_2, u\}$ has a 1-factor F.

Since for any $v \in B$, $d_\Delta(v) \geq 3$, by VAL, $3((2n+1) - 5) \leq 5(\Delta - 2)$. This, together with (5), implies that

$$\Delta \geq n + 1. \tag{10}$$

Hence $\delta(G') \geq (\Delta - 1) - 3 \geq (n-1) - 2$. By Lemma 2.10, G' has a 1-factor F except when $G' \in \{G_0, G_0', 3K_3 + K_1\}$. However, since $\Delta(G') - \delta(G') \leq 4 < 6 = \Delta(3K_3 + K_1) - \delta(3K_3 + K_1)$, we have $G' \neq 3K_3 + K_1$. Suppose $G' = G_0$ or $G' = G_0'$. Then $2s + 2 = (2n+1) - 3$ and $\Delta - 4 \leq \delta(G') \leq s$, and it follows that $s = n - 2$ and $\Delta \leq s + 4$. Let $Y = G - V(O_{s+2})$. Then $|Y| = s + 3$ and $w_1, w_2 \in V(Y)$. Since $w_1w_4, w_1w_5 \notin E(G)$, we have $d_{G'}(w_i) \geq \Delta - 2 \geq s + 1$, $i = 4, 5$. Thus $w_4, w_5 \in V(J_s) \subset V(Y)$. If $w_3 \in V(O_{s+2})$, then $\Delta - 3 \leq d_{G'}(w_3) \leq s = n - 2 \leq \Delta - 3$ (by (10)), which implies that w_3 is adjacent to all the vertices of $V(Y)$ in G. Hence $d_\Delta(w_2) \geq d_\Delta(w_3) = 4$. By VAL, $3((2n+1) - 5) \leq 3(\Delta - 2) + 2(\Delta - 4)$. This, together with (5), implies that $\Delta \geq n + 2$, a contradiction. Hence $w_3 \in V(Y)$. We now have $A \subset V(Y)$ and $|V(Y)\backslash A| = s - 2$. Since for any $v \in V(Y)\backslash A$, $d_\Delta(v) \geq 3$, and by VAL, $e(G_\Delta) \geq 5$, we have $e(Y) \geq 3|V(Y)\backslash A| + e(G_\Delta) \geq 3(s-2) + 5 = 3s - 1$. Now by counting the number of edges joining $V(O_{s+2})$ and $V(Y)$ in two different ways, we have

$$(s+2)(\Delta - 1) = e_G(V(O_{s+2}), V(Y)) \leq 5\Delta + (s-2)(\Delta - 1) - 2e(Y).$$

It follows that $\Delta \geq 2e(Y) - 4 \geq 2(3s - 1) - 4 = 6s - 6$, which, together with the fact that $\Delta \leq s + 4$, yields $6s \leq 10$, contradicting the fact that $s = n - 2 \geq 3$. Hence G' has a 1-factor F.

Clearly, $G^* = G - (F \cup \{w_1w_2\})$ is of class 2, $\Delta(G^*) = \Delta - 1$, and $A \cup \{u\}$ is the set of major vertices of G^*. Since w_1 is adjacent to only one major vertex w_3 in G^*, and $\Delta(G^* - w_1) = \Delta(G^*)$, by Lemma 2.5, $G^* - w_1$ is of class 2. By Lemma 2.3, $G^* - w_1$ contains a $(\Delta - 1)$-critical subgraph H, which has at most four major vertices u, w_2, w_4, w_5. By (8), Δ is even, and so by Lemma 2.6 and Lemma 2.7, $|H| = \Delta(H) + 2 = \Delta + 1$ and $\delta(H) = \Delta(H) - 1$. Since for any $v \in N(w_1)\backslash A$, $d_{G^*-w_1}(v) = \Delta - 3 = \Delta(H) - 2$, we have $v \notin V(H)$. Thus $(N(w_1)\backslash A) \cap V(H) = \phi$. Hence $\Delta + 1 = |H| \leq |G^* - w_1| - |N(w_1)\backslash A| = 2n - (\Delta - 2)$, and it follows that $\Delta \leq n$, which contradicts (10).

Next suppose $\delta(G_\Delta) = 3$. Then $G_\Delta \cong 3^2 4^3$ or $G_\Delta \cong 3^4 4^1$. In either case we can rename the major vertices, if necessary, so that $w_1w_2 \notin E(G)$

(and $w_3w_4 \notin E(G)$ if $G_\Delta \cong 3^4 4^1$). By (8), there exist two minor vertices, say x and y, of G such that $xw_1, yw_1 \notin E(G)$. Since $d_\Delta(y) \geq 3$, by symmetry of w_3 and w_4, we may assume that $yw_4 \in E(G)$. Clearly, $e(G_\Delta) \geq 8$. Now by VAL, $3(2n-4) \leq 5\Delta - 2e(G_\Delta)$, and it follows that

$$\Delta \geq n+2. \tag{11}$$

Furthermore, if $w_3w_4 \in E(G)$, then $e(G_\Delta) = 9$. By VAL again, we have the following better inequality:

$$\Delta \geq n+3. \tag{12}$$

Now by (8) and (11), we have $n \geq 6$, and thus

$$\Delta \geq n+2 \geq 8. \tag{13}$$

We next show that $G-x$ has a 1-factor F_1 containing w_1w_4 and w_3w_5, and $G-F_1-y$ has a 1-factor F_2 containing w_1w_3.

Let $G' = G - \{x, w_1, w_3, w_4, w_5\}$. By (11), $\delta(G') \geq (\Delta-1)-5 \geq (n-2)-2$. Thus by Lemma 2.10, G' has a 1-factor F' except when $G' \in \{G_0, G_0', 3K_3+K_1\}$. However, since $\Delta(G') - \delta(G') \leq 3$, we have $G' \neq 3K_3+K_1$. Suppose $G' = G_0$ or $G' = G_0'$. Then $s = n-3$ and $\Delta - 6 \leq \delta(G') \leq s$. This, together with (11), implies that $s+5 \leq \Delta \leq s+6$. Let $Y = G - V(O_{s+2})$. Then $|Y| = s+5$ and $\{x, w_1, w_3, w_4, w_5\} \subset V(Y)$. Since $w_1w_2 \notin E(G)$, we have $d_{G'}(w_2) \geq \Delta - 4 \geq s+1$. Hence $w_2 \in V(Y)$. Thus $A \subset V(Y)$ and $|V(Y)\backslash A| = (s+5)-5 = s$. As $e(G_\Delta) \geq 8$ and $d_\Delta(v) \geq 3$ for any $v \in V(Y)\backslash A$, we have $e(Y) \geq 3|V(Y)\backslash A| + e(G_\Delta) \geq 3s+8$. Thus

$$(s+2)(\Delta-1) = e_G(V(O_{s+2}), V(Y)) = 5\Delta + s(\Delta-1) - 2e(Y),$$

and it follows that $6s+14 \leq 3\Delta$, which, together with the inequality $\Delta \leq s+6$, implies that $3s \leq 4$, contradicting the fact that $s = n-3 \geq 2$. Hence $G' \neq G_0, G_0'$, and G' has a 1-factor F'. Let $F_1 = F' \cup \{w_1w_4, w_3w_5\}$ and let $z \in B$ (z could be y) be such that $zw_2 \in F_1$.

We now show that $G-F_1-y$ has a 1-factor F_2 containing w_1w_3. Let $G'' = G - F_1 - \{y, w_1, w_3\}$. By (11), $\delta(G'') \geq \Delta - 5 \geq (n-1)-2$. Thus by Lemma 2.10, G'' has a 1-factor F'' except when $G'' \in \{G_0, G_0', 3K_3+K_1\}$. Since $\Delta(G'') - \delta(G'') \leq (\Delta-1)-(\Delta-5) = 4$, $G'' \neq 3K_3+K_1$. Suppose $G'' = G_0$ or $G'' = G_0'$. Then $s = n-2$ and $\Delta - 5 \leq \delta(G'') \leq s$. This, together with (11), implies that $s+4 \leq \Delta \leq s+5$. Let $Y = G - V(O_{s+2})$. Then $|Y| = s+3$ and $w_1, w_3 \in V(Y)$. Since $xw_1, w_1w_2, w_1w_4, w_3w_5 \notin E(G-F_1)$, $d_{G''}(v) \geq \Delta - 3 \geq s+1$ for any $v \in \{x, w_2, w_4, w_5\}$. Hence $x, w_2, w_4, w_5 \in V(J_s) \subset V(Y)$. We now have $(A \cup \{x\}) \subset V(Y)$ and $|V(Y)\backslash A| = (s+3)-5 = s-2$. As $e_{G-F_1}(A, A) \geq 6$, and for any $v \in$

$V(Y)\backslash A$, $e_{G-F_1}(v, A) \geq 2$, we have $e(Y) \geq 2|V(Y)\backslash A| + e_{G-F_1}(A, A) \geq 2(s-2)+6 = 2s+2$. Thus

$$(s+2)(\Delta-2) = e_{G-F_1}(V(O_{s+2}), V(Y)) = 6(\Delta-1)+(s-3)(\Delta-2)-2e(Y),$$

and it follows that $\Delta \geq 2e(Y)-4 \geq 2(2s+2)-4 = 4s$, which, together with the inequality $\Delta \leq s+5$, implies that $3s \leq 5$, contradicting the fact that $s = n-2 \geq 3$. Hence $G' \neq G_0$, G_0', and G'' has a 1-factor F''. Clearly , $F_2 = F'' \cup \{w_1w_3\}$ is a 1-factor of $G-F_1-y$. Let $z', w, w' \in V(G'')$ (each of z', w, w' could be any vertex in $V(G'')$) be such that $z'w_2, ww_4, w'w_5 \in F_2$.

Let $G^* = G - F_1 - F_2$. Then G^* is of class 2, and $A \cup \{x, y\}$ is the set of major vertices of G^*. Since w_1 is adjacent to only one major vertex w_5 in G^* and $\Delta(G^* - w_1) = \Delta(G^*)$, by Lemma 2.5, $G^* - w_1$ is of class 2. By Lemma 2.3, $G^* - w_1$ has a $(\Delta-2)$-critical subgraph H with at most five major vertices w_2, w_3, w_4, x, and y.

Suppose H has five major vertices. Since $e_{G^*}(w_5, \{w_2, w_4\}) \geq 1$, we have $w_5 \in V(H)$. From the choice of F_1 and F_2, it follows that, for any $v \in B\backslash\{z, z', w, w'\}$, $e_{G^*}(v, A) \geq 3$. Thus $(B\backslash\{z, z', w, w'\}) \subset V(H)$, and $2n = |G^* - w_1| \geq |H| \geq |G^* - \{w_1, z, z', w, w'\}| \geq 2n-4$. By Theorem 1.1, $|H|$ is odd, and so either $|H| = 2n-1$ or $|H| = 2n-3$. By the induction hypothesis on Δ, we have $e(H) = (\Delta-2)\frac{|H|-1}{2}+2$. Suppose $|H| = 2n-1$. Then $e(H) = (\Delta-2)(n-1)+1$ and so $e(G) \geq e(H)+|F_1|+|F_2|+2(\Delta-3) \geq n\Delta+\Delta-3 \geq n\Delta+5$ (by (13)), contradicting the assumption that $e(G) \leq n\Delta$. Next suppose $|H| = 2n-3$. Then $e(H) = (\Delta-2)(n-2)+1$ and so $e(G) \geq e(H)+|F_1|+|F_2|+4(\Delta-3)-6 \geq n\Delta-2\Delta-13 \geq n\Delta+3$ (by (13)), again contradicting the assumption that $e(G) \leq n\Delta$. Hence H has at most four major vertices.

Suppose H has four major vertices. By (8), Δ is even, and so by Lemma 2.7 and Lemma 2.8, $|H| = \Delta(H)+1 = \Delta-1$. Since $w_2w_3, yw_4 \in E(G^*)$, we have $w_2, w_3, w_4 \in V(H)$ (otherwise H would have at most three major vertices), and at least one of w_3 and w_4, say w_3, is a major vertex in H. As $|H| = \Delta(H)+1$, it follows that w_3 is adjacent to all the other vertices in H. In particular, w_3 is adjacent to w_4 in H. Thus $w_3w_4 \in E(G)$, and so by (12), $\Delta \geq n+3$. However, since for any $v \in N(w_1)\backslash A$, $d_{G^*-w_1}(v) = \Delta-4 = \Delta(H)-2$, by Lemma 2.8, H contains at most one vertex $v \in N(w_1)\backslash A$. Thus $\Delta-1 = |H| \leq |G^*-w_1| - (|N(w_1)\backslash A|-1) = 2n-(\Delta-4)$, and it follows that $\Delta \leq n+2$, which is a contradiction.

Suppose H has three major vertices. Then by Lemma 2.6, $|H| = \Delta(H)+1 = \Delta-1$ and $\delta(H) = \Delta(H)-1 = |H|-2$. Since for any $v \in N(w_1)\backslash A$, $d_{G^*-w_1}(v) = \Delta-4 = \Delta(H)-2$, we have $v \notin V(H)$. Thus

$(N(w_1)\backslash A) \cap V(H) = \phi$ and so $\Delta - 1 = |H| \le |G^* - w_1| - |N(w_1)\backslash A| = 2n - (\Delta - 4)$. This, together with (11), implies that $\Delta = n + 2$ and $|H| = |G^* - w_1| - |N(w_1)\backslash A|$. Hence $w_2, w_3, w_4, w_5 \in V(H)$. As $w_3w_5 \in F_1$, $w_3w_5 \notin E(H)$. Since $\delta(H) = \Delta(H) - 1 = |H| - 2$, w_3 is adjacent to any other vertex of H except w_5. In particular, w_3 is adjacent to w_4 in H. Thus $w_3w_4 \in E(G)$. By (12), $\Delta \ge n + 3$, which contradicts the fact that $\Delta = n + 2$. ∎

Claim 2. *If $6 \le \Delta \le n$ and $G[B]$ has a maximum matching M with $|M| \ge n - 3$, then $\Delta(G_\Delta) \ge 3$. In particular, $e(G_\Delta) \ge 6$.*

Proof. Suppose otherwise that $\Delta(G_\Delta) = 2$. Then by VAL, $\Delta(G_\Delta) = \delta(G_\Delta) = 2$, and so $G_\Delta \cong C_5$. Assume that $G_\Delta \cong (w_1, w_2, w_3, w_4, w_5)$. We first show that $G[B]$ has a matching M' of size $n - 3$ such that one of the two M'-unsaturated vertices is adjacent to at most four major vertices in G.

Since $|M| \ge n - 3$, if $|M| = n - 2$, then by (8), there exists $x \in B$ such that $xw_1 \notin E(G)$. Let $xy \in M$. Then $M' = M - \{xy\}$ is a required matching because $d_\Delta(x) \le 4$. On the other hand, if $|M| = n - 3$, let u and v be the two M-unsaturated vertices. Then $uv \notin E(G)$. If $min\{d_\Delta(u), d_\Delta(v)\} \le 4$, then $M' = M$ is a required matching of $G[B]$. Hence we assume that $d_\Delta(u) = d_\Delta(v) = 5$. Let $x_1x_1', \cdots, x_tx_t' \in M$ and $y_1y_1', \cdots, y_sy_s' \in M$ be such that $ux_i, ux_i', uy_j \in E(G)$ and $uy_j' \notin E(G)$, where $2t + s = |N_G(u)\backslash A| = (\Delta - 1) - 5 = \Delta - 6$, $i = 1, 2, \cdots, t$, $j = 1, 2, \cdots, s$. Let $C = \{x_1, x_1', \cdots, x_t, x_t', y_1', \cdots, y_s'\}$. Since M is a maximum matching of $G[B]$, v is not adjacent to any vertex of C. Suppose $d_\Delta(w) = 5$ for any $w \in C$. Then by VAL, $2((2n - 4) - |C \cup \{u, v\}|) + 5|C \cup \{u, v\}| \le 5(\Delta - 2)$, which implies that $\Delta \ge 2n - 5$, contradicting the fact that $6 \le \Delta \le n$. Hence there exists $w \in C$ such that $d_\Delta(w) \le 4$. Let $ww' \in M$. Then $uw' \in E(G)$. Now $M' = (M - \{ww'\}) \cup \{uw'\}$ is a required matching of $G[B]$.

The above shows that in either case, $G[B]$ has a matching M' of size $n - 3$ such that one of the M'-unsaturated vertices is adjacent to at most four major vertices in G. Let $x, y \in B$ be the two M'-unsaturated vertices and let $d_\Delta(x) \le 4$. As $G_\Delta \cong C_5$, assume that $xw_1 \notin E(G)$. We next show that $G - x$ has a 1-factor.

Since $d_\Delta(y) \ge 2$ and $G_\Delta \cong C_5$, we may choose $w_k (\ne w_1) \in A$ such that $yw_k \in E(G)$ and $d_G(w_1, w_k) = mim\{d_G(w_1, w_j) : j = 2, 3, 4, 5\}$. Clearly, $G_\Delta - w_k$ has a 1-factor $\{w_1w_j, e\}$, where $j \in \{2, 5\}$, and $e \in E(G_\Delta)$. By the symmetry of w_2 and w_5, assume that $j = 2$. Now $F = M' \cup \{w_1w_2, e, yw_k\}$ is a 1-factor of $G - x$.

Finally, let $G^* = G - F$. Then G^* is of class 2 and $A \cup \{x\}$ is the set of major vertices. As $d_\Delta(w_1) = 2$, $w_1w_2 \in F$, and $xw_1 \notin E(G)$, it follows that w_1 is adjacent to only one major vertex, namely w_5, in G^*. Since $\Delta(G^* - w_1) = \Delta(G^*)$, by Lemma 2.5, $G^* - w_1$ is of class 2, and so by Lemma 2.3, $G^* - w_1$ contains a $(\Delta - 1)$-critical subgraph H with at most four major vertices y, w_2, w_3, w_4. By (8), Δ is even, and so by Lemma 2.6, Lemma 2.7, and Lemma 2.8, $|H| \neq \Delta(H) + 1 = (\Delta - 1) + 1 = \Delta$. Thus by Lemma 2.6 and Lemma 2.8, $|H| = \Delta(H) + 2 = \Delta + 1$ and H has exactly four major vertices y, w_2, w_3, w_4. Note that $w_2w_4 \notin E(H)$. Thus $yw_2 \in E(H)$ (because $d_H(w_2) = \Delta(H) = |H| - 2$). By the assumption of w_k, it follows that $w_k = w_5$. Now $yw_5 \in F$. This, together with the fact that w_4 is a major vertex in H, implies that $w_5w_4 \in E(H)$. Thus $w_5 \in V(H)$ (otherwise H would have at most three major vertices). Hence $\{w_2, w_3, w_4, w_5\} \subset V(H)$. As w_2 is a major vertex in H and $w_1w_2 \in F$, it follows that $(N_G(w_2)\backslash A) \subset V(H)$. Hence $|H| \geq |N_G(w_2)\backslash A| + |\{w_2, w_3, w_4, w_5\}| = (\Delta - 2) + 4 = \Delta + 2$, which contradicts the fact that $|H| = \Delta + 1$. ■

In what follows, we use m_i ($i = 2, 3, 4, 5$) to denote the number of minor vertices of G each of which is adjacent to exactly i major vertices in G. We are now in a position to prove Case 2.

Subcase 2.1. $min\{d_\Delta(v) : v \in B\} = 4$.

By Claim 1, $G_\Delta \cong K_5$. By VAL and (8), $4m_4 + 5m_5 = e_G(B, A) = 5(\Delta - 4) \leq 5((2n - 4) - 4)$. This, together with $m_4 + m_5 = |B| = 2n - 4$, implies that $m_4 \geq 20$. Now from $2n - 4 = |B| = m_4 + m_5$, it follows that $n \geq \frac{1}{2}m_5 + 12 \geq 12$. By VAL again, $4(2n - 4) \leq e_G(B, A) = 5(\Delta - 4)$. This, together with the inequality $n \geq 12$, yields

$$\Delta \geq n + 8. \tag{14}$$

By (8), there exists $x_i \in B$ ($i = 1, 2, 3$) satisfying $w_1x_i \notin E(G)$. By (14) and Dirac's theorem (see Lemma 2.9), $G - \{x_1, w_1, w_2, w_3, w_4\}$ has a 1-factor F_1'. Let $F_1 = F_1' \cup \{w_1w_2, w_3w_4\}$. By (14) and Dirac's theorem, $G - F_1 - \{x_2, w_1, w_3\}$ has a 1-factor F_2'. Let $F_2 = F_2' \cup \{w_1w_3\}$. Then by (14) and Dirac's theorem again, $G - F_1 - F_2 - \{x_3, w_1, w_4\}$ has a 1-factor F_3'. Let $F_3 = F_3' \cup \{w_1w_4\}$. Clearly, $G^* = G - F_1 - F_2 - F_3$ is of class 2 and $A \cup \{x_1, x_2, x_3\}$ is the set of major vertices of G^*. Since w_1 is adjacent to only one major vertex w_5 in G^* and $\Delta(G^* - w_1) = \Delta(G^*)$, by Lemma 2.5, $\chi'(G^* - w_1) = \chi'(G^*)$, and so $G^* - w_1$ is of class 2 with six major vertices $w_2, w_3, w_4, x_1, x_2, x_3$. However, from the choice of F_1 ,F_2 and F_3, it follows that $w_3w_4 \notin E(G^* - w_1)$ and $w_3x_3, w_4x_2 \in E(G^* - w_1)$. Therefore the core of $G^* - w_1$ has a maximal matching containing w_3x_3, w_4x_2. Since $G^* - w_1$ has six major vertices and by (14), $\delta(G^* - w_1) \geq \delta(G^*) - 1 =$

$(\delta(G)-3)-1=\Delta-5\geq n+6-3$. Thus by Lemma 2.12, G^*-w_1 is of class 1, which is a contradiction.

Subcase 2.2. $min\{d_\Delta(v):v\in B\}=3$.

By Claim 1, $G_\Delta\cong K_5$. By VAL, $3((2n+1)-5)\leq e_G(B,A)=5(\Delta-4)$. This, together with (5), implies that

$$\Delta\geq n+3. \tag{15}$$

By (8) and (15), we have

$$n\geq 7. \tag{16}$$

Let $x\in B$ satisfying $d_\Delta(x)=3$. Suppose there exists a vertex $y\in B\backslash N[x]$ such that $d_\Delta(y)=3$. Since $G_\Delta\cong K_5$, we may assume that $xw_i\in E(G)$ $(i=1,2,3)$ and $yw_1,yw_j,yw_k\in E(G)$, where $w_j,w_k\in A$ and $w_j\neq w_2$. By (8), there exists another vertex $z(\neq y)\in B$ such that $xz\notin E(G)$. We next show that $G-z$ has a 1-factor F_1 containing xw_2,yw_j, $G-F_1-y$ has a 1-factor F_2 containing xw_3, and $G-F_1-F_2-\{x,y,z\}$ has a 1-factor F_3.

Let $G_1=G-\{x,y,z,w_2,w_j\}$. By (15), $\delta(G_1)\geq(\Delta-1)-5\geq(n-2)-1$. By Lemma 2.10, G_1 has a 1-factor F_1' except when $G_1=G_0$. Suppose $G_1=G_0$. Then $s=n-3$ and $\Delta-6\leq\delta(G_1)\leq s$. Thus $\Delta\leq s+6$. This, together with (15), implies that $\Delta=s+6$ and $\delta(G_1)=\Delta-6=s$. As $d_\Delta(w_2)=4$ and $xw_2\in E(G)$, it follows that w_2 is adjacent to at most $\Delta-5=s+1$ vertices of $B\backslash\{x,y,z\}$ in G_1. Thus G_1 has at most $s+1$ vertices of degree $\delta(G)=\Delta-6=s$, which contradicts the fact that $G_1=G_0$ has $s+2$ vertices of degree s. Hence $G_1\neq G_0$ and thus G_1 has a 1-factor F_1'. Clearly, $F_1=F_1'\cup\{xw_2,yw_j\}$ is a 1-factor of $G-z$.

Let $G_2=G-F_1-\{x,y,w_3\}$. By (15), $\delta(G_2)\geq(\Delta-2)-3\geq(n-1)-1$. By Lemma 2.10 again, G_2 has a 1-factor F_2' except when $G_2=G_0$. Suppose $G_2=G_0$. Then $s=n-2$ and $\Delta-5\leq\delta(G_2)=\delta(G_0)=s$. Thus $\Delta\leq s+5$. This, together with (15), implies that $\Delta=s+5$ and $\delta(G_2)=\Delta-5=s$. As $xw_3\in E(G_2)$ and $e_{G_2}(w_3,\{w_1,w_2,w_4,w_5\})\leq 4$, it follows that w_3 is adjacent to at most $\Delta-5=s$ vertices of $B\backslash\{x,y,z\}$ in G_2. Thus G_2 has at most s vertices of degree $\delta(G_2)=\Delta-5=s$, which contradicts the fact that $G_2=G_0$ has $s+2$ vertices of degree s. Hence $G_2\neq G_0$ and thus G_2 has a 1-factor F_2' and $F_2=F_2'\cup\{xw_3\}$ is a 1-factor of $G-y$.

Let $G_3=G-F_1-F_2$ and $G_4=G_3-\{x,y,z\}$. By (15), $\delta(G_4)\geq(\Delta-3)-3\geq(n-1)-2$. By (16) and Lemma 2.10, G_4 has a 1-factor F_3 except when $G_4\in\{G_0,G_0'\}$. Suppose $G_4\in\{G_0,G_0'\}$. Then $s=n-2$ and $\Delta-6\leq\delta(G_4)\leq s$. Thus $\Delta\leq s+6$. By (15), we have $s+5\leq\Delta\leq s+6$. Since $xv\notin E(G_3)$ for any $v\in\{w_2,w_3,w_4,w_5\}$, we have

$d_{G_4}(v) \geq \Delta - 4 \geq s+1$, and so $w_2, w_3, w_4, w_5 \in V(J_s)$. Suppose $w_1 \in V(J_s)$. Then $A \subset V(J_s)$. Observe that for any $u \in V(O_{s+2})$, $d_{G_4}(u) \geq \Delta - 6 \geq (s+5)-6 = s-1$. This, together with $A \subset V(J_s)$, implies that each vertex of O_{s+2} is adjacent to at least four vertices of A in G_4. Hence $d_\Delta(u) \geq 4$ for any $u \in V(O_{s+2})$. As $min\{d_\Delta(v) : v \in B\} = 3$, by VAL, it follows that $3|B \backslash V(O_{s+2})| + 4|O_{s+2}| \leq e_G(B, A) = 5(\Delta - 4)$. But then $\Delta \geq s+7$ (because $s = n-2 \geq 5$ (by (16))), contradicting the fact that $\Delta \leq s+6$. Suppose $w_1 \in V(O_{s+2})$. Then $\Delta - 5 \leq d_{G_4}(w_1) \leq s = n-2 \leq \Delta - 5$ (by (15)), which implies that $s = \Delta - 5$ and w_1 is adjacent to all the vertices of J_s. Thus there exists $u \in V(O_{s+2})$ such that $w_1 u \in F_1$. Since $d_{G_4}(u) \geq \delta(G_4) \geq \Delta - 6 = s-1$, $|J_s| = s$, and $w_2, w_3, w_4, w_5 \in V(J_s)$, it follows that u is adjacent to at least three vertices of w_2, w_3, w_4, w_5 in G. This, together with the fact that $uw_1 \in E(G)$, implies that $d_\Delta(u) \geq 4$. As $d_\Delta(v) \geq 3$ for any $v \in B$, by VAL, $3(2n-4-1)+4 \leq e_G(B,A) = 5(\Delta-4)$, which, by (16), implies that $\Delta \geq n+4 = s+6$ (because $s = n-2$), contradicting the fact that $s = \Delta - 5$. Hence $G_4 \notin \{G_0, G_0'\}$. Consequently, G_4 has a 1-factor F_3'.

Finally, let $G^* = G_3$ if $yz \notin E(G)$ or $G^* = G_3 - (F_3 \cup \{yz\})$ if $yz \in E(G)$. Then G^* is of class 2. As x is adjacent to only one major vertex w_1 in G^* and $\Delta(G^* - x) = \Delta(G^*)$, by Lemma 2.5, $G^* - x$ is of class 2 with six major vertices y, z, w_2, w_3, w_4, w_5. From the choice of F_1, F_2 and F_3, it follows that y is adjacent to only one major vertex w_k in $G^* - x$. Observe that $\Delta(G^* - \{x,y\}) = \Delta(G^* - x)$. By Lemma 2.5, $G^* - \{x,y\}$ is of class 2 with four major vertices. By Lemma 2.3, $G^* - \{x,y\}$ contains a Δ'-critical subgraph H with at most four major vertices, where $\Delta' = \Delta - 2$ if $G^* = G_3$, and $\Delta' = \Delta - 3$ if $G^* = G_3 - (F \cup \{yz\})$. Note that, if $\Delta' = \Delta - 2$, then $|H| \geq \Delta(H) + 1 = (\Delta - 2) + 1 = \Delta - 1$. If $\Delta' = \Delta - 3$, by (8), Δ is even, and so by Lemma 2.6 and Lemma 2.8, $|H| = \Delta(H) + 2 = (\Delta - 3) + 2 = \Delta - 1$. Hence in both cases, we have $|H| \geq \Delta - 1$. Observe that for any $v \in N_{G^* - \{x,y\}}(x) \backslash A$, $d_{G^* - \{x,y\}}(v) \leq \Delta' - 2 = \Delta(H) - 2$. By Lemma 2.6 and Lemma 2.8 again, H contains at most one vertex $v \in N(x) \backslash A$. Thus $\Delta - 1 \leq |H| \leq |G^* - \{x,y\}| - (|N_{G^* - \{x,y\}}(x) \backslash A| - 1) \leq (2n-1) - (\Delta - 5)$, and it follows that $\Delta \leq n+2$, contradicting (15). Hence for any $v \in B \backslash N[x]$, $d_\Delta(v) \geq 4$ and $G[C]$ is a complete subgraph of order m_3 in G, where $C = \{v \in B : d_\Delta(v) = 3\}$.

As $d_\Delta(v) \geq 4$ for any $v \in B \backslash N[x]$, we have $m_3 \leq |N[x] \cap B| = \Delta - 3$. By VAL,

$$3(\Delta - 3) + 4(2n - \Delta - 1) \leq e_G(B, A) = 5(\Delta - 4). \tag{17}$$

This, together with (16), implies that $\Delta \geq n+4$. Suppose $\Delta = n+4$. Then (8) and (17) imply that $n = 8$ and $\Delta = n + 4 = 12$. Since $m_3 =$

$(2n-4)-(m_4+m_5)$ and by VAL, $3m_3+4m_4+5m_5 = 5(\Delta-4) = 5n$, we have $m_4+2m_5 = 4$. As $m_3 \le \Delta-3 = 9$, it follows that $m_3 = 9$, $m_4 = 2$, and $m_5 = 1$ or $m_3 = 8$, $m_4 = 4$, and $m_5 = 0$. In both cases, as $G[C] = K_{m_3}$, by counting the number of edges joining C with $B \backslash C$ in two different ways, we have $14 = 2(11-4-2)+(11-5-2) \le e_G(B\backslash C, C) \le 8$, which is a contradiction. Hence

$$\Delta \ge n+5. \tag{18}$$

Since $xw_i \in E(G)$ $(i = 1,2,3)$, we have $xw_5 \notin E(G)$. By (8), there exist another two vertices $y, z \in B$ such that yw_5, $zw_5 \notin E(G)$. As $d_\Delta(y)$, $d_\Delta(z) \ge 3$, we may assume that $yw_2, zw_2 \in E(G)$. We next show that $G-z$ has a 1-factor F_1 containing w_5w_2, $G-F_1-y$ has a 1-factor F_2 containing w_5w_3, and $G-F_1-F_2-x$ has a 1-factor containing w_5w_4.

Let $G_1 = G-\{w_5, w_2, x, y, z\}$ if $xy \in E(G)$ (or $G_1 = G-\{w_5, w_2, w_3, w_4, z\}$ if $xy \notin E(G)$). By (18), $\delta(G_1) \ge (\Delta-1)-5 \ge n-1$. By Dirac's Theorem, G_1 has a 1-factor F_1'. Let $F_1 = F_1' \cup \{w_5w_2, xy\}$ (or $F_1 = F_1' \cup \{w_5w_2, w_3w_4\}$) and let $G_2 = G-F_1-\{w_5, w_3, x, y, z\}$ if $xz \in E(G)$ (or $G_2 = G-F_1-\{w_5, w_3, w_2, w_4, y\}$ if $xz \notin E(G)$). By (18), $\delta(G_2) \ge (\Delta-2)-5 \ge n-2$. By Dirac's Theorem again, G_2 has a 1-factor F_2'. Let $F_2 = F_2' \cup \{w_5w_3, xz\}$ (or $F_2 = F_2' \cup \{w_5w_3, w_2w_4\}$) and let $G_3 = G-F_1-F_2-\{w_5, w_4, x\}$. By (18), $\delta(G_3) \ge (\Delta-3)-3 \ge n-1$. Thus by Dirac's Theorem, G_3 has a 1-factor F_3'. Let $F_3 = F_3' \cup \{w_5w_4\}$.

Next, let $G^* = G-F_1-F_2-F_3$. Then G^* is of class 2, and $A \cup \{x, y, z\}$ is the set of major vertices of G^*. As w_5 is adjacent to only one major vertex w_1 in G^* and $\Delta(G^*-w_5) = \Delta(G^*)$, by Lemma 2.5, G^*-w_5 is of class 2, and x is adjacent to only two major vertices w_2 and w_3 in G^*-w_5. We now show that G^*-w_5 has a 1-factor F_4 containing xw_2.

Let $G_4 = G^*-\{w_5, w_2, x\}$. By (18), $\delta(G_4) \ge (\Delta-4)-3 \ge (n-1)-1$. By Lemma 2.10, G_4 has a 1-factor F_4' except when $G_4 = G_0$. Suppose $G_4 = G_0$. Then $s = n-2$ and $\Delta-7 \le \delta(G_4) = \delta(G_0) \le s$. Thus $\Delta \le s+7$. This, together with (18), implies that $\Delta = s+7$ and $\delta(G_4) = \Delta-7 = s$. As w_2 is adjacent to at most $\Delta-7 = s$ vertices of $B\backslash\{x, y, z\}$ in G^*-w_5, it follows that G_4 has at most s vertices of degree $\delta(G_4) = \Delta-7 = s$, which contradicts the fact that $G_4 = G_0$ has $s+2$ vertices of degree s. Hence $G_4 \ne G_0$ and thus G_4 has a 1-factor F_4'.

Let $F_4 = F_4' \cup \{xw_2\}$ and let $G^{**} = G^*-w_5-F_4$. Then G^{**} is of class 2 and x is adjacent to only one major vertex w_3 in G^{**}. Since $\Delta(G^{**}-x) = \Delta(G^{**}) = \Delta-4$, by Lemma 2.5, $G^{**}-x$ is of class 2, and y, z, w_2, w_4 are the four major vertices of $G^{**}-x$. By Lemma 3, $G^{**}-x$ contains a $(\Delta-4)$-critical subgraph H with at most four major vertices y, z, w_2, w_4. Observe that for any $v \in N_{G^{**}}(x)\backslash A$, $d_{G^{**}-x}(v) = (\Delta - 5) - 1 = \Delta(H) - 2$. By Lemma 2.6 and Lemma 2.8, $N_G(x)\backslash A$ and $V(H)$ have at most one vertex in common. Thus $\Delta - 3 = \Delta(H) + 1 \leq |H| \leq |G^{**} - x| - (|(N_{G^{**}}(x)\backslash A)| - 1) \leq (2n-1) - ((\Delta - 4) - 1)$, and so $\Delta \leq n + 3$, which contradicts (18).

Subcase 2.3. $min\{d_\Delta(v) : v \in B\} = 2$.

In this case, we first show that there exist $x \in B$ and $y \in B\backslash N[x]$ such that $d_\Delta(x) = 2$ and $G - y$ has a 1-factor F containing xw, where $w \in A$. We consider the following three cases.

Case i: $\Delta \geq n + 1$

Since $min\{d_\Delta(v) : v \in B\} = 2$, let $x \in B$ be such that $d_\Delta(x) = 2$. By (8), there exists $y \in B$ such that $xy \notin E(G)$. Assume that $xw_i \in E(G)$, where $i = 1, 2$. By symmetry of w_1 and w_2, assume that $d_\Delta(w_1) \geq d_\Delta(w_2)$. Let $G' = G - \{x, y, w_1\}$. As $\Delta \geq n + 1$, it follows that $\delta(G') \geq \Delta - 4 \geq (n-1) - 2$. By Lemma 2.10, G' has a 1-factor F' except when $G' \in \{G_0, G_0', 3K_3 + K_1\}$. However, as $\Delta(G') - \delta(G') \leq \Delta - (\Delta - 4) = 4$, $G' \neq 3K_3 + K_1$. Suppose $G' = G_0$ or $G' = G_0'$. Then $s = n - 2$ and $\Delta - 4 \leq \delta(G') \leq s$. Thus $s + 3 = n + 1 \leq \Delta \leq s + 4$. Let $Y = G - V(O_{s+2})$. Then $|Y| = s + 3$, $w_1 \in V(Y)$ and $V(J_s) \subset V(Y)$. Since $xw_i \notin E(G)$, $i = 3, 4, 5$, we have $d_{G'}(w_i) \geq \Delta - 2 \geq s + 1$. Thus $w_i \in V(J_s)(\subset V(Y))$. Suppose $w_2 \in V(J_s)$. Then $A \subset V(Y)$. This, together with the fact that $d_\Delta(v) \geq 2$ for any $v \in V(Y)$, yields that $e(Y) \geq e(G_\Delta) + 2|Y\backslash A| = 5 + 2((s+3) - 5) = 2s + 1$. Now $(s+2)(\Delta - 1) = e_G(V(O_{s+2}), V(Y)) = 5\Delta + (s-2)(\Delta - 1) - 2e(Y)$ implies that $\Delta \geq 4s - 2 \geq s + 7$ because $s = n - 2 \geq 3$, contradicting the fact that $\Delta \leq s + 4$. Next, suppose $w_2 \in V(O_{s+2})$. As $d_G(w_2) = \Delta \geq s + 3$ and $|Y| = s + 3$, it follows that w_2 is adjacent to all the vertices of Y in G and thus $\Delta = s + 3$. In particular, w_2 is adjacent to w_1, w_3, w_4, and w_5 in G. Thus $d_\Delta(w_2) = 4$. Since $d_\Delta(w_1) \geq d_\Delta(w_2)$, we have $d_\Delta(w_1) = d_\Delta(w_2) = 4$ and so $e(G[A - w_2]) \geq 3$. This, together with the fact that $d_\Delta(v) \geq 2$ for any $v \in V(Y)$, implies that $e(Y) \geq e(G[A - w_2]) + |Y\backslash A| \geq 3 + ((s+3) - 4) = s + 2$. Now $(s+1)(\Delta - 1) + \Delta = e_G(V(O_{s+2}), V(Y)) = 4\Delta + (s-1)(\Delta - 1) - 2e(Y)$ yields that $\Delta \geq 2s + 2 \geq s + 5$ (because $s = n - 2 \geq 3$), which contradicts the fact that $\Delta = s + 3$. Hence $G' \notin \{G_0, G_0'\}$ as desired and G' has a 1-factor F'. Clearly, $F = F' \cup \{xw_1\}$ is a 1-factor of $G - y$.

Case ii: $\Delta = n$ and $G[B] \cong J_{n-2} \cup J_{n-2}'$, where $m_2 \geq n$, and J_{n-2} and J_{n-2}' are graphs of order $n - 2$ with $\Delta(J_{n-2}) = \Delta(J_{n-2}') = n - 3$.

In this case, $n = \Delta \geq 6$ (by (9)). Let $x \in V(J_{n-2})$ and $y \in V(J_{n-2}')$ be such that $d_\Delta(x) = d_\Delta(y) = 2$. Clearly, $xy \notin E(G)$. Assume that $xw_1, xw_2 \in E(G)$, and $yw_k, yw_j \in E(G)$, where $w_k, w_j \in A$. By symmetry of w_1 and w_2 and symmetry of x and y, assume that $d_\Delta(w_1) \geq d_\Delta(w_2)$, and $d_\Delta(w_1) \leq max\{d_\Delta(w_k), d_\Delta(w_j)\}$. Let $G' = G - \{x, y, w_1\}$. As $\Delta = n$ and $(N_G(x) \cap N_G(y)) \cap B = \phi$, it follows that $\delta(G') = \Delta - 3 = (n-1) - 2$. By Lemma 2.10, G' has a 1-factor F' except when $G' \in \{G'_0, 3K_3 + K_1\}$. However, as $\Delta(G') - \delta(G') \leq \Delta - (\Delta - 3) = 3$, $G' \neq 3K_3 + K_1$. Suppose $G' = G'_0$. Then $2s + 2 = 2n - 2$. Thus $s = n - 2$ and $\Delta = n = s + 2$. Let $Y = G - V(O_{s+2})$. Then $|Y| = s + 3$ and $w_1 \in V(Y)$. Observe that for any $v \in V(O_{s+2}) - A$, $d_G(v) = \Delta - 1 = s + 1$ and v is adjacent to at most one of x and y. If $|A \cap V(Y)| \geq 4$, then $|A \cap V(O_{s+2})| \leq 1$ and each vertex $v \in V(O_{s+2}) - A$ is adjacent to at least three vertices of A in G. Thus $m_3 + m_4 + m_5 \geq |V(O_{s+2}) - A| = |O_{s+2}| - |A \cap V(O_{s+2})| \geq (s+2) - 1 = n-1$, and so $m_2 \leq |B| - (m_3 + m_4 + m_5) \leq (2n-4) - (n-1) = n - 3$, which contradicts the fact that $m_2 \geq n$. Hence $|A \cap V(Y)| \leq 3$. On the other hand, by VAL, $d_\Delta(v) \geq 2$ for any $v \in V(O_{s+2}) - A$, we have $|A \cap V(Y)| \geq 2$. Thus $2 \leq |A \cap V(Y)| \leq 3$.

Suppose $|A \cap V(Y)| = 2$. By VAL, for any $w \in A \cap V(O_{s+2})$, $d_\Delta(w) = 2$ and $ww_1 \in E(G)$. Thus $d_\Delta(w_1) = 3$. As $xw_i \notin E(G)$ $(i = 3, 4, 5)$, it follows that for any $w \in \{w_3, w_4, w_5\} \cap V(O_{s+2})$, $xw \notin E(G)$. Now $d_G(w) = \Delta = s + 2$ and $|Y - x| = s + 2$ imply that $yw \in E(G)$. Thus $w_k, w_j \in V(O_{s+2})$ and $max\{d_\Delta(w_k), d_\Delta(w_j)\} = 2$, which contradicts the assumption that $3 = d_\Delta(w_1) \leq max\{d_\Delta(w_k), d_\Delta(w_j)\}$.

Suppose $|A \cap V(Y)| = 3$. We claim that $e(Y) \geq s$. Since $N_G(x) \backslash A$ and $N_G(y) \backslash A$ have no vertices in common, if $w_2 \notin V(O_{s+2})$, then $e_G(\{x, y\}, V(O_{s+2})) \leq s + 2$ and so $e(Y) \geq d_G(x) + d_G(y) - e_G(\{x, y\}, V(O_{s+2})) \geq 2(s+1) - (s+2) = s$. On the other hand, if $w_2 \in V(O_{s+2})$, then $e_G(\{x, y\}, V(O_{s+2})) \leq s+3$. Let $\{w, w_2\} = A \cap V(O_{s+2})$. Then $xw \notin E(G)$. Now $d_G(w) = \Delta = s+2$ and $|Y - x| = s+2$ imply that $yw \in E(G)$ and w is adjacent to the other three vertices of $A \cap V(Y)$. Thus $d_\Delta(w) = 3$. Observe that $d_\Delta(w_1) \geq d_\Delta(w_2)$. This, together with the fact that the number of vertices of odd degrees in G_Δ is even, implies that $e(G[A \cap V(Y)]) \geq 1$. Hence $e(Y) \geq (d_G(x) + d_G(y) - e_G(\{x, y\}, V(O_{s+2}))) + e(G[A \cap V(Y)]) \geq 2(s+1) - (s+3) + 1 = s$. Thus, in both cases, we have $e(Y) \geq s$ as claimed. Now by counting the number of edges joining $V(Y)$ and $V(O_{s+2})$ in two different ways, we have $2\Delta + s(\Delta - 1) = e_G(V(O_{s+2}), V(Y)) = 3\Delta + s(\Delta - 1) - 2e(Y) \leq 3\Delta - s(\Delta - 1) - 2s$, and it follows that $\Delta \geq 2s \geq s + 4$ (because $s = n - 2 \geq 6 - 2 = 4$), which contradicts the fact that $\Delta = n = s + 2$. Hence $G' \neq G'_0$ and G' has a 1-factor F' and $F = F' \cup \{xw_1\}$ is a 1-factor of $G - y$.

Case iii: $\Delta \leq n - 1$ or $\Delta = n$ and $G[B] \not\cong J_{n-2} \cup J'_{n-2}$, where $m_2 \geq n$, and J_{n-2} and J'_{n-2} are graphs of order $n - 2$ with $\Delta(J_{n-2}) = \Delta(J'_{n-2}) = n - 3$.

In this case, $n \geq \Delta \geq 6$ (by (9)). We first show that $G[B]$ has a maximal matching M of size $n-3$ and one of the two M-unsaturated vertices is adjacent to exactly two major vertices in G.

Since $min\{d_\Delta(v) : v \in B\} = 2$, $\Delta(G[B]) = \Delta - 3$. By Lemma 2.1, $\chi'(G[B]) \leq \Delta - 2$. Let π be a $(\Delta-2)$-colouring of $G[B]$ and let $E_1, E_2, \cdots, E_{\Delta-2}$ be the colour classes, where $|E_1| \geq |E_2| \geq \cdots \geq |E_{\Delta-2}|$. Suppose $|E_1| \leq n-4$. By VAL, $e(G_\Delta) \geq 5$ and $e_G(A,B) = 5\Delta - 2e(G_\Delta) \leq 5\Delta - 10$. Now $(2n-4)(\Delta-1) - e_G(A,B) = 2e(G[B]) = 2\sum_{i=1}^{\Delta-2}|E_i| \leq 2(\Delta-2)(n-4)$ implies that $\Delta \geq 2n-2$, contradicting (8). Hence $|E_1| \geq n-3$. Let M' be a maximum matching of $G[B]$. Then $|M'| \geq |E_1| \geq n-3$. Since $|B| = 2n-4$, we have $n-3 \leq |M'| \leq n-2$.

Suppose $|M'| = n-3$. Let u and v be the two M'-unsaturated vertices. Then $uv \notin E(G)$. Clearly, if $d_\Delta(u) = 2$ or $d_\Delta(v) = 2$, then $M = M'$ is a maximal matching of $G[B]$ as required. Hence we assume that $d_\Delta(u), d_\Delta(v) \geq 3$. Let $x_1x_1', \cdots, x_tx_t' \in M'$ and $y_1y_1', \cdots, y_sy_s' \in M'$ be such that $ux_i, ux_i', uy_j \in E(G)$ and $uy_j' \notin E(G)$, where $2t+s = |N_G(u)\backslash A| = (\Delta-1) - d_\Delta(u)$, $i = 1, \cdots, t$, $j = 1, \cdots, s$. Let $C = \{x_1, x_1', \cdots, x_t, x_t', y_1', \cdots, y_s'\}$. Since M' is a maximum matching of $G[B]$, v is not adjacent to any vertex of C in G. Suppose $d_\Delta(w) \geq 3$ for any $w \in C$. Then by VAL and Claim 2, $2((2n-4) - |C \cup \{u,v\}|) + 3|C \cup \{v\}| + d_\Delta(u) \leq e_G(B,A) = 5\Delta - 2e(G_\Delta) \leq 5\Delta - 12$, and it follows that $\Delta \geq n+1$, which contradicts the fact that $\Delta \leq n$. Hence there exists $w \in C$ such that $d_\Delta(w) = 2$. Let $ww' \in M'$. Then $uw' \in E(G)$ and $vw \notin E(G)$. Now $M = (M' - \{ww'\}) \cup \{uw'\}$ is a maximal matching of $G[B]$ as required.

Next suppose $|M'| = n-2$. By Claim 2, $e(G_\Delta) \geq 6$. Thus by VAL, $2(2n-4) \leq e_G(B,A) = 5\Delta - 2e(G_\Delta) \leq 5\Delta - 12$ and $2m_2 + 3(2n-4-m_2) \leq e_G(B,A) = 5\Delta - 2e(G_\Delta) \leq 5n - 12$. It follows that

$$\Delta \geq \frac{4n+4}{5} \quad \text{and} \quad m_2 \geq n \geq 6. \tag{19}$$

Since $m_2 \geq n$ and $|M'| = n-2$, there exists $uv \in M'$ such that $d_\Delta(u) = d_\Delta(v) = 2$. Let $x_1x_1', \cdots, x_tx_t' \in M'$ and $y_1y_1', \cdots, y_sy_s' \in M'$ be such that $ux_i, ux_i', uy_j \in E(G)$ and $uy_j' \notin E(G)$, where $2t+s = |N_G(u)\backslash(A \cup \{v\})| = (\Delta-1) - (2+1) = \Delta - 4$, $i = 1, \cdots, t$, $j = 1, \cdots, s$. Let $C = \{x_1, x_1', \cdots, x_t, x_t', y_1', \cdots, y_s'\}$. Suppose there exists $w \in C$ such that $vw \notin E(G)$. Let $ww' \in M'$. Then $uw' \in E(G)$. Now $(M' - \{uv, ww'\}) \cup \{uw'\}$ is a maximal matching of $G[B]$ as required. Hence we assume that v is adjacent to all the vertices of C, and thus v is not adjacent to any vertex of $\{y_1, \cdots, y_{\Delta-4-2t}\}$ in G. Let $D = (N(u) \cup N(v)) \cap B$. If there exist $y \in B\backslash D$ and $z \in D$ such that $yz \in E(G)$, let $yy', zz' \in M'$. Then $y'u, y'v \notin E(G)$. By symmetry of u and v, we may assume that $vz' \in E(G)$. Now $M = (M' - \{uv, yy', zz'\}) \cup \{vz', yz\}$ is a maximal matching of $G[B]$

as required. Hence $e_G(B\backslash D, D) = 0$. Since $2t + s = \Delta - 4$, either $s \geq 1$ or $s = 0$.

Suppose $s \geq 1$. We claim that there exists $y_j \in \{y_1, \cdots, y_s\}$ (or $y'_j \in \{y'_1, \cdots, y'_s\}$) such that y_j (or y'_j) is adjacent to some vertex $z \in \{x_1, x'_1, \cdots, x_t, x'_t, y_1, \cdots, y_s\}$ (or $z \in \{x_1, x'_1, \cdots, x_t, x'_t, y'_1, \cdots, y'_s\}$) in G. Suppose otherwise. Then $(N_G(y_j)\backslash A) \subseteq \{u, y'_1, y'_2, \cdots, y'_s\}$ and $(N_G(y'_j)\backslash A) \subseteq \{v, y_1, y_2, \cdots, y_s\}$, where $1 \leq j \leq s$. Let $l = \min\{d_\Delta(v) : v \in \{y_1, y'_1, \cdots, y_s, y'_s\}\}$. Without loss of generality, we assume that $d_\Delta(y_1) = l$. Then $2 \leq l \leq 5$ and $|\{y'_1, \cdots, y'_s\}| \geq |N_G(y_1)\backslash(A \cup \{u\})| = (\Delta - 1) - (l + 1) = \Delta - l - 2$. Thus $|\{y_1, y'_1, \cdots, y_s, y'_s\}| = 2|\{y'_1, \cdots, y'_s\}| \geq 2(\Delta - l - 2) = 2\Delta - 2l - 4$. We claim that $l = 2$.

Suppose $5 \geq l \geq 3$. Then $t \geq 1$ (otherwise if $t = 0$, then $s = \Delta - 4$ and $m_3 + m_4 + m_5 \geq |\{y_1, y'_1, \cdots, y_s, y'_s\}| = 2s = 2(\Delta - 4)$. By (19), $n \leq m_2 = |B| - (m_3 + m_4 + m_5) \leq (2n - 4) - (2\Delta - 8)$, and it follows that $\Delta \leq \frac{n+4}{2}$, which contradicts (19)). Thus $ux_1, ux'_1 \in E(G)$ and so $\Delta - 1 = d_G(u) \geq d_\Delta(u) + |\{v, x_1, x'_1, y_1\}| = 6$. This, together with (8), implies that

$$n \geq \Delta \geq 8. \tag{20}$$

On the other hand, by VAL and Claim 2, $l(2\Delta - 4 - 2l) + 2((2n - 4) - (2\Delta - 4 - 2l)) \leq e_G(B, A) \leq 5\Delta - 12$, and it follows that

$$(l - 2)(2\Delta - 4 - 2l) + 4n - 8 \leq 5\Delta - 12. \tag{21}$$

If $l = 5$, then (21) implies that $\Delta + 4n \leq 38$, which contradicts (20). If $l = 4$, then (21) again implies that $\Delta \geq 4n - 20$, which contradicts (20). If $l = 3$, then $2\Delta - 10 = 2\Delta - 4 - 2l \leq |\{y_1, y'_1, \cdots, y_s, y'_s\}| \leq m_3 + m_4 + m_5 = |B| - m_2 = (2n - 4) - n = n - 4$, and it follows that $\Delta \leq \frac{n+6}{2}$. However, by (21), $\Delta \geq n + \frac{n-6}{3}$, which contradicts (20).

Hence $l = 2$ as claimed and so $s = \Delta - 4$ and $t = 0$. Thus $|D| = |\{u, v, y_1, y'_1, \cdots, y_{\Delta-4}, y'_{\Delta-4}\}| = 2\Delta - 6$ and $|B\backslash D| = (2n - 4) - (2\Delta - 6) = 2n - 2\Delta + 2$. Let $w \in B\backslash D$ be such that $d_\Delta(w) = \min\{d_\Delta(v) : v \in B\backslash D\}$. Then $2n - 2\Delta + 2 = |B\backslash D| \geq |N[w]| - d_\Delta(w) = \Delta - d_\Delta(w)$, and it follows that

$$\Delta \leq \frac{2n + 2 + d_\Delta(w)}{3}. \tag{22}$$

By VAL and Claim 2, $d_\Delta(w)|B\backslash D| + 2|D| \leq e_G(B, A) \leq 5\Delta - 12$. Thus

$$d_\Delta(w)(2n - 2\Delta + 2) + 2(2\Delta - 6) \leq e_G(B, A) \leq 5\Delta - 12. \tag{23}$$

If $d_\Delta(w) = 2$, then by (22), $\Delta \leq \frac{2n+4}{3}$, which contradicts (19). If $d_\Delta(w) = 3$, by (22), $\Delta \leq \frac{2n+5}{3}$. However, by (23), $\Delta \geq \frac{6n+6}{7}$, which contradicts

(20). If $d_\Delta(w) = 4$, by (23), $\Delta \geq \frac{8n+8}{9}$. However, by (22), $\Delta \leq \frac{2n+6}{3}$, which contradicts (20). If $d_\Delta(w) = 5$, by (23), $\Delta \geq \frac{10n+10}{11}$. However, by (22), $\Delta \leq \frac{2n+7}{3}$, which, again, contradicts (20).

Hence by symmetry of u and v, we can claim from above that there exists $y_j \in \{y_1, \cdots, y_s\}$ such that y_j is adjacent to some vertex $z \in \{x_1, x'_1, \cdots, x_t, x'_t, y_1, \cdots, y_s\}$ in G. Let $zz' \in M'$. Then $vz' \in E(G)$. Now $M = (M' - \{uv, y_jy'_j, zz'\}) \cup \{y_jz, vz'\}$ is a maximal matching of $G[B]$ as required.

Suppose $s = 0$. Then $t = \frac{\Delta-4}{2}$ and $|D| = \Delta - 2 \leq n - 2$. As $m_2 \geq n$, there exists $w \in B \backslash D$ such that $d_\Delta(w) = 2$. If $\Delta = n$, then $|D| = n - 2$ and $|B \backslash D| = (2n-4) - (n-2) = n - 2$. Since $e_G(D, B\backslash D) = 0$, we have $G \cong J_{n-2} \cup J'_{n-2}$, which contradicts our assumption. Hence $\Delta \leq n-1$. By VAL, $2m_2 + 3((2n-4) - m_2) \leq 5\Delta - 12 \leq 5(n-1) - 12$, and it follows that $m_2 \geq n+5$. Thus $n + 5 \leq m_2 \leq |B| = 2n - 4$, which implies that $n \geq 9$. If there exists $u'v' \in M' \cap E(G[B\backslash D])$ such that $d_\Delta(u') = d_\Delta(v') = 2$, then as similar to the case that $uv \in M'$ with $d_\Delta(u) = d_\Delta(v) = 2$ as shown above, we have $|D'| = \Delta - 2$ and $e_G(D', B\backslash(D' \cup D)) = 0$, where $D' = (N(u') \cup N(v')) \cap B$. We claim that $D' = B\backslash D$. Suppose otherwise. Since for any $w \in B\backslash(D \cup D')$, $d_G(w) = \Delta - 1$ and $d_\Delta(w) \leq 5$, we have $(2n-4) - 2(\Delta - 2) = |B\backslash(D \cup D')| \geq |N_G[w]| - d_\Delta(w) \geq \Delta - 5$, and it follows that $\Delta \leq \frac{2n+5}{3}$, which contradicts (19) because $n \geq 9$. Hence $D' = B\backslash D$ as claimed. Now $(\Delta - 2) + (\Delta - 2) = |D| + |D'| = |B| = 2n - 4$ implies that $\Delta = n$, which contradicts the fact that $\Delta \leq n - 1$. Hence for any $u'v' \in M' \cap E(G[B\backslash D])$, $max\{d_\Delta(u'), d_\Delta(v')\} \geq 3$ and so $m_3 + m_4 + m_5 \geq |M' \cap E(G[B\backslash D])| = (n-2) - \frac{\Delta-2}{2}$. Thus $n + 5 \leq m_2 = (2n-4) - (m_3 + m_4 + m_5) \leq (2n-4) - ((n-2) - \frac{\Delta-2}{2}$, and it follows that $\Delta \geq 16$, and so $n \geq \Delta + 1 \geq 17$. By VAL and Claim 2, $2((2n-4) - (m_3 + m_4 + m_5)) + 3(m_3 + m_4 + m_5) \leq e_G(B, A) \leq 5\Delta - 12$. This, together with the inequality $m_3 + m_4 + m_5 \geq n - 2 - \frac{\Delta-2}{2}$, implies that $\Delta \geq \frac{10n+6}{11}$. As $m_2 \geq n + 5$ and $\Delta \leq n - 1$, there exist $w, w' \in B\backslash D$ such that $d_\Delta(w) = 2$ and $ww' \notin E(G)$. Observe that $\delta(G[B\backslash D] - \{w, w'\}) \geq (\Delta - 1) - 5 - 2 = \Delta - 8 \geq \frac{2n-\Delta-4}{2}$ (because $\Delta \geq \frac{10n+6}{11}$ and $n \geq 17$). By Dirac's Theorem, $G[B\backslash D] - \{w, w'\}$ has a 1-factor F', and thus $M = (M' \cap E(G[D])) \cup F'$ is a maximal matching of $G[B]$ as required.

We thus conclude that $G[B]$ has a maximal matching M of size $n - 3$ and one of the two M-unsaturated vertices is adjacent to exactly two major vertices in G. Let $x, y \in B$ be the two M-unsaturated vertices and let $d_\Delta(x) = 2$. Since M is maximal, $xy \notin E(G)$. Assume that $xw_i \in E(G)$ $(i = 1, 2)$. By symmetry of w_1 and w_2, we may assume that $d_\Delta(w_1) \leq d_\Delta(w_2)$. We next show that $G_\Delta - w_1$ has a perfect matching.

As $d_\Delta(w) \geq 2$ for any $w \in A$ and $d_\Delta(w_1) \leq d_\Delta(w_2)$, it follows that

$G_\Delta - w_1$ has a perfect matching $\{e_1, e_2\}$ except when $G_\Delta \cong 2^3 4^2$ or $G_\Delta \cong 2^3 3^2$, and in both cases w_1 and w_2 are the two major vertices in G_Δ. Suppose $G_\Delta \cong 2^3 4^2$ or $G_\Delta \cong 2^3 3^2$. We first show that $G - x$ has a 1-factor F_0. If y is adjacent to one vertex of w_3, w_4, w_5, say w_3, in G, then $F_0 = M \cup \{w_1 w_4, y w_3, w_2 w_5\}$ is a 1-factor of $G - x$. On the other hand, if y is adjacent to only two major vertices w_1 and w_2 in G, let $x_1 x_1', \cdots, x_t x_t', y_1 y_1', \cdots, y_s y_s' \in M$ be such that $y x_i, y x_i', y y_j \in E(G)$ and $y y_j' \notin E(G)$, where $2t + s = |N_G(y) \backslash A| = (\Delta - 1) - 2 = \Delta - 3$, $i = 1, \cdots, t$, $j = 1, \cdots, s$. Let $C = \{x_1, x_1', \cdots, x_t, x_t', y_1', \cdots, y_s'\}$. Clearly, $|C| = 2t + s = \Delta - 3$. Since $d_\Delta(w_1) \geq 3$ and $x w_1, y w_1 \in E(G)$, there must exist a vertex $z \in C$ such that $z w_1 \notin E(G)$ (otherwise $d_G(w_1) \geq d_\Delta(w_1) + |C \cup \{x, y\}| \geq 3 + (\Delta - 3) + 2 = \Delta + 2$, which is false). As $d_\Delta(z) \geq 2$, z is adjacent to one of w_3, w_4, w_5, say w_3, in G. Let $zz' \in M$. Then $yz' \in E(G)$. Now $F_0 = (M - \{zz'\}) \cup \{yz', z w_3, w_1 w_4, w_2 w_5\}$ is a 1-factor of $G - x$. In either case, let $G^* = G - F_0$. Then G^* is of class 2. Observe that w_4 is adjacent to only one major vertex w_2 in G^* and w_5 is adjacent to only one major vertex w_1 in G^*. Since $\Delta(G^* - w_4) = \Delta(G^*)$ and $\Delta(G^* - \{w_4, w_5\}) = \Delta(G^* - w_4)$, by Lemma 2.5, it follows that $G^* - \{w_4, w_5\}$ is of class 2 with only two major vertices w_3 and x, which, by VAL, is false. Hence $G_\Delta \ncong 2^3 4^2$ and $G_\Delta \ncong 2^3 3^2$, and so $G_\Delta - w_1$ has a perfect matching $\{e_1, e_2\}$ as desired.

Let $F = M \cup \{x w_1, e_1, e_2\}$ and let $G^* = G - F$. Then F is a perfect matching of $G - y$ containing $x w_1$ and G^* is of class 2. Since x is adjacent to only one major vertex w_2 in G^* and $\Delta(G^* - x) = \Delta(G^*)$, by Lemma 2.5, $G^* - x$ is of class 2. Observe that $\{w_1, w_3, w_4, w_5, y\}$ is the set of major vertices of G^*. By Lemma 2.3, G^* has a $(\Delta - 1)$-critical subgraph H which has at most five major vertices.

Suppose H has five major vertices. By VAL, $5 \geq \Delta(H) - \delta(H) + 2$, and so $\delta(H) \geq \Delta(H) - 3 = \Delta - 4$. As $e_{G^*}(w_2, A) \geq 1$, we have $w_2 \in V(H)$ (otherwise H has at most four major vertices) and so $A \subset V(H)$. This, together with the fact that $d_\Delta(v) \geq 2$ for any $v \in V(G^*) \backslash V(H)$, implies that $v w_2 \in E(G^*)$ (otherwise H would have at most four major vertices). Thus $|V(G^*) \backslash V(H)| \leq d_{G^*}(w_2) - d_H(w_2) \leq (\Delta - 1) - (\Delta - 4) = 3$, and it follows that $2n \geq |G^* - x| \geq |H| \geq |G^*| - |V(G^*) \backslash V(H)| \geq (2n + 1) - 3 = 2n - 2$. By Theorem 1.1, $|H| = 2n - 1$. By the induction hypothesis on Δ, $e(H) = (n - 1)(\Delta - 1) + 1$. Let $\{z\} = V(G^* - x) \backslash V(H)$. Then $e(G) \geq e(H) + |F| + d_{G^*}(x) + d_{G^* - x}(z) \geq ((n - 1)(\Delta - 1) + 1) + n + (\Delta - 2) + (\Delta - 3) > n\Delta + 1$ (by (9)), which contradicts the assumption that $e(G) \leq n\Delta$.

Suppose H has at most four major vertices. By (8), Δ is even, and so by Lemma 2.6, Lemma 2.7, and Lemma 2.8, $|H|$ is odd and $|H| \neq \Delta(H) + 1 = \Delta$. By Lemma 2.6 and Lemma 2.8 again, $|H| = \Delta(H) + 2 = \Delta + 1$,

$\delta(H) = \Delta(H) - 1 = \Delta - 2$, and H has exactly four major vertices. Thus $3 \le |A \cap V(H)| \le 5$.

Suppose $|A \cap V(H)| = 5$. Then $A \subset V(H)$. As $d_\Delta(v) \ge 2$ for any $v \in N[x]\backslash A$, $e_{G^*}(N[x]\backslash A, A) \ge |N[x]\backslash A| = \Delta - 2 \ge 6 - 2 = 4$ (by (9)). However, since $\delta(H) = \Delta(H) - 1$ and H has exactly four major vertices, we have $e_{G^*}(A, N(x)\backslash A) \le 2$, which is a contradiction.

Suppose $|A \cap V(H)| = 4$. As w_2 is a minor vertex in $G^* - x$, $\delta(H) = \Delta(H) - 1$, and H has exactly four major vertices, we have $w_2 \notin V(H)$. Thus $d_\Delta(w_2) = 2$ and there exists $w_j \in A$ such that $w_2w_j \in F$ (otherwise H would have at most three major vertices). This, together with the fact that $xw_1 \in F$ and $\delta(H) = \Delta(H) - 1$, implies that $e_G(A \cap V(H), N(x)\backslash A) \le 2$. However, since $d_\Delta(v) \ge 2$ for any $v \in N(x)\backslash A$, we have $e_G(N(x)\backslash A, A \cap V(H)) \ge |N(x)\backslash A| = \Delta - 3 \ge 6 - 3 = 3$, which is a contradiction.

Suppose $|A \cap V(H)| = 3$. As H has exactly four major vertices, it follows that y is a major vertex in H and $e_{G^*}(V(G^*)\backslash V(H), A \cap V(H)) = 0$. Let $\{w_k, w_j\} = A\backslash V(H)$. Then by VAL, $d_\Delta(w_k) = 2 = d_\Delta(w_j)$ and $w_kw_j \in E(G)$. Since $xw_1 \in F$, from the choice of F, there exist $w'_k, w'_j \in A - \{w_1, w_k, w_j\}$ such that $w_kw'_k, w_jw'_j \in F$ (otherwise H would have at most three major vertices), and thus $yw_k, yw_j \notin E(G)$. Clearly, $xw'_k, xw'_j \notin E(G)$. Since $xw_1 \in F$, $d_\Delta(x) = 2$, and $e_{G^*}(V(G^*)\backslash V(H), A \cap V(H)) = 0$, we may assume that $xw_k \in E(G)$. Now $xw_1, w_kw'_k, w_jw'_j \in F$ and w_1, w'_k, w'_j are major vertices in H. Thus for any $v \in N[x]$, $d_\Delta(v) = 2$. If $w'_kw'_j \notin E(G)$, then $w_1w'_k, w_1w'_j \in E(G)$ and thus $G_\Delta \cong C_5$. Since $\delta(H) = \Delta(H) - 1$ and $e_{G^*}(V(G^*)\backslash V(H), A \cap V(H)) = 0$, by Lemma 2.10, there exists exactly one vertex $z \in V(H)\backslash A$ such that $d_\Delta(z) = 2$ and for any $v \in V(H)\backslash(A \cup \{z\})$, $d_\Delta(v) = 3$. Thus, by VAL, $5\Delta - 11 = 2((\Delta - 2) + 1) + 3((\Delta + 1) - 4) = 2|N[x] \cup \{z\}| + 3(|V(H)\backslash(A \cup \{z\})| \le e_G(B, A) = 5\Delta - 2e(G_\Delta) = 5\Delta - 10$, which is false. Hence $w'_kw'_j \in E(G)$. Now replace F by $F^* = (F - \{w_kw'_k, w_jw'_j\}) \cup \{w_kw_j, w'_kw'_j\}$. Then in $G^{**} - x$ (where $G^{**} = G - F^*$), w_k is adjacent to only one major vertex w'_k, and w_j is adjacent to only one major vertex w'_j. Observe that $\Delta(G^{**} - \{x, w_k, w_j\}) = \Delta(G^{**} - x)$. By Lemma 2.5, $G^* - \{x, w_k, w_j\}$ is of class 2 with only two major vertices w_1 and y, which, by VAL, is false. The proof of Theorem 1.2 is thus completed. ■

Corollary 3.1. *Let G be a graph of order $2n+1 \ge 7$ with $|G_\Delta| = 5$, where $\Delta = \Delta(G) \ge 3$. Then G is Δ-critical if and only if*

(i) $G \cong (2n-3)^{2n-4}(2n-2)^5$;

(ii) $G \cong (2n-3)(2n-2)^{2n-5}(2n-1)^5$;

(iii) $G \cong (2n-2)^2(2n-1)^{2n-6}(2n)^5$; *or*

(iv) $G \cong (2n-3)(2n-1)^{2n-5}(2n)^5$.

Proof. Necessity. Let $\delta = \delta(G)$. Then by Lemma 2.4, $\delta \le \Delta - 1$. As G is Δ-critical, by VAL, $5 = |G_\Delta| \ge \Delta - \delta + 2$. It follows that $\Delta - 1 \ge \delta \ge \Delta - 3$.

Suppose $\delta = \Delta - 1$. Then $G \cong (\Delta-1)^{2n-4}\Delta^5$. By Theorem 1.2, $2(n\Delta+1) = 2e(G) = \sum_{v \in V(G)} d_G(v) = (2n-4)(\Delta-1) + 5\Delta$, which implies that $\Delta = 2n-2$. Thus $G \cong (2n-3)^{2n-4}(2n-2)^5$.

Suppose $\delta = \Delta - 2$. Then $G \cong (\Delta-2)^x(\Delta-1)^{2n-4-x}\Delta^5$, where $x \ge 1$. By Theorem 1.2, $2(n\Delta+1) = 2e(G) = \sum_{v \in V(G)} d_G(v) = x(\Delta-2) + (2n-4-x)(\Delta-1) + 5\Delta$, and it follows that $\Delta = 2n-2+x$. As $\Delta \le |G|-1 = 2n$ and $x \ge 1$, we have $1 \le x \le 2$. Hence $G \cong (2n-3)(2n-2)^{2n-5}(2n-1)^5$ or $G \cong (2n-2)^2(2n-1)^{2n-6}(2n)^5$.

Suppose $\delta = \Delta - 3$. Then $G \cong (\Delta-3)^x(\Delta-2)^y(\Delta-1)^{2n-4-x-y}\Delta^5$, where $x \ge 1$ and $y \ge 0$. By Theorem 1.2 again, $2(n\Delta+1) = 2e(G) = \sum_{v \in V(G)} d_G(v) = x(\Delta-3) + y(\Delta-2) + (2n-4-x-y)(\Delta-1) + 5\Delta$, which implies that $\Delta = 2n-2+2x+y$. As $\Delta \le |G|-1 = 2n$ and $x \ge 1$, we have $x = 1$ and $y = 0$. Thus $\Delta = 2n$ and $G \cong (2n-3)(2n-1)^{2n-5}(2n)^5$.

Sufficiency. Suppose G satisfies one of (i), (ii), (iii) and (iv). Then $\Delta \ge 2n-2$ and $e(G) = n\Delta + 1$. Thus G is of class 2. We next show that G is Δ-critical.

Suppose otherwise. Then by Lemma 2.3, G contains a Δ-critical subgraph H with at most five major vertices. Since G is of class 2 and G is not Δ-critical, we have $e(H) < e(G)$. By Theorem 1.1, Lemma 2.6 and Lemma 2.7, $|H|$ is odd. Observe that $|H| \ge \Delta(H) + 1 \ge (2n-2) + 1 \ge 2n-1$. Thus either $|H| = 2n+1$ or $|H| = 2n-1$ (in this case, $\Delta = 2n-2$).

Suppose $|H| = 2n+1$. Then by Lemma 2.6 , Lemma 2.8, and Theorem 1.2, $e(H) = n\Delta + 1$. But then $e(G) > e(H) = n\Delta + 1$, contradicting the fact that $e(G) = n\Delta + 1$.

Suppose $|H| = 2n-1$. Then $\Delta = 2n-2$. Thus $G \cong (2n-3)^{2n-4}(2n-2)^5$. Let $\{x, y\} = V(G) \backslash V(H)$. Then $d_G(x), d_G(y) \ge 2n-3 = \Delta - 1$. By Lemma 2.6, Lemma 2.8, and Theorem 1.2 again, $e(H) = (n-1)\Delta + 1$. Therefore $e(G) > e(H) + d_G(x) + d_{G-x}(y) \ge ((n-1)\Delta + 1) + (\Delta - 1) + (\Delta - 2) = n\Delta + 1 + (\Delta - 3) > n\Delta + 1$ (because $\Delta = 2n-2 \ge 5$), which again contradicts the fact that $e(G) = n\Delta + 1$. ■

Theorem 3.2. *Let G be a connected graph of order $2n+1 \ge 7$ with $|G_\Delta| = 5$, where $\Delta = \Delta(G) \ge 3$. Then G is of class 2 if and only if*

(i) $G \cong (2n-3)^{2n-4}(2n-2)^5$;

(ii) $G \cong (2n-3)(2n-2)^{2n-5}(2n-1)^5$;

(iii) $G \cong (2n-2)^2(2n-1)^{2n-6}(2n)^5$;

(iv) $G \cong (2n-3)(2n-1)^{2n-5}(2n)^5$;

(v) $G \cong (2n-1)^{2n-4}(2n)^5$;

(vi) for some $m < n$, *G contains a cut-edge e such that* $G - e$ *is the union of two disjoint graphs* G_1 *and* G_2, *where* $\Delta(G_1) \leq \Delta$ *and, in G, e is incident with a vertex of degree in* G_2 *at most* $\Delta - 1$; *and* G_2 *is* Δ*-critical and isomorphic to one of the following:* $(2m-1)^{2m-2}(2m)^3$, $(2m-2)^{2m-3}(2m-1)^4$, $(2m-2)(2m-1)^{2m-4}(2m)^4$, $(2m-3)(2m-2)^{2m-5}(2m-1)^5$, $(2m-2)^2(2m-1)^{2m-6}(2m)^5$, *and* $(2m-3)(2m-1)^{2m-5}(2m)^5$;

(vii) for some $m < n$, *G contains a cut set of two edges* e_1 *and* e_2 *such that* $G - \{e_1, e_2\}$ *is the union of two disjoint graphs* G_1 *and* G_2, *where* $\Delta(G_1) \leq \Delta - 1$ *and, in G,* e_i $(i = 1, 2)$ *is incident with a vertex of degree at most* $\Delta - 1$ *in* G_2; *and* G_2 *is* Δ*-critical and isomorphic to one of the following:* $(2m-1)^{2m-2}(2m)^3$, $(2m-2)(2m-1)^{2m-4}(2m)^4$, $(2m-2)^2(2m-1)^{2m-6}(2m)^5$, *and* $(2m-3)(2m-1)^{2m-5}(2m)^5$.

Proof. Sufficiency. Suppose G satisfies one of (i), (ii), (iii), (iv), (v), (vi), and (vii). Then either G is overfull or G contains an overfull subgraph with the same maximum degree. Thus G is of class 2.

Necessity. Suppose G is of class 2. If G is Δ-critical, then by Corallory 3.1, G satisfies one of (i), (ii), (iii), and (iv). On the other hand, if G is not Δ-critical, then by Lemma 2.3, G contains a Δ-critical subgraph G_2 with at most five major vertices and $e(G_2) < e(G)$. Observe that $|G_2| \leq |G|$. If $|G_2| = |G|$, as G has exactly five major vertices and $e(G) > e(G_2)$, by Lemma 2.6, Lemma 2.8, and Corollary 3.1, $G_2 \cong (2n-1)^{2n-2}(2n)^3$, $G_2 \cong (2n-2)(2n-1)^{2n-4}(2n)^4$ or $G_2 \cong (2n-2)^2(2n-1)^{2n-6}(2n)^5$. In either case, $G \cong (2n-1)^{2n-4}(2n)^5$. If $|G_2| < |G|$, let $G_1 = G - V(G_2)$. Since G is connected, we have $e_G(V(G_1), V(G_2)) \geq 1$. On the other hand, as G_2 has at most five major vertices and G has exactly five major vertices, by Lemma 2.6, Lemma 2.8 and Corollary 3.1, we have $e_G(V(G_1), V(G_2)) \leq 2$. Thus $1 \leq e_G(V(G_1), V(G_2)) \leq 2$.

Suppose $e_G(V(G_1), V(G_2)) = 1$. Then G has a cut-edge e connecting G_1 and G_2 in G. If G_2 has three major vertices, then $\Delta(G_1) \leq \Delta$, and by Lemma 2.6, $G_2 \cong (2m-1)^{2m-2}(2m)^3$ for some $m < n$. Thus either G_1 has exactly one vertex of degree Δ and e is incident with a vertex of degree at most $\Delta - 2$ in G_1 and a vertex of degree $(2m-1)$ in G_2, or $\Delta(G_1) \leq \Delta - 1$ and e is incident with a vertex of degree $\Delta - 1$ in G_1 and a vertex of degree $(2m-1)$ in G_2. If G_2 has exactly four major vertices, by Lemma 2.8, for some $m < n$, $G_2 \cong (2m-2)^{2m-3}(2m-1)^4$ (in this case $\Delta(G_1) \leq \Delta - 1$ and e is incident with a vertex of degree at most $\Delta - 2$ in G_1 and a vertex of degree $(2m-2)$ in G_2) or $G_2 \cong (2m-2)(2m-1)^{2m-4}(2m)^4$ (in this case, $\Delta(G_1) \leq \Delta$, and if $\Delta(G_1) = \Delta$, then G_1 has exactly one major vertex and e is incident with a vertex of degree at most $\Delta - 2$ in G_1 and a vertex of degree $(2m-1)$ in G_2. If $\Delta(G_2) \leq \Delta - 1$, then either e is incident with a vertex of degree $\Delta - 1$ in G_1 and a vertex of degree $(2m-2)$ in G_2, or e is incident with a vertex of degree at most $\Delta - 2$ in G_1 and a vertex of degree $2m-1$ in G_2). If G_2 has five major vertices, by Corollary 3.1, for some $m < n$, $G_2 \cong (2m-3)(2m-2)^{2m-5}(2m-1)^5$, $G_2 \cong (2m-2)^2(2m-1)^{2m-6}(2m)^5$ or $G_2 \cong (2m-3)(2m-1)^{2m-5}(2m)^5$. In either case, since G has exactly five major vertices, it follows that $\Delta(G_1) \leq \Delta - 2$ and e is incident with a vertex of degree at most $(2m-2)$ in G_2.

Suppose $e_G(V(G_1), V(G_2)) = 2$. Then G has two cut-edges e_1 and e_2 connecting G_1 and G_2 in G. If G_2 has three major vertices, by Lemma 2.6, $G_2 \cong (2m-1)^{2m-2}(2m)^3$ for some $m < n$. Thus $\Delta(G_1) \leq \Delta - 1$ and in G, e_1 and e_2 are incident with different vertices of degree $(2m-1)$ in G_2. If G_2 has exactly four major vertices, by Lemma 2.8, for some $m < n$, $G_2 \cong (2m-2)(2m-1)^{2m-4}(2m)^4$. Thus $\Delta(G_1) \leq \Delta - 1$ and either e_1 and e_2 are both incident with the vertex of degree $(2m-2)$ in G_2 or e_1 and e_2 are incident with two vertices of degree $(2m-2)$ and $(2m-1)$ respectively in G_2. If G_2 has five major vertices, by Corollary 3.1, for some $m < n$, $G_2 \cong (2m-2)^2(2m-1)^{2m-6}(2m)^5$ (in this case e_1 and e_2 in G are incident with two vertices of degree $(2m-2)$ respectively in G_2) or $G_2 \cong (2m-3)(2m-1)^{2m-5}(2m)^5$ (in this case both e_1 and e_2 in G are incident with the vertex of degree $(2m-3)$ in G_2). In either case, since G has exactly five major vertices, $\Delta(G_1) \leq \Delta - 1$. ■

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