A remark on q-dominating cycles

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Abstract

A cycle C of a graph G is called a q-dominating cycle if every vertex of G which is not contained in C is adjacent to at least q vertices of C. Let G be a k-connected graph with $k \geq 2$. We present a sufficient condition, in terms of the degree sum of $k+1$ independent vertices, for G to have a q-dominating cycle. This is an extension of a 1987 result by J.A. Bondy and G. Fan. Furthermore, examples will show that the given condition is best possible.

Keywords: Dominating cycle; Dominating set, q-dominating cycle; q-dominating set

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$. For $u \in V(G)$, let $N_G(u) = N(u)$ be the neighborhood of u. We denote by $d_G(u) = d(u) = |N(u)|$ the degree of a vertex u and by $\delta(G)$ the minimum degree of G. If H is a subgraph and v a vertex of G, then $N_H(v)$ denotes the set of neighbors of v in H. If q is a positive integer, then a subset D of $V(G)$ is a q-dominating set if each vertex of $V(G) - D$ is adjacent to at least q vertices of D. The concept of q-domination was introduced by Fink and Jacobson [2], [3]. Thus, we call a cycle C of a graph G a q-dominating cycle if every vertex of G which is not contained in C is adjacent to at least q vertices of C. In this situation we also say that every vertex in $G - C$ is q-dominated by C. Note that a 1-dominating cycle is the classical dominating cycle introduced by Lesniak and Williamson [5] in 1977. For other graph theory terminology we follow the book by Haynes, Hedetniemi, and Slater [4].

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In 1987, J.A. Bondy and G. Fan [1] presented the following sufficient condition for the existence of a dominating cycle.

Theorem 1 (Bondy, Fan [1]) Let G be a k-connected graph on n vertices, where $k \geq 2$. If

$$\sum_{i=0}^{k} d(x_i) \geq n - 2k$$

for any $k+1$ independent vertices $x_0, x_1, \ldots, x_k$ with $N(x_i) \cap N(x_j) = \emptyset$ for $0 \leq i \neq j \leq k$, then G has a dominating cycle.

The proof of this theorem can also be found in [4] on the pages 177-179. Similarly to Bondy and Fan [1], we derive in this note the following extension of Theorem 1.

Theorem 2 Let k and q be positive integers and let G be a k-connected graph on n vertices, where $k \geq 2$. If for any $k+1$ independent vertices $x_0, x_1, \ldots, x_k$ with $|N(x_i) \cap N(x_j)| \leq 2q - 2$ for $0 \leq i \neq j \leq k$,

$$\sum_{i=0}^{k} d(x_i) \geq n + (k+1)(q-3) + 2,$$

then G has a q-dominating cycle.

Proof. Suppose that G satisfies the hypotheses of the theorem, but has no q-dominating cycle. Since G is k-connected, G has a cycle of length at least k. Let C be a cycle in G with $|V(C)| \geq k$ such that

(i) C q-dominates as many vertices as possible.

Since C is not a q-dominating cycle, there is a component H of $G - C$ containing a vertex x_0 such that $|N_C(x_0)| \leq q - 1$. Choose C, subject to (i), in such a way that

(ii) H has as few vertices as possible.

Since G is k-connected and $|V(C)| \geq k$, there are k paths $P_1, P_2, \ldots, P_k$ from x_0 to C, pairwise disjoint except for x_0 and with all internal vertices in H. Let u_i be the end-vertices of the paths P_i for $1 \leq i \leq k$, taken in order around C, and define $u_{k+1} = u_1$ and $P_{k+1} = P_1$.

Let $c_1, c_2, \ldots, c_s$ be the vertices in order around C and define $C(c_i, c_j) = \{c_t : i < t < j\}$ and $C[c_i, c_j] = \{c_t : i \leq t \leq j\}$, where the subscripts are taken modulo s. The sets $C(c_i, c_j]$ and $C[c_i, c_j)$ are defined analogously.

If, for some i and every $v \in C(u_i, u_{i+1})$, $N_{G-C}(v) \cup \{v\}$ is q-dominated by $C - C(u_i, u_{i+1})$, then by deleting $C(u_i, u_{i+1})$ and adding the path $P_i x_0 P_{i+1}$, we obtain a new cycle which has length at least k and q-dominates more vertices than C, contradicting (i). Thus, for each i, there is at least one vertex $y_i \in C(u_i, u_{i+1})$ such that

(a) $N_{G-C}(y_i) \cup \{y_i\}$ is not q-dominated by $C - C(u_i, u_{i+1})$.

Now subject to (a), choose y_i as close to u_i along $C(u_i, u_{i+1})$ as possible. Thus we have

(b) for each $v \in C(u_i, y_i)$, $N_{G-C}(v) \cup \{v\}$ is q-dominated by $C - C(u_i, u_{i+1})$.

Using (b), we shall show that

(c) there is no edge joining H and $C(u_i, y_i]$ for $1 \leq i \leq k$.

Suppose to the contrary, that there is an edge $e = hw$, where $h \in V(H)$ and $w \in C(u_i, y_i]$. Choose e such that w is as close to u_i along $C(u_i, y_i]$ as possible. Since H is connected, there is a path P from u_i to w, containing e, with all internal vertices in H. By adding the path P and deleting $C(u_i, w)$, we obtain a new cycle C' such that $|V(C')| \geq k$. If C' q-dominates x_0, then by (b), C' q-dominates more vertices than C, contradicting (i). If C' does not q-dominate x_0, then by (b), C' q-dominates at least as many vertices as C. However, in this case, the component of $G - C'$ containing x_0 has fewer vertices than H and so contradicts (ii). This completes the proof of (c).

Next we show for $1 \leq i \neq j \leq k$ that

(d) there is no path from $C(u_i, y_i]$ to $C(u_j, y_j]$ with all internal vertices in $G - C$.

Suppose that there is a such a path P from $w \in C(u_i, y_i]$ to $z \in C(u_j, y_j]$. By (c), $V(P) \cap V(H) = \emptyset$, and so

$$wPz \dots y_j \dots u_{j+1} \dots u_i P_i x_0 P_j u_j \dots u_{i+1} \dots y_i \dots w$$

is a cycle C'' with $|V(C'')| \geq k$. We choose P in such a way that w is as close to u_i along $C(u_i, y_i]$ as possible. By this choice, there is no path joining $C(u_i, w)$ and $C(u_j, z)$, and so, for every vertex $v \in C(u_i, w) \cup C(u_j, z)$, $N(v) \cup \{v\}$ is q-dominated by C'', by (b). Therefore, C'' q-dominates more vertices than C. This contradiction established (d).

It follows from (a) that

(e) for $1 \leq i \leq k$, there exists a vertex $x_i \in N_{G-C}(y_i) \cup \{y_i\}$ which is not q-dominated by $C - C(u_i, u_{i+1})$.

In view of (c) and (d),

(1) the x_i for $0 \leq i \leq k$ are $k+1$ independent vertices.

Next, we shall show that

(2) $|N(x_i) \cap N(x_j)| \leq 2q - 2$ for $0 \leq i \neq j \leq k$.

Suppose that there is exists a vertex $v \in N(x_i) \cap N(x_j)$ which is not contained in C. For $i, j \neq 0$, the path $y_i x_i v x_j y_j$ yields a contradiction to (d). The case $i = 0$ and $j \geq 1$ leads to $v \in V(H)$. If $x_j \neq y_j$, then also $x_j \in V(H)$, contradicting (c). If $x_j = y_j$, then we have also a contradiction to (c). Consequently, $N(x_i) \cap N(x_j) \subset V(C)$ for $0 \leq i \neq j \leq k$. Since C does not q-dominate x_0, it follows therefore from (e)

$$\begin{aligned} |N(x_i) \cap N(x_j)| &\leq |N(x_i) \cap (C - C(u_i, u_{i+1}))| \\ &+ |N(x_i) \cap N(x_j) \cap C(u_i, u_{i+1})| \\ &\leq |N(x_i) \cap (C - C(u_i, u_{i+1}))| \\ &+ |N(x_j) \cap (C - C(u_j, u_{j+1})| \\ &\leq q - 1 + q - 1 = 2q - 2. \end{aligned}$$

Now let $J = \{i : x_i = y_i,\ 1 \leq i \leq k\}$. Since x_i is not q-dominated by $C - C(u_i, u_{i+1})$ for $1 \leq i \leq k$, we obtain for $i \in J$

$$|N_{G-C}(x_i)| \geq d(x_i) - |C(u_i, u_{i+1})| - (q-1) + 1$$

and for $i \notin J$

$$|N_{G-C}(x_i)| \geq d(x_i) - |C(u_i, u_{i+1})| - (q-1).$$

Because of $N(x_i) \cap N(x_j) \subset V(C)$, we have $N_{G-C}(x_i) \cap N_{G-C}(x_j) = \emptyset$ for $0 \leq i \neq j \leq k$. This implies

$$\begin{aligned} n - |V(C)| &= |V(G - C)| \geq |\{x_0\}| + \sum_{i=0}^{k} |N_{G-C}(x_i)| + \sum_{i \notin J} |\{x_i\}| \\ &\geq |\{x_0\}| + |N(x_0)| - (q-1) \\ &+ \sum_{i \notin J} |N_{G-C}(x_i)| + \sum_{i \in J} |N_{G-C}(x_i)| + \sum_{i \notin J} |\{x_i\}| \\ &\geq 1 + d(x_0) - (q-1) + \sum_{i \notin J} (d(x_i) - |C(u_i, u_{i+1})| - (q-1)) \\ &+ \sum_{i \in J} (d(x_i) - |C(u_i, u_{i+1})| - (q-1) + 1) + (k - |J|) \\ &= \sum_{i=0}^{k} d(x_i) - \sum_{i=1}^{k} |C(u_i, u_{i+1})| - (k+1)(q-2) \\ &= \sum_{i=0}^{k} d(x_i) - (|V(C)| - k) - (k+1)(q-2). \end{aligned}$$

In view of (1) and (2), this is a contradiction to the hypotheses and the proof is complete. □

Corollary 3 Let k and q be positive integers and let G be a k-connected graph on n vertices, where $k \geq 2$. If

$$\delta(G) \geq \frac{n+2}{k+1} + q - 3,$$

then G has a q-dominating cycle.

Corollary 4 (Bondy, Fan [1]) Let G be a k-connected graph on n vertices, where $k \geq 2$. If $\delta(G) \geq (n-2k)/(k+1)$, then G has a dominating cycle.

Example 5 Let k and q be positive integers such that $2 \leq q \leq k+1$, and let $H_0, H_1, \ldots, H_{k+1}$ be $k+2$ disjoint copies of the complete graph K_k. For each $0 \leq i \leq k$, we firstly join the vertices of H_i with the vertices of H_{k+1} by exactly k non-incident edges, and then we join every vertex of H_i with $q-2$ further vertices of H_{k+1}. Certainly, the resulting graph H is k-connected. In addition, H has $k+1$ independent vertices $y_i \in V(H_i)$ for $0 \leq i \leq k$ with $|N_H(y_i) \cap H_H(y_j)| \leq q-1 \leq 2q-2$ for $0 \leq i \neq j \leq k$ and

$$\sum_{i=0}^{k} d_H(y_i) = (k+1)(k+q-2) = |V(H)| + (k+1)(q-3) + 1.$$

Clearly, a q-dominating cycle of H contains at least one vertex from H_i for $0 \leq i \leq k$. However, since such a cycle doesn't exist, the graph H has no q-dominating cycle. Thus, the graph H shows that Theorem 2 is best possible.

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