On tree factorizations of K_{10}

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We use here undirected graphs without loops and multiple edges. For undefined terms see [5].

We shall call a *tree r-packing* in a graph K_n the collection P of r mutually edge disjoint trees of the graph. Then the mentioned trees are called *components* of the tree r-packings. A tree r-packing P is called a *tree decomposition* of K_n if every edge of K_n belongs to some component.

The problem is considered how to select the set $\mathbf{T}_n$ of all trees of order n such that for every $T \in \mathbf{T}_n$ there exists a tree decomposition $R = R(T)$ of the graph K_n into the components isomorphic to T. Such decompositions we call *T-factorizations.*

We must point out the well-known fact that $P_n \in \mathbf{T}_n$ where P_n denotes n-vertex path with n even.

It is known [1] that $T \in \mathbf{T}_n \Rightarrow$ (a) n is even and (b) $\Delta(T) \leq n/2$ where $\Delta(T)$ is the greatest vertex degree in the tree T. The trees satisfying the conditions (a), (b) are called *admissible.*

For $n \leq 8$ the problem is completely solved in [2, 3], and in [3, 4, 6] all nonisomorphic tree factorizations for these orders are enumerated. In this paper we give the complete solution in the case $n = 10$.

The necessary conditions of factorizability

For $n = 2k$ every admissible tree T defines the vector $\mathbf{d}(T) = (d_1, d_2, \ldots, d_k)$ where d_i is the quantity of vertices of degree i in the tree T. Let us define the *type* of a vertex x in the factorization R as the vector $(s_1, s_2, \ldots, s_k)$ where s_j is the quantity of components in R having the degree j in x.

The possible types of vertices can be determined from the following correlations.

$$s_1 + s_2 + \cdots + s_k = k, \tag{1}$$

$$s_1 + 2s_2 + \cdots + ks_k = n - 1, \tag{2}$$

JCMCC **41** (2002), pp. 193-202

$$s_j \text{ is integer, } s_j \geq 0 \; (j = 1, \ldots, k). \tag{3}$$

We denote by $w(n)$ the number of solutions of the system (1) - (3), i.e. the number of *possible types* of vertices for order n.

Thus for $n = 10$ there are just 5 possible types of vertices (writing the types we omit commas and brackets), namely $t_1 = 14000$, $t_2 = 22100$, $t_3 = 30200$, $t_4 = 31010$, $t_5 = 40001$.

For $n = 12$ we have $w(12) = 7$ and $t_1 = 150000$, $t_2 = 231000$, $t_3 = 312000$, $t_4 = 320100$, $t_5 = 401100$, $t_6 = 410010$, $t_7 = 500001$.

For $n = 14$ we have $w(14) = 11$ possible types, namely $t_1 = 1600000$, $t_2 = 2410000$, $t_3 = 3220000$, $t_4 = 3301000$, $t_5 = 4030000$, $t_6 = 4111000$, $t_7 = 4200100$, $t_8 = 5002000$, $t_9 = 5010100$, $t_{10} = 5100010$, $t_{11} = 6000001$.

We also present the full list of 15 possible types for $n = 16$: $t_1 = 17000000$, $t_2 = 25100000$, $t_3 = 33200000$, $t_4 = 34010000$, $t_5 = 41300000$, $t_6 = 42110000$, $t_7 = 43001000$, $t_8 = 50210000$, $t_9 = 51020000$, $t_{10} = 61101000$, $t_{11} = 52000100$, $t_{12} = 60011000$, $t_{13} = 60100100$, $t_{14} = 61000010$, $t_{15} = 7000000l$.

Let us denote a_j the number of vertices of type t_j in the factorization R. Let $\mathbf{a} = \mathbf{a}(R) = (a_1, a_2, a_3, \ldots, a_{w(n)})$, and let $\mathbf{S} = (s_{ij})$ be a matrix with $w(n)$ rows and k columns in which i-th row is i-th possible vertex type for order n.

Counting in two different ways the total number of the vertices of degree i in R, for every i, leads us to the matrix equality

$$\mathbf{aS} = \mathbf{d}(T). \tag{4}$$

From the above it follows

Theorem 1. *For the existence of a tree factorization of the graph K_{2k} into tree factors isomorphic to T it is necessary that the matrix equation (4) has a solution with nonnegative integer components.*

Remark that the notion of vertex types, for the case $n = 8$, was introduced by C. Huang and A. Rosa [2]. Just in their paper the system of scalar equations was used equivalent to (4).

Below we demonstrate that Theorem 1 makes it possible to affirm the nonexistence of the T-factorizations for some admissible T.

The necessary condition for the case $n = 10$

Consider the partial case $n = 10$. In this case the relation (4) can be written in the scalar form

$$a_1 + 2a_2 + 3a_3 + 3a_4 + 4a_5 = 5d_1; \quad (5)$$
$$4a_1 + 2a_2 + a_4 = 5d_2; \quad (6)$$
$$a_2 + 2a_3 = 5d_3; \quad (7)$$
$$a_4 = 5d_4; \quad (8)$$
$$a_5 = 5d_5. \quad (9)$$

Subtracting (8) from (6), we obtain $4a_2+2a_2 = 5(d_2-d_4)$, hence $d_2 \geq d_4$. So we prove the following

Theorem 2. *If $T \in \mathbf{T}_{10}$ then $d_2 \geq d_4$.*

For example, every tree T with $d(T) = 70120$ does not belong to $\mathbf{T}_{10}$ because the necessary condition $d_2 \geq d_4$ is not fulfilled.

In F. Harary's monograph [5] the diagrams of all nonisomorphic trees of order 10 are presented. There are exactly 106 such trees. We will denote T_m the tree drawn in m-th diagram in the list.

The examination of the list shows that the trees T_{20} - T_{26}, T_{69} - T_{72} and T_{81} are not admissible ($\Delta > 5$), and so do not belong to $\mathbf{T}_{10}$. The trees T_{102}, T_{103} have $\mathrm{d}(T) = 70120$, and due to Theorem 2 do not belong to $\mathbf{T}_{10}$. The remaining trees are investigated in the following sections.

The sufficient conditions in the case $n = 10$ and constructions

As we may see from the following theorem the predominate number of admissible 10-vertex trees belongs to $\mathbf{T}_{10}$.

Theorem 3. *If for 10-vertex tree T the conditions $\Delta(T) \leq 4$ and $d(T) \neq 70120$ take place then $T \in \mathbf{T}_{10}$.*

Proof: We have constructed the corresponding factorizations of K_{10} for every such tree. In Table 1 the column 'Num' contains the number m of the tree T_m and the column 'Base component' presents the edge list of the component from which the T_m-factorization develops under the action of the permutations α^s $(s = 0, 1, 2, 3, 4)$ where $\alpha = (12345)(6789A)$.

The factorizations having the automorphism α, and isomorphic to them, are called *bicyclic.*

Table 1

Num	Base component	Num	Base component
1.	12 24 35 39 46 59 5A 78 7A	2.	17 23 25 36 46 48 5A 78 79
3.	15 19 25 27 3A 48 4A 69 6A	4.	18 19 26 27 34 35 56 78 7A
5.	13 19 27 29 45 4A 59 68 6A	6.	12 17 35 38 46 47 59 69 6A
7.	13 15 26 46 47 4A 5A 79 89	8.	12 13 16 2A 37 4A 57 68 89
9.	12 13 16 19 4A 57 59 78 8A	10.	12 13 16 17 46 58 59 7A 9A
11.	12 26 35 39 46 49 58 79 9A	12.	12 14 17 19 26 38 57 68 9A
13.	12 14 16 17 29 36 59 78 8A	14.	13 15 2A 46 56 59 5A 78 79
27.	12 13 16 47 48 56 57 89 8A	28.	12 13 16 28 46 47 59 79 9A
29.	12 13 16 47 48 56 57 79 9A	30.	12 24 2A 37 39 46 5A 69 89
31.	12 13 18 36 48 49 56 79 9A	32.	12 13 16 46 47 4A 59 89 8A
33.	12 13 16 46 48 4A 58 79 9A	34.	12 24 38 39 46 47 59 78 8A
35.	12 25 37 3A 49 56 58 78 79	36.	12 13 16 46 4A 58 59 67 8A
37.	12 24 26 36 38 56 57 79 9A	38.	12 24 39 46 47 78 7A 59 5A
39.	13 16 18 26 34 47 56 8A 9A	40.	12 24 36 46 56 59 5A 78 7A
41.	12 24 37 39 46 47 5A 78 8A	42.	12 24 36 46 4A 59 5A 7A 89
43.	12 24 36 46 4A 59 5A 78 7A	44.	12 25 26 38 3A 4A 58 7A 9A
45.	12 25 26 29 36 38 4A 78 8A	46.	12 24 26 38 4A 57 58 89 8A
47.	12 26 35 38 39 3A 47 67 79	48.	12 13 16 3A 4A 58 59 7A 9A
49.	12 26 35 38 39 3A 47 67 68	50.	12 24 39 46 47 59 5A 79 89
51.	12 16 18 1A 24 36 56 79 9A	52.	12 24 36 38 39 46 59 79 9A
53.	12 25 37 38 3A 47 56 67 79	54.	12 26 35 46 47 4A 5A 8A 9A
55.	12 26 35 46 47 4A 5A 8A 9A	56.	12 24 36 37 38 46 56 69 9A
57.	12 24 38 39 46 48 58 67 8A	58.	12 24 36 46 56 59 5A 67 8A
59.	12 24 37 39 46 47 5A 78 7A	60.	12 16 17 1A 24 36 57 8A 9A
73.	12 13 16 17 48 57 58 89 8A	74.	16 23 25 28 29 37 47 67 7A
75.	12 16 17 1A 24 36 57 78 79	76.	12 24 39 46 47 49 59 89 8A
77.	12 26 35 36 38 3A 4A 7A 9A	83.	12 24 36 37 39 46 5A 6A 8A
84.	12 24 36 37 46 4A 5A 68 9A	85.	12 24 26 36 56 57 5A 89 8A
86.	12 24 26 36 38 3A 56 7A 89	87.	12 24 39 46 47 59 5A 78 79
88.	12 26 33 36 38 46 4A 7A 9A	89.	12 24 26 38 39 3A 58 67 69
90.	12 24 26 36 38 39 57 79 9A	91.	12 25 26 39 49 57 58 8A 9A
92.	12 24 26 36 38 39 57 78 7A	93.	12 24 26 36 56 57 5A 68 89
94.	12 24 26 36 56 57 5A 68 9A	95.	12 24 26 38 39 3A 58 68 78
96.	12 24 26 39 47 49 57 78 7A	97.	12 25 37 49 56 57 58 79 9A
98.	12 24 26 29 39 58 5A 7A 9A	99.	12 24 26 27 3A 4A 58 89 8A
100.	12 13 16 48 56 57 58 89 8A	101.	12 13 16 46 47 48 56 69 6A
104.	12 16 17 25 29 37 47 68 6A	105.	12 24 26 36 56 57 5A 8A 9A

So the problem under consideration is solved for the trees T of order 10 with $\Delta(T) < 5$. For 11 from the remaining 18 admissible trees with $\Delta(T) = 5$ the corresponding bicyclic factorizations are cited in the Table 2.

Table 2

Num	Base component	Num	Base component
16.	12 13 16 18 19 4*A* 59 67 7*A*	18.	12 13 16 17 19 46 59 78 8*A*
63.	12 24 26 3*A* 4*A* 58 5*A* 7*A* 9*A*	64.	16 23 24 28 29 2*A* 59 67 79
65.	12 24 36 46 56 59 5*A* 67 68	67.	12 24 38 39 46 48 58 78 8*A*
68.	12 13 16 46 47 56 59 68 6*A*	79.	12 25 38 48 56 57 58 89 8*A*
80.	12 24 26 28 38 57 58 89 8*A*	82.	12 13 19 *lA* 16 46 56 67 68
106.	12 24 26 3*A* 47 4*A* 5*A* 8*A* 9*A*		

For trees T_{15}, T_{17}, T_{19}, T_{61}, T_{62}, T_{66}, T_{78} the question, do they belong to $\mathbf{T}_{10}$ or not, will be answered in the next section.

The nonexistence theorems

It is easy to establish that the system (5) - (9) has a solution in real numbers under the condition

$$4d_1 = d_2 + 6d_3 + 11d_4 + 16d_5, \tag{10}$$

and in this case its general solution looks like

$$\begin{aligned} a_1 &= 5d_1 - l0d_3 - 15d_4 - 20d_5 + h, \\ a_2 &= 5d_3 - 2h, \\ a_3 &= h, \\ a_4 &= 5d_4, \\ a_5 &= 5d_5. \end{aligned} \tag{11}$$

Also, it is easy to check that the condition (10) fulfils for all admissible trees of order 10.

Of course we are interested in the solutions with nonnegative integer a_i's.

Remark 1: In particular, for $T \in \{T_{15}, T_{17}, T_{19}\}$ we have $d(T) = 54001$, and the system (5) - (9) have the unique solution

$$a_1 = a_5 = 5, \ a_2 = a_3 = a_4 = 0.$$

Theorem 4. *None of the trees T_{15}, T_{19} belongs to $\mathbf{T}_{10}$.*

Proof: Suppose that there exist the decomposition R of K_{10} into trees isomorphic to T, $T \in \{T_{15}, T_{19}\}$. Due to Remark 1,there are 5 vertex of type t_1 and 5 vertex of type t_5 in R.

Let A be the set of the vertices of type t_5, and B be the set of other vertices. Let $X(A, B)$ be the complete bipartite graph with parts A, B.

Consider the case

$$T = T_{15}: 12\,13\,14\,15\,16\,67\,78\,89\,9A.$$

On one hand, the vertex of degree 2 in every component of R must lie in the set B. But, on the other hand, then the 5 vertices in B will be joined by 15 edges (3 edges from every component). But it is impossible because there are only ten such edges. The contradiction proves our theorem in the case $T = T_{15}$.

Now, let us consider the case

$$T = T_{19}: 12\,13\,14\,15\,16\,27\,38\,49\,5A.$$

As all the 2-vertices must lie in B, the edges of components joining the centers with 2-vertices must lie in $X(A, B)$. Those edges cover 20 edges of $X(A, B)$. Then at most 5 from the 20 edges incident with 2-vertices lie in $X(A, B)$. So at least 15 component endvertices must lie in B. But, on the other hand, B contains only 5 such vertices, a contradiction.

The Theorem is proved.

By straight computer search we had obtained the following result.

Theorem 5. *None of the trees T_{17}, T_{61}, T_{62}, T_{66}, T_{78} belongs to $\mathbf{T}_{10}$.*

It will be interesting to find a visual reason of the nonexistence of T-factorizations in these 5 cases.

The above consideration covers all trees of order 10. Now we may unite our results in

Theorem 6. *From 106 trees of order 10 there are exactly 85 admitting T-factorizations. Moreover, for every such tree there exists a bicyclic T-factorization.*

The more general problem

Let us formulate a more general problem and generalize Theorems 1 and 2.

Let $T^1, T^2, \ldots, T^k$ be a k-tuple of arbitrary trees of order $n = 2k$, not necessary distinct. We say that we have a $\{T^1, T^2, \ldots, T^k\}$*-factorization* R if there are k edge-disjoint trees in K_n such that i-th tree is isomorphic to T^i, $i = 1, 2, \ldots, k$. (In fact, the order of trees in the k-tuple is not substantial.)

As an examples we present the $\{T_{17}, T_{17}, T_{17}, T_{17}, T_{18}\}$-factorization

$12\,23\,34\,45\,56\,67\,48\,49\,4A$ $\quad$ $13\,27\,35\,47\,57\,68\,69\,78\,7A$
$15\,26\,29\,39\,46\,58\,79\,89\,9A$ $\quad$ $14\,16\,17\,18\,19\,25\,28\,36\,3A$
$1A\,24\,2A\,37\,38\,59\,5A\,6A\,8A$

and the $\{T_{17}, T_{17}, T_{17}, T_{17}, T_{19}\}$-factorization

12 23 34 45 56 67 48 49 4A	13 27 28 35 47 57 68 79 7A
14 16 17 18 1A 25 36 39 58	19 24 26 37 38 59 69 89 9A
15 29 2A 3A 46 5A 6A 78 8A	

.

The problem is: for what k-tuples $\{T^1, T^2, \ldots, T^k\}$ there exists a $\{T^1, T^2, \ldots, T^k\}$-factorization? Let, further,

$$\mathbf{d}(T^i) = (d_{i1}, d_{i2}, \ldots, d_{ik}),\ i = 1, 2, \ldots, k.$$

Moreover, let us introduce the quantities

$$D_j = d_{1j} + d_{2j} + \cdots + d_{kj}\ (j = 1, 2, \ldots, k)$$

and the vector $\mathbf{D} = (D_1, D_2, \ldots, D_k)$.

It is obvious that the notion of vertex types and vector $\mathbf{a} = \mathbf{a}(R)$ can be simply carried over the general case.

In this notation we can write the matrix equation

$$\mathbf{aS} = \mathbf{D}, \tag{12}$$

which is the analog and generalization of (4), and formulate the theorem absolutely analogous to Theorem 1. The corresponding analog of Theorem 2 can be formulated so.

Theorem 7. *If there exists a $\{T^1, T^2, \ldots, T^5\}$-factorization then $D_2 \geq D_4$.*

It seems like the relations (4) and (12) are able to produce new necessary conditions and give other important information about the tree factorizations.

Further nonexistences

Now we shall present an auxiliary nonexistence result. A tree r-packing into K_n is called *completable* if it can be embedded in any $\{T^1, T^2, \ldots, T^k\}$-factorization, and *completeless* otherwise.

Let P be a tree r-packing in K_n consisting of trees $T^1, T^2, \ldots, T^r$, and let $G(P) = T^1 \cup T^2 \cup \cdots \cup T^r$. Then the following statement takes place.

Theorem 8. *If P is completable then, for every vertex x of K_n, the inequality $\deg(x) \leq k + r - 1$ holds. If, for a packing P, there exists a vertex x with $\deg(x) > k + r - 1$ then P is completeless.*

Proof: Let P be a completable r-packing, and R be a completion of P. Let x be an arbitrary vertex of K_n. Then every one of $k - r$ components in R not belonging to P has in the vertex x degree ≥ 1. So at least $k - r$ edges incident to x do not take part in P. This gives

$$\deg(x) + k - r \leq 2k - 1, \tag{13}$$

and hence $\deg(x) \leq k + r - 1$, and the first assertion of the theorem is proved. The second assertion directly follows from the first one, and so the proof is over.

In the case $r = 1$ the completability condition (13) coincides with the condition of admissibility of a tree. So the Theorem 8 is a generalization of the condition discovered by Beineke [1]. Being applied to constructing the tree factorizations 'component by component', the condition helps us to avoid a heap of unperspective material to look through.

For the general problem Beineke's result [1] is: only admissible trees can be components of a $\{T^1, T^2, \ldots, T^k\}$-factorization. It is interesting to select from the ordered quintuples $\{T^1, T^2, \ldots, T^5\}$ of admissible trees such ones that do not satisfy the condition $D_2 \geq D_5$, and so to obtain a nonexistence result.

The superficial investigation in the case $n = 12$ gives us the following necessary condition, similar to Theorem 2.

Theorem 9. *$T \in \mathbf{T}_{12}$ implies $d_2 \geq d_5$.*

It is easy to establish that for the case $n = 8$ we have $d_2 \geq d_3$. By analogy, the following generalization was conjectured and later on the simple graph theoretical proof was found.

Theorem 10. *The condition $d_2 \geq d_{k-1}$ is necessary for $T \in \mathbf{T}_{2k}$.*

Proof: If $d_{k-1} = 0$ then the assertion is obvious. Let $d_{k-1} > 0$, and let there exists a T-factorization R. Let, further, x be an arbitrary vertex of K_{2k}, in which some component of R has degree $k - 1$. It is obvious that for x such component is unique. Other $k - 1$ components must cover k remaining edges incident to x, and every one of them must have in x a degree ≥ 1. This situation can be realized in the unique way, namely when $k - 2$ components have the degree 1 in x and one component has degree 2.

So, every vertex with degree $k - 1$ in a component has the degree 2 in some other component. As in R there are in total $k \cdot d_{k-1}$ vertices having component degrees $k - 1$, and $k \cdot d_2$ vertices with component degree 2 then we obtain $k \cdot d_{k-1} \leq k \cdot d_2$. Hence the theorem assertion follows.

A similar reflection leads us to the following

Theorem 11. *For $k \geq 6$, the condition $d_2 + 2d_3 \geq 2d_{k-2}$ is necessary for $T \in \mathbf{T}_{2k}$.*

Corollary. *If $T \in T_{12}$ then $d_2 + 2d_3 \geq 2d_4$.*

The problems and prospects

Now we shall list some open problems connected with the topic of this paper.

To every T-factorization R there corresponds a vector $\mathbf{a}(R)$. We call the vector the type of the factorization R. All possible nonnegative integer solutions $\mathbf{a}$ of (4) are called possible types.

For example, the tree T_{66} has $\mathbf{d} = 62101$, the system (5) - (9) has three solutions and accordingly we have three possible types of T_{66}-factorizations.

	a_1	a_2	a_3	a_4	a_5
Type 1	0	5	0	0	5
Type 2	1	3	1	0	5
Type 3	2	1	2	0	5

Problem 1. For every possible type $\mathbf{a}$ answer if there exists a T-factorization of the type $\mathbf{a}$.

Problem 2. For every $T \in \mathbf{T}_n$, enumerate all T-factorizations, up to isomorphism.

Remark that Problem 2 is completely solved in [3,4] for the cases $n \leq 8$. Now the problem is open for $n \geq 10$.

We can formulate the following partial result in this area obtained with a computer.

Theorem 12. *The T_{64}-factorization presented in Table 2 is unique, up to isomorphism.*

Problem 3. Enumerate, up to isomorphism, $\{T^1, T^2, \ldots, T^k\}$-factorizations for every k-tuple of admissible trees of order n.

For $n = 6$ the problem is solved in [3]. For $n = 8$ the problem is partialy solved in [4].

Acknowledgement. The author thanks the anonymous referee for helpful suggestions.

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Summary

We consider the problem of existence of T-factorizations, i.e. the decompositions of K_n into mutually isomorphic spanning trees. It occurs that from 106 nonisomorphic 10-vertex trees exactly 85 trees admit the T-factorization. The necessary condition for the existence is produced, and a generalization of the problem is proposed and considered.