Cordial Labelings Of Some Wheel Related Graphs

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Abstract

Let G be a graph with vertex set V and edge set E. A vertex labelling $f : V \longrightarrow \{0,1\}$ induces an edge labelling $\overline{f} : E \longrightarrow \{0,1\}$ defined by $\overline{f}(uv) = | f(u) - f(v) |$. Let $v_f(0), v_f(1)$ denote the number of vertices v with $f(v) = 0$ and $f(v) = 1$ respectively. Let $e_f(0), e_f(1)$ be similarly defined. A graph is said to be **cordial** if there exists a vertex labeling f such that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$. In this paper, we show that for every positive integer t and n the following families are cordial: (1) Helms H_n. (2) Flower graphs FL_n. (3) Gear graphs G_n. (4) Sunflower graphs SFL_n. (4) Closed helms CH_n. (5) Generalised closed helms $CH(t,n)$. (6) Generalised webs $W(t,n)$.

Introduction

In this paper all graphs are finite, simple and undirected. Let $G = G(V, E)$ be a graph with the vertex set V and the edge set E. A mapping $f : V \longrightarrow \{0,1\}$ is called a **vertex labeling** or a **binary vertex labeling** of the graph G. For each $v \in V$, $f(v)$ is called the vertex-label of v. For an edge $e = uv$, the induced edge labeling $\overline{f} : E \longrightarrow \{0,1\}$ is given by $\overline{f}(e) = | f(u) - f(v) |$. Let $v_f(0), v_f(1)$ be the number of vertices of G having label zero and one respectively and let $e_f(0), e_f(1)$ be the number of edges having label zero and one respectively.

Definition: A binary vertex labeling of a graph G is called a **cordial labeling** if $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$. A graph G is called **cordial** if it admits a cordial labeling.

Cordial graphs were first introduced by Cahit, as a weaker version of both graceful and harmonious graphs [1]. In the same paper, Cahit proved that a wheel W_n is cordial iff $n \equiv 0, 1, 2 \bmod 4$. In this paper, we show that following wheel related families are cordial: Helms, generalised helms, generalised webs, flower graphs, sunflower graphs and gear graphs.

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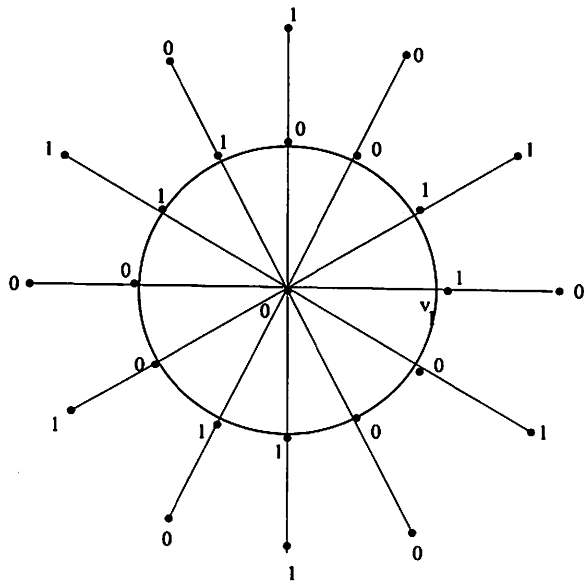
Helms, gears and flower graphs

A **wheel** W_n, is the Cartesian product $K_1 \times C_n$. A helm H_n is obtained from the wheel by attaching a pendant vertex to each of the vertices on C_n in W_n. The helm H_n has $2n+1$ vertices and $3n$ edges.

Theorem: The helm H_n is cordial for all $n \geq 3$.

Proof: Let the vertex set of H_n be $V = \{u, v_1, \cdots, v_n, w_1, \cdots, w_n\}$ and let the edge set of H_n be $E = \{uv_i, v_iw_i \mid 1 \leq i \leq n\} \bigcup \{v_nv_1, v_iv_{i+1} \mid 1 \leq i \leq n-1\}$. Here u is the central vertex. Let $n = 4q + r$.



H_{12}

Define a binary labeling f of H_n as follows:

$$f(u) = 0, \quad f(v_i) = \begin{cases} 1, & i \equiv 1,2 \bmod 4 \\ 0, & i \equiv 0,3 \bmod 4 \end{cases}$$

$$f(w_i) = \begin{cases} 0, & i \equiv 1 \bmod 2 \\ 1, & i \equiv 0 \bmod 2 \end{cases}$$

The following table shows that f is a cordial labeling.

n	$v_f(0)$	$v_f(1)$	$e_f(0)$	$e_f(1)$
$4q$	$4q+1$	$4q$	$6q$	$6q$
$4q+1$	$4q+2$	$4q+1$	$6q+1$	$6q+2$
$4q+2$	$4q+2$	$4q+3$	$6q+3$	$6q+3$
$4q+3$	$4q+4$	$4q+3$	$6q+4$	$6q+5$

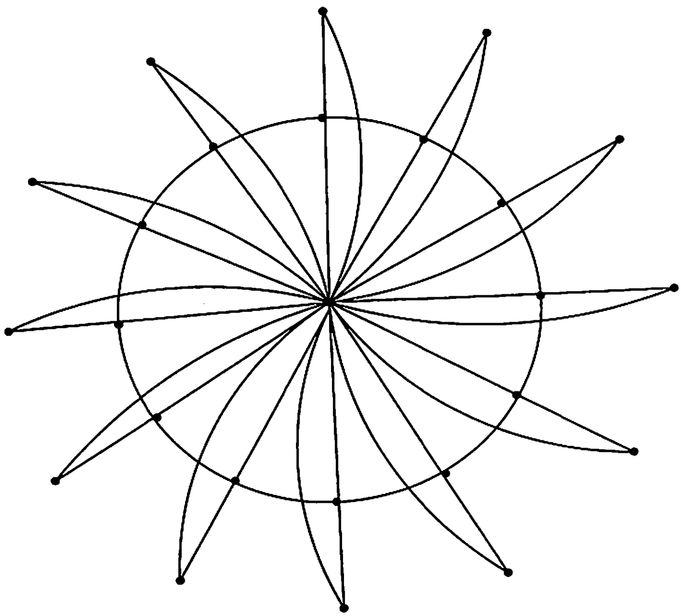
□

A closed **helm** CH_n is obtained by taking a helm H_n and by adding edges $\{w_i w_{i+1} \mid 1 \leq i \leq n\}$ to the edge set $E(H_n)$.

Theorem: All closed helms are cordial.

Proof: Define a binary labeling f of the closed helm CH_n as follows: $f(u) = 0,\ f(v_i) = 1,\ f(w_i) = 0, 1 \leq i \leq n$. One can easily see that this is a cordial labeling. □

A **flower** FL_n is a graph obtained from the helm H_n by attaching each of its pendant vertices to its central vertex by an edge. Thus $E(FL_n) = E(H_n) \bigcup \{uw_i \mid 1 \leq i \leq n\}$ and $V(FL_n) = V(H_n)$.



FL_{12}

Corollary: All flowers FL_n are cordial.

Proof: Consider the flower FL_n obtained from H_n. Let f be the binary labeling given before. We only have to check that $\mid e_f(0) - e_f(1) \mid \leq 1$. One just notes that for FL_n with this labeling $e_f(0) = e_f(1) = 2n$. □

A **gear** graph G_n is obtained from a wheel W_n by inserting a vertex on each of the cyclic edges of C_n in W_n. Consider the wheel W_n with u as the central vertex. Let $C_n = \{v_1, \cdots, v_n, v_1\}$. For $1 \leq i \leq n$, let w_i be the additional vertex inserted on the edge $v_i v_{i+1}$. Here $i+1$ is taken modulo n. So the gear graph has $2n+1$ vertices and $3n$ edges.

Theorem: All gears are cordial.

Proof: Let G_n be the gear graph given before. Define a binary labeling f

of G_n as follows:
$f(u)=0,\ f(v_i)=f(w_i)=1,\quad i\equiv 1 \bmod 2,$
$f(v_i)=f(w_i)=0,\quad i\equiv 0 \bmod 2.$

It is easy to see that for odd values of n, $v_f(0)=n, v_f(1)=n+1$, $e_f(0)=(3n+1)/2$ and $e_f(1)=(3n-1)/2$ and for even values of n, $v_f(0)=n+1, v_f(1)=n, e_f(0)=e_f(1)=(3n)/2$. Hence G_n is cordial. □

A **sunflower graph** SF_n has vertex set and edge set as follows:
$V(SF_n)=\{u,v_i \mid 1\le i\le n\}\bigcup\{w_i \mid 1\le i\le n\}$,
$E(SF_n)=\{uv_i, v_iv_{i+1}, w_iv_i, w_iv_{i+1} \mid 1\le i\le n\}$.
Here $i+1$ is taken modulo n. This means that, SF_n is obtained by replacing each cyclic edge of C_n in W_n by K_3.

Theorem: All sunflowers are cordial.

Proof: Define a binary labeling f of SF_n as follows: $f(u)=0, f(v_i)=0$, $f(w_i)=1, 1\le i\le n$. One can see that $v_f(0)=n+1, v_f(1)=n, e_f(0)=e_f(1)=2n$. Hence SF_n is cordial. □

Generalised closed helms and webs

A t-ply generalised helm $CH(t,n)$ is obtained by taking t copies of C_n in a concentric manner. If the central vertex is u and the jth cycle is denoted by $C_{n,j}$ with vertices $x_{1,j},\cdots,x_{n,j}$ then

$$E(CH(t,n))=\bigcup_{j=1}^{t}E(C_{n,j})\bigcup_{r=1}^{n}\{ux_{r,1},x_{r,1}x_{r,2},\cdots,x_{r,t-1}x_{r,t}\}.$$

We show that $CH(t,n)$ is cordial for all positive integers t,n.

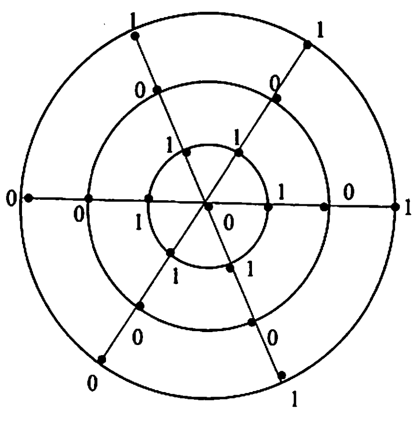
Theorem: For all positive integers $t\ge 2$ and $n\ge 3$ the generalised closed helms $CH(t,n)$ are cordial.

Proof: Case 1: Let t be even. Label the central vertex u by 0.
For $1\le r\le n$, define $f(x_{r,j})=1$ if j is odd and $f(x_{r,j})=0$ if j is even. One can easily see that $v_f(0)=1+(nt)/2, v_f(1)=(nt)/2, e_f(0)=e_f(1)=nt$. Hence $CH(t,n)$ is cordial in this case.

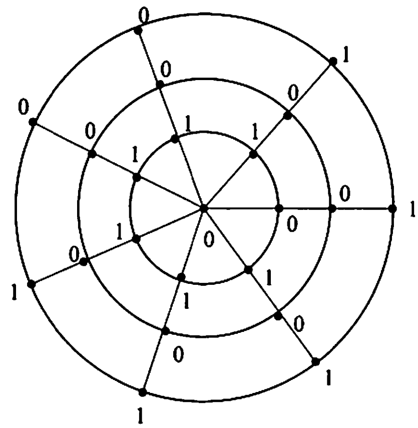
The labeling given here coincides with the labeling of the closed helm CH_n given earlier, if we take $t=2$.

Case 2: Let t be odd. Let $n\equiv s \bmod 4$. First suppose $s\ne 3$. Label $CH(t-1,n)$ as in Case 1. Let $f(x_{r,t})=1$ if $r\equiv 1,2 \bmod 4$ and $f(x_{r,t})=0$ if $r\equiv 0,3 \bmod 4$. One can see that $e_f(0)=e_f(1)=nt$ for $r=0,1,2$. On the other hand,
(1) For $s=0$, $v_f(0)=1+(nt)/2, v_f(1)=nt/2$,
(2) For $s=1$, $v_f(0)=(1+nt)/2, v_f(1)=(1+nt)/2$,
(3) For $s=2$, $v_f(0)=nt/2, v_f(1)=1+(nt)/2$.

Now suppose $s=3$. Let $f(x)$ be same as before for all $x\ne x_{1,1},x_{n,t}$.



CH(3.6)



CH(3.7)

Let $f(x_{1,1}) = 0, f(x_{n,t}) = 1$. One can see that $v_f(0) = v_f(1) = (1+nt)/2$, $e_f(0) = e_f(1) = nt$. This shows that $CH(t,n)$ is cordial in this case also.□

A **generalised web** $W(t,n)$ is the graph obtained from $CH(t,n)$ by attaching a pendant vertex to each of the vertices of the outer cycle of $CH(t,n)$.

Corollary: All generalised webs are cordial.

Proof: Take $CH(t,n)$ with its cordial labeling defined before. Let the pendant vertex attached to $x_{r,t}$ be denoted by $x_{r,t+1}, 1 \leq r \leq n$. Let $f(x_{r,t+1}) = 1$ if r is odd and $f(x_{r,t+1}) = 0$ if r is even. Let $n \equiv s \bmod 4$. The following table shows that $W(t,n)$ is cordial for all $t \geq 2, n \geq 3$.

t	s	$v_f(0)$	$v_f(1)$	$e_f(0)$	$e_f(1)$
Even	$0,2$	$\frac{2+n(t+1)}{2}$	$\frac{n(t+1)}{2}$	$\frac{n(2t+1)}{2}$	$\frac{n(2t+1)}{2}$
	$1,3$	$\frac{1+n(t+1)}{2}$	$\frac{1+n(t+1)}{2}$	$\frac{n(2t+1)-1}{2}$	$\frac{n(2t+1)+1}{2}$
Odd	0	$\frac{2+n(t+1)}{2}$	$\frac{n(t+1)}{2}$	$\frac{n(2t+1)}{2}$	$\frac{n(2t+1)}{2}$
	$1,3$	$\frac{n(t+1)}{2}$	$\frac{2+n(t+1)}{2}$	$\frac{n(2t+1)+1}{2}$	$\frac{n(2t+1)-1}{2}$
	2	$\frac{n(t+1)}{2}$	$\frac{2+n(t+1)}{2}$	$\frac{n(2t+1)}{2}$	$\frac{n(2t+1)}{2}$

□

References

1. I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, Ars Combin., 23(1987) 201-207.
2. J. A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, 5(1999).