# Valuations of plane quartic graphs

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**Abstract**

We deal with $(a,d)$-face antimagic labelings of a certain class of plane quartic graphs. A connected plane graph $G = (V, E, F)$ is said to be $(a,d)$-*face antimagic* if there exist positive integers $a$ and $d$, and a bijection $g : E(G) \to \{1, 2, \ldots, |E(G)|\}$ such that the induced mapping $\varphi_g : F(G) \to N$, defined by $\varphi_g(f) = \sum\{g(e) : e \in E(G)$ adjacent to face $f\}$, is injective and $\varphi_g(F) = \{a, a+d, \ldots, a+(|F(G)|-1)d\}$.

## 1 Introduction and Definitions

The graphs considered here are finite, undirected, without loops and multiple edges. The symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph $G$.

Hartsfield and Ringel [7] introduced the concept of an antimagic graph. An *antimagic graph* is a graph whose edges can be labeled with the integers $1, 2, \ldots |E(G)|$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is, no two vertices have

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the same sum. Hartsfield and Ringel conjecture that every tree different from $K_2$ is antimagic, and, more strongly, that every connected graph other than $K_2$ is antimagic.

Bodendiek and Walther started looking for a new edge labeling arising from antimagic labeling. They defined the concept of an $(a,d)$-antimagic graph and in [5,6] described $(a,d)$-antimagic labelings for the special graphs called parachutes.

In [1], there is a complete characterization of $(a,d)$-antimagic graphs of prisms $D_n$ when $n$ is even and it is shown that if $n$ is odd, the prisms $D_n$ are $(\frac{5n+5}{2},2)$-antimagic.

Furthermore, $(a,d)$-antimagic labelings for the antiprisms $Q_n$ are given in [2].

In this paper we concentrate on finite connected plane graphs. A graph is said to be *plane* if it is drawn on the Euclidean plane in such a way that edges do not cross each other except at vertices of the graph. Assume that all the plane graphs considered in this paper possess no vertices of degree one so that it makes sense to determine the face set $F(G)$ of a plane graph $G=(V,E,F)$.

The *weight* $w(f)$ of a face $f \in F(G)$ under an edge labeling $g : E(G) \to \{1,2,\ldots,|E(G)|\}$ is the sum of the labels of the edges surrounding that face.

A connected plane graph $G=(V,E,F)$ is said to be *$(a,d)$-face antimagic* if there exist positive integers $a$ and $d$, and a bijection $g : E(G) \to \{1,2,\ldots,|E(G)|\}$ such that the induced mapping $\varphi_g : F(G) \to W$ is also a bijection, where $W = \{w(f) : f \in F(G)\} = \{a, a+d, a+2d, \ldots, a+(|F(G)|-1)d\}$ is the set of weights of faces.

If $G=(V,E,F)$ is $(a,d)$-face antimagic and $g : E(G) \to \{1,2,\ldots|E(G)|\}$ is a corresponding bijective mapping of $G$ then $g$ is said to be an *$(a,d)$-face antimagic labeling* of $G$.

Notice that $(a,d)$-face antimagic labeling of a convex polytope $G$ is equivalent to $(a,d)$-antimagic labeling of its dual graph $G^*$.

Now, we provide the construction of a certain class of convex polytopes $\mathcal{D}_n^m$ whose $(a,d)$-face antimagic labelings will be investigated in this paper.

The $m$-prism $D_n^m$, $n \geq 3$, $m \geq 1$, is a trivalent graph of a convex polytope which can be defined as the cartesian product $P_{m+1} \times C_n$ of a path on $m+1$ vertices with a cycle on $n$ vertices, embedded in the plane.

Let $I = \{1, 2, \ldots, n\}$ and $J = \{1, 2, \ldots, m\}$ be index sets. Let us denote the vertex set of $m$-prism $D_n^m$ by

$V(D_n^m) = \{x_{j,i} : i \in I \text{ and } j \in J \cup \{m+1\}\}$ and the edge set by $E(D_n^m) = \{x_{j,i}x_{j,i+1} : i \in I \text{ and } j \in J \cup \{m+1\}\} \cup \{x_{j,i}x_{j+1,i} : i \in I \text{ and } j \in J\}$.

We make the convention that $x_{j,n+1} = x_{j,1}$ and $x_{j,n+2} = x_{j,2}$ for $j \in J \cup \{m+1\}$.

The face set $F(D_n^m)$ contains $nm$ 4-sided faces, an internal $n$-sided face and an external $n$-sided face.

Now, we insert exactly one vertex $y$ (respectively $z$) into the internal (respectively external) $n$-sided face of $D_n^m$.

Suppose that $n$ is even, $n \geq 4$, and consider the graph $\mathcal{D}_n^m$ with vertex set $V(\mathcal{D}_n^m) = V(D_n^m) \cup \{y, z\}$ and

(i) if $m$ is odd, the edge set $E(\mathcal{D}_n^m) = E(D_n^m) \cup \{x_{1,2k-1}y : k = 1, 2, \ldots, \frac{n}{2}\} \cup \{x_{m+1,2k}z : k = 1, 2, \ldots, \frac{n}{2}\}$ ; and

(ii) if $m$ is even, the edge set $E(\mathcal{D}_n^m) = E(D_n^m) \cup \{x_{1,2k-1}y : k = 1, 2, \ldots, \frac{n}{2}\} \cup \{x_{m+1,2k-1}z : k = 1, 2, \ldots \frac{n}{2}\}$.

Then $\mathcal{D}_n^m$, $n \geq 4$, $m \geq 1$, is the plane graph of the convex polytope on $|V(\mathcal{D}_n^m)| = n(m+1)+2$ vertices, $|E(\mathcal{D}_n^m)| = 2n(m+1)$ edges and consisting of $|F(\mathcal{D}_n^m)| = n(m+1)$ 4-sided faces. Let vertices of $\mathcal{D}_n^m$, if $m$ is odd, be labeled as in Figure 1.

## 2 Necessary Conditions

In this section we present several necessary conditions for the graph $\mathcal{D}_n^m$ to be $(a, d)$-face antimagic.

**Theorem 1.** If $\mathcal{D}_n^m$ is $(a, d)$-face antimagic then either $d = 2$ and $a = 3n(m+1)+3$, or $d = 4$ and $a = 2n(m+1)+4$, or $d = 6$ and $a = n(m+1)+5$.

**Proof.** Assume that $\mathcal{D}_n^m$ is $(a, d)$-face antimagic and $W = \{w(f) : f \in F(\mathcal{D}_n^m)\} = \{a, a+d, \ldots, a+[n(m+1)-1]d\}$ is the set of weights of faces.

The sum of weights in the set $W$ is $n(m+1)[a + \frac{d}{2}(nm+n-1)]$. The edges of $\mathcal{D}_n^m$ are labeled by the set of consecutive integers $\{1, 2, \ldots, 2n(m+1)\}$ and each of these labels is used twice in the computation of the weights of faces.

Thus, the sum of all the edge labels used to calculate the weights of faces

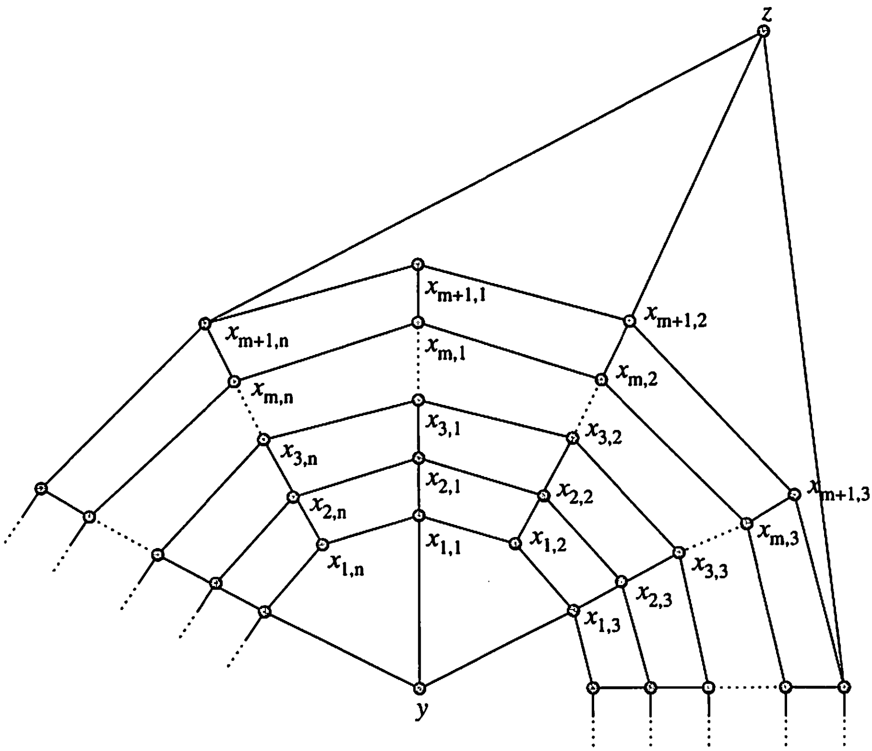


Figure 1: Plane graph $\mathcal{D}_n^m$

is equal to

$$n(m+1)(2nm+2n+1).$$

We have the following equation

$$2(2nm+2n+1) \quad = \quad a+\frac{d}{2}(nm+n-1) \tag{1}$$

or, equivalently,

$$d \quad = \quad \frac{8n(m+1)+4-2a}{nm+n-1}. \tag{2}$$

The minimal value of weight which can be assigned to a 4-sided face is equal to 10.

From (2), if $a \geq 10$, we get the upper bound on the parameter $d$, i.e., $0 < d < 8$. Since $n$ is even then from (1) it follows that $d$ is also even. This implies that the equation (1) has exactly the three solutions

$$(a,d) = (3n(m+1)+3,2) \text{ or}$$
$$(a,d) = (2n(m+1)+4,4) \text{ or}$$
$$(a,d) = (n(m+1)+5,6).$$

# 3 $(a,2)$-Face Antimagic Labelings

Armed with the results from the previous section, we are now ready to study the $(a,d)$-face antimagic labelings of $\mathcal{D}_n^m$. Let us start with the case $d=2$.

An $(3n(m+1)+3,2)$-face antimagic labeling of $\mathcal{D}_n^m$ for $m=1$, $n \geq 4$ was described in [3] and for $m=2$, $n \geq 4$, in [4]. Here we shall consider $m \geq 3$.

We will make use of the following functions to simplify later notations.

$$\psi(u) = \begin{cases} 1 & \text{if } u \equiv 1 \pmod 2 \\ 0 & \text{if } u \equiv 0 \pmod 2 \end{cases}$$

$$\lambda(p,q)=\begin{cases}1 & \text{if } p\le q\\ 0 & \text{otherwise}\end{cases}$$

$$\alpha(u)=\begin{cases}1 & \text{if } u\equiv 1 \pmod 4\\ 0 & \text{if } u\equiv 3 \pmod 4\end{cases}$$

**Theorem 2.** If $n\equiv 0 \pmod 2$, $n\ge 4$ and $m\equiv 1 \pmod 2$, $m\ge 3$, or if $n\equiv 2 \pmod 4$, $n\ge 6$ and $m\equiv 0 \pmod 2$, $m\ge 4$, then the graph of the convex polytope $\mathcal{D}_n^m$ has $(3n(m+1)+3,2)$-face antimagic labeling.

**Proof.** First suppose $m$ is odd, $m\ge 3$, and construct the edge labeling $g_1$ of $\mathcal{D}_n^m$ in the following way.

$$\begin{aligned}g_1(x_{j,i}x_{j,i+1}) = \quad & ([(j-1)n+i+1]\lambda(j,\tfrac{m+1}{2})\\ & +[(m-j+1)n+i+1]\lambda(\tfrac{m+1}{2}+1,j))\psi(j)\\ & +[((m+2-j)n\lambda(n-1,i)\\ & +[(m+2-j)n-1-i]\lambda(i,n-3))\lambda(j,\tfrac{m+1}{2})\\ & +[nj\lambda(n-1,i)\\ & +(nj-i-1)\lambda(i,n-3)]\lambda(\tfrac{m+1}{2}+1,j)]\psi(j+1)\end{aligned}$$

for $i\in I$ odd and $j\in J\cup\{m+1\}$,

$$\begin{aligned}g_1(x_{j,i}x_{j,i+1}) = \quad & (([(m+2-j)n-1]\lambda(n,i)\\ & +[(m+2-j)n-1-i]\lambda(i,n-2))\lambda(j,\tfrac{m+1}{2})\\ & +[(nj-1)\lambda(n,i)\\ & +(nj-1-i)\lambda(i,n-2)]\lambda(\tfrac{m+1}{2}+1,j))\psi(j)\\ & +[([(j-1)n+1]\lambda(n,i)\\ & +[(j-1)n+1+i]\lambda(i,n-2))\lambda(j,\tfrac{m+1}{2})\\ & +([(m+1-j)n+1]\lambda(n,i)\\ & +[(m+1-j)n+1+i]\lambda(i,n-2))\lambda(\tfrac{m+1}{2}+1,j)]\\ & \psi(j+1)\end{aligned}$$

for $i\in I$ even and $j\in J\cup\{m+1\}$.

$$g_1(x_{j,i}x_{j+1,i}) = \quad [(m+1)n+1]\lambda(n,i)+[(m+1)n+1+i]\lambda(i,n-1)$$

for $i\in I$ and $j=\frac{m+1}{2}$,

$$\begin{aligned}g_1(x_{j,i}x_{j+1,i}) = \quad & [[n(2m+1-2j)+1]\lambda(n,i)\\ & +([n(2m+1-2j)+1+i]\psi(i+1)\\ & +[n(2m+2-2j)+1+i]\psi(i))\lambda(i,n-1)]\\ & \lambda(j,\tfrac{m+1}{2}-1)\\ & +((2jn+1)\lambda(n,i)+[(2jn+1+i)\psi(i+1)\\ & +[n(2j-1)+1+i]\psi(i)]\lambda(i,n-1))\lambda(\tfrac{m+1}{2}+1,j)\end{aligned}$$

for $i \in I$ and $j \in J$.

$$g_1(x_{1,i}y) = \quad n(2m+1)+i$$

for $i \in I$ odd.

$$g_1(x_{m+1,i}z) = \quad [n(2m+1)+2]\lambda(n,i) + [n(2m+1)+2+i]\lambda(i,n-2)$$

for $i \in I$ even.

It is a matter for routine checking to see that the edge labeling $g_1$ uses each integer $1, 2, \dots, 2n(m+1)$ exactly once and this implies that the labeling $g_1$ is a bijection from the edge set $E(\mathcal{D}_n^m)$ onto the set $\{1, 2, \dots, 2n(m+1)\}$.

Let us denote the weights of the 4-sided faces of $\mathcal{D}_n^m$, $m \geq 3$, $m \equiv 1 \pmod 2$, (under an edge labeling $g$) by

$$\begin{aligned} w_g(f_{j,i}) = \quad & g(x_{j,i}x_{j,i+1}) + g(x_{j+1,i}x_{j+1,i+1}) + g(x_{j,i}x_{j+1,i}) + \\ & g(x_{j,i+1}x_{j+1,i+1}) \end{aligned}$$

if $i \in I$ and $j \in J$,

$$w_g(f_{0,i}) = \quad g(x_{1,i}x_{1,i+1}) + g(x_{1,i+1}x_{1,i+2}) + g(x_{1,i}y) + g(x_{1,i+2}y)$$

if $i \in I$ is odd,

$$\begin{aligned} w_g(f_{m+1,i}) = \quad & g(x_{m+1,i}x_{m+1,i+1}) + g(x_{m+1,i+1}x_{m+1,i+2}) \\ & + g(x_{m+1,i}z) + g(x_{m+1,i+2}z) \end{aligned}$$

if $i \in I$ is even.

The weights of faces under the labeling $g_1$ constitute the sets

$$\begin{aligned} W_1 &= \{w_{g_1}(f_{j,i}) : i \in I \text{ and } j \in J\} \\ &= \{3n(m+1)+3, 3n(m+1)+5, \dots, 5mn+3n+1\} \end{aligned}$$

and

$$\begin{aligned} W_2 &= \{w_{g_1}(f_{0,i}) : i \in I \text{ odd }\} \cup \{w_{g_1}(f_{m+1,i}) : i \in I \text{ even }\} \\ &= \{5mn+3n+3, 5mn+3n+5, \dots, 5n(m+1)+1\}. \end{aligned}$$

We can see that each 4-sided face of $\mathcal{D}_n^m$ receives exactly one value of weight from $W_1 \cup W_2 = \{a, a+d, \dots, a+(|F(\mathcal{D}_n^m)|-1)d\}$, where $a = 3n(m+1)+3$ and $d = 2$, is used exactly once as a value of weight, which implies that the edge labeling $g_1$ is $(3n(m+1)+3, 2)$-face antimagic.

If $n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 4$ we define the bijective function

$g_2 : E(\mathcal{D}_n^m) \to \{1, 2, \dots, 2n(m+1)\}$, where

$$g_2(x_{j,i}x_{j,i+1}) = \begin{array}{l}([(m+1-j)n+i]\psi(i)+[(j-1)n+i]\psi(i+1))\psi(j)\\ +[(jn-i)\psi(i)+([(m+2-j)n-i]\lambda(i,n-2)\\ +(m+2-j)n\lambda(n,i))\psi(i+1)]\psi(j+1)\end{array}$$

for $i \in I$ and $j \in J \cup \{m+1\}$,

$$g_2(x_{j,i}x_{j+1,i}) = \begin{array}{l}([(2m-2j+1)n+i]\psi(i)\\ +[(2m-2j+2)n+i]\psi(i+1))\lambda(j,\frac{m}{2})\\ +((2jn+i)\psi(i)+[(2j-1)n+i]\psi(i+1))\lambda(\frac{m}{2}+1,j)\end{array}$$

for $i \in I$ and $j \in J$.

$$g_2(x_{1,i}y) = \begin{array}{l}[\frac{1}{4}(8mn+5n-2i+4)\psi(i,\frac{n}{2})\\ +\frac{1}{4}(8mn+9n-2i+4)\psi(\frac{n}{2}+4,i)]\alpha(i)\\ +\frac{1}{4}(8mn+7n-2i+4)\alpha(i+2)\end{array}$$

for $n \neq 6$ and $i \in I$,

$$g_2(x_{1,i}y) = [(2m+2)n-\tfrac{i+1}{2}]\alpha(i)+[(2m+1)n+i-2]\alpha(i+2)$$

for $n = 6$ and $i \in I$.

$$g_2(x_{m+1,i}z) = \tfrac{1}{2}(4mn+3n-i+3)\alpha(i)+\tfrac{1}{2}(4mn+4n-i+3)\alpha(i+2)$$

for $i \in I$.

If $n \equiv 2 \pmod 4$ and $m \equiv 2 \pmod 4$ we construct the bijective function $g_3 : E(\mathcal{D}_n^m) \to \{1, 2, \ldots, 2n(m+1)\}$ as follows.

$$g_3(x_{j,i}x_{j,i+1}) = g_2(x_{j,i}x_{j,i+1})$$

,

$$g_3(x_{j,i}x_{j+1,i}) = \begin{array}{l}([(2m+3-2j)n-i+1]\psi(i)\\ +[(2m+2-2j)n-i+1]\psi(i+1))\lambda(j,\frac{m}{2})\\ +[(2jn-i+1)\psi(i)\\ +[(2j+1)n-i+1]\psi(i+1)]\lambda(\frac{m}{2}+1,j)\end{array}$$

for $i \in I$ and $j \in J$,

$$g_3(x_{1,i}y) = g_2(x_{1,i}y),$$

$$g_3(x_{m+1,i}z) = g_2(x_{m+1,i}z).$$

For $m \geq 4$, $m \equiv 0 \pmod 2$, we denote the weights of the 4-sided faces of $\mathcal{D}_n^m$ (under an edge labeling $g$) by

$v_g(f_{j,i}) = w_g(f_{j,i})$,

$v_g(f_{0,i}) = w_g(f_{0,i})$,

$$v_g(f_{m+1,i}) = \begin{array}{l} g(x_{m+1,i}x_{m+1,i+1}) + g(x_{m+1,i+1}x_{m+1,i+2}) \\ +g(x_{m+1,i}z) + g(x_{m+1,i+2}z) \end{array} \quad \text{if } i \in I \text{ is odd.}$$

By direct computation we obtain that the weights of the 4-sided faces under the labelings $g_2$ and $g_3$ constitute the sets of consecutive integers

$$\begin{aligned} W_3 &= \{v_{g_2}(f_{j,i}) : i \in I \text{ and } j \in J\} \\ &= \{v_{g_3}(f_{j,i}) : i \in I \text{ and } j \in J\} \\ &= \{3n(m+1)+3, 3n(m+1)+5, \ldots, 5mn+3n+1\} \end{aligned}$$

and

$$\begin{aligned} W_4 &= \{v_{g_2}(f_{0,i}) : i \in I \text{ is odd }\} \cup \{v_{g_2}(f_{m+1,i}) : i \in I \text{ is odd }\} \\ &= \{v_{g_3}(f_{0,i}) : i \in I \text{ is odd }\} \cup \{v_{g_3}(f_{m+1,i}) : i \in I \text{ is odd }\} \\ &= \{5mn+3n+1+2i : i \in I\}. \end{aligned}$$

We see that $W_3 \cap W_4 = \emptyset$ and the set $W_3 \cup W_4 = \{a, a+d, \ldots, a+(n(m+1)-1)d\}$, where $a = 3n(m+1)+3$ and $d = 2$, which implies that the induced mappings $\varphi_{g_k} : F(\mathcal{D}_n^m) \to W_3 \cup W_4$, $k = 2, 3$, are bijections.

# 4 $(a, 4)$-Face Antimagic Labelings

This section deals with $(a, d)$-face antimagic labelings of $\mathcal{D}_n^m$ for $d = 4$. An $(4n+4, 4)$-face antimagic labeling of $\mathcal{D}_n^1$, $n \equiv 0 \pmod 2$, $n \geq 4$, can be found in [3] and $(6n+4, 4)$-face antimagic labeling of $\mathcal{D}_n^2$, $n \geq 4$ even, is given in [4].

**Theorem 3.** If $n$ is even, $n \geq 4$, and $m$ is odd, $m \geq 3$, or if $n \equiv 2 \pmod 4$, $n \geq 6$, and $m$ is even, $m \geq 4$, then the plane graph $\mathcal{D}_n^m$ has $(2n(m+1)+4, 4)$-face antimagic labeling.

**Proof.** Let us distinguish three cases.

Case 1. $n \equiv 0 \pmod 2$ and $m \equiv 1 \pmod 2$.

For $n \geq 4$ and $m \geq 3$ we define the edge labeling $g_4$ of a plane graph $\mathcal{D}_n^m$ as follows.

$$\begin{aligned} g_4(x_{j,i}x_{j,i+1}) = \quad & ([2(j-1)n+2i+2]\lambda(j,\tfrac{m+1}{2}) \\ & +[2(m-j+1)n+2i+2]\lambda(\tfrac{m+1}{2}+1,j))\psi(j) \\ & +[(2n(m-j+2)\lambda(n-1,i) \\ & +[2n(m-j+2)-2i-2]\lambda(i,n-3))\lambda(j,\tfrac{m+1}{2}) \\ & +(2jn\lambda(n-1,i) \\ & +(2jn-2i-2)\lambda(i,n-3))\lambda(\tfrac{m+1}{2}+1,j)] \\ & \psi(j+1) \end{aligned}$$

for $i \in I$ odd and $j \in J \cup \{m+1\}$,

$$\begin{aligned} g_4(x_{j,i}x_{j,i+1}) = \quad & [([2(m+2-j)n-2]\lambda(n,i) \\ & +[2(m+2-j)n-2i-2]\lambda(i,n-2))\lambda(j,\tfrac{m+1}{2}) \\ & +((2jn-2)\lambda(n,i) \\ & +(2jn-2i-2)\lambda(i,n-2))\lambda(\tfrac{m+1}{2}+1,j)]\psi(j) \\ & +[([(2j-2)n+2]\lambda(n,i) \\ & +[(2j-2)n+2i+2]\lambda(i,n-2))\lambda(j,\tfrac{m+1}{2}) \\ & +([2(m+1-j)n+2]\lambda(n,i) \\ & +[2(m+1-j)n+2i+2]\lambda(i,n-2))\lambda(\tfrac{m+1}{2}+1,j)]\psi(j+1) \end{aligned}$$

for $i \in I$ even and $j \in J \cup \{m+1\}$.

$$g_4(x_{j,i}x_{j+1,i}) = \quad \lambda(n,i) + (2i+1)\lambda(i,n-1)$$

for $i \in I$ and $j = \frac{m+1}{2}$,

$$\begin{aligned} g_4(x_{j,i}x_{j+1,i}) = \quad & ([2n(m+1-2j)+2i+1]\lambda(j,\tfrac{m+1}{2}-1) \\ & +[2n(2j-m-2)+2i+1]\lambda(\tfrac{m+1}{2}+1,j))\psi(i) \\ & +[([2(m-2j)n+1]\lambda(n,i) \\ & +[2(m-2j)n+2i+1]\lambda(i,n-2))\lambda(j,\tfrac{m+1}{2}-1) \\ & +([2n(2j-m-1)+1]\lambda(n,i) \\ & +[2n(2j-m-1)+2i+1]\lambda(i,n-2)) \\ & \lambda(\tfrac{m+1}{2}+1,j)]\psi(i+1) \end{aligned}$$

for $i \in I$ and $j \in J$.

$$g_4(x_{1,i}y) = \quad 2mn+2i-1 \text{ for } i \in I \text{ odd},$$

$$g_4(x_{m+1,i}z) = \quad (2mn+3)\lambda(n,i) + (2mn+2i+3)\lambda(i,n-2)$$

for $i \in I$ even.

Case 2. $n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 4$.

For $n \geq 6$ and $m \geq 4$ use the following labeling $g_5$.

$$g_5(x_{j,i}x_{j,i+1}) = \quad 2g_2(x_{j,i}x_{j,i+1}).$$

$$\begin{aligned} g_5(x_{j,i}x_{j+1,i}) = \; & ([(2m-4j)n+2i-1]\psi(i) \\ & +[(2m-4j+2)n+2i-1]\psi(i+1))\lambda(j,\tfrac{m}{2}) \\ & +([(4j-2m-2)n+2i-1]\psi(i) \\ & +[(4j-2m-4)n+2i-1]\psi(i+1)) \\ & \lambda(\tfrac{m}{2}+1,j) \end{aligned}$$

for $i \in I$ and $j \in J$.

$$\begin{aligned} g_5(x_{1,i}y) = \; & ([\tfrac{n}{2}(4m+1)-i+1]\lambda(i,\tfrac{n}{2}) \\ & +[\tfrac{n}{2}(4m+5)-i+1]\lambda(\tfrac{n}{2}+4,i))\alpha(i) \\ & +[\tfrac{n}{2}(4m+3)-i+1]\alpha(i+2) \end{aligned}$$

for $n \neq 6$ and $i \in I$,

$$g_5(x_{1,i}y) = [(2m+1)n-i]\alpha(i)+[(2m+2)n-i]\alpha(i+2)$$

for $n = 6$ and $i \in I$.

$$g_5(x_{m+1,i}z) = [n(2m+1)-i+2]\alpha(i)+[n(2m+2)-i+2]\alpha(i+2)$$

for $n \neq 6$ and $i \in I$,

$$g_5(x_{m+1,i}z) = 2mn+2i+1$$

for $n = 6$ and $i \in I$ odd.

Case 3. $n \equiv 2 \pmod 4$ and $m \equiv 2 \pmod 4$.

For $n \geq 6$ and $m \geq 6$ we construct the edge labeling $g_6$ of $\mathcal{D}_n^m$ in the following way.

$$g_6(x_{j,i}x_{j,i+1}) = 2g_3(x_{j,i}x_{j,i+1}).$$

$$\begin{aligned} g_6(x_{j,i}x_{j+1,i}) = \; & ([(2m-4j+4)n-2i+1]\psi(i) \\ & +[(2m-4j+2)n-2i+1]\psi(i+1))\lambda(j,\tfrac{m}{2}) \\ & +([(4j-2m-2)n-2i+1]\psi(i) \\ & +[(4j-2m)n-2i+1]\psi(i+1))\lambda(\tfrac{m}{2}+1,j) \end{aligned}$$

for $i \in I$ and $j \in J$.

$$g_6(x_{1,i}y) = g_5(x_{1,i}y)\ .$$

$$g_6(x_{m+1,i}z) = g_5(x_{m+1,i}z)\ .$$

It is not difficult to check that the labelings $g_4$, $g_5$ and $g_6$ are bijections from the edge set $E(\mathcal{D}_n^m)$ onto the set $\{1, 2, ..., 2n(m+1)\}$.

The weights of the 4-sided faces of $\mathcal{D}_n^m$ under the labeling $g_k$, $4 \leq k \leq 6$, constitute the sets

$$
\begin{aligned}
W_5 &= \{w_{g_4}(f_{j,i}) : i \in I \text{ and } j \in J\} \\
&= \{v_{g_5}(f_{j,i}) : i \in I \text{ and } j \in J\} \\
&= \{v_{g_6}(f_{j,i}) : i \in I \text{ and } j \in J\} \\
&= \{2n(m+1)+4, 2n(m+1)+8, \dots, 6nm+2n\}
\end{aligned}
$$

and

$$
\begin{aligned}
W_6 &= \{w_{g_4}(f_{0,i}) : i \in I,\ i \text{ odd}\} \cup \{w_{g_4}(f_{m+1,i}) : i \in I,\ i \text{ even}\} \\
&= \{v_{g_5}(f_{0,i}) : i \in I,\ i \text{ odd}\} \cup \{v_{g_5}(f_{m+1,i}) : i \in I,\ i \text{ odd}\} \\
&= \{v_{g_6}(f_{0,i}) : i \in I,\ i \text{ odd}\} \cup \{v_{g_6}(f_{m+1,i}) : i \in I,\ i \text{ odd}\} \\
&= \{6nm+2n+4, 6nm+2n+8, \dots, 6n(m+1)\}.
\end{aligned}
$$

The set $W_5 \cup W_6 = \{a, a+d, \dots, a+(|F(\mathcal{D}_n^m)|-1)d\}$, where $a = 2n(m+1)+4$ and $d = 4$, is the set of weights of all the 4-sided faces of $\mathcal{D}_n^m$. This proves that the edge labelings $g_4$, $g_5$ and $g_6$ are $(2n(m+1)+4, 4)$-face antimagic. □

# 5 Conclusion

We have shown that there exist $(a, 2)$-face antimagic and $(a, 4)$-face antimagic labelings for the graph of the convex polytope $\mathcal{D}_n^m$ for all $n \not\equiv 0 \pmod 4$ and $m \not\equiv 0 \pmod 2$.

We conjecture that

**Conjecture 1** There are $(3n(m+1)+3, 2)$-face antimagic and $(2n(m+1)+4, 4)$-face antimagic labelings for the plane graph $\mathcal{D}_n^m$ for $m \equiv 0 \pmod 4$, $n \geq 4$, and $m \equiv 0 \pmod 2$, $m \geq 4$.

Then to completely characterize the $(a, d)$-face antimagic graphs of $\mathcal{D}_n^m$ it only remains to consider the case of $(n(m+1)+5, 6)$-face antimagic labeling. This prompts us to propose the following conjecture.

**Conjecture 2.** If $n$ is even, $n \geq 4$, $m \geq 1$, then the plane graph $\mathcal{D}_n^m$ has a $(n(m+1)+5, 6)$-face antimagic labeling.

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