The number of 6-cycles in 2-factorizations of K_n, n odd

Peter Adams[1], Elizabeth J. Billington[1]
Centre for Discrete Mathematics and Computing,
Department of Mathematics,
The University of Queensland,
Queensland 4072
Australia

and

C.C. Lindner
Department of Discrete and Statistical Sciences
120 Math Annex
Auburn University
Auburn
Alabama 36849–5307
U.S.A.

Abstract

In this paper we construct 2-factorizations of K_n (n odd) containing a specified number, k, of 6-cycles, for all integers k between 0 and the maximum possible expected number of 6-cycles in any 2-factorization, and for all odd n, with no exceptions.

1 Introduction and necessary conditions

We start with some definitions. A 2-factor in a graph is a spanning subgraph, regular of degree 2. If the graph is simple then necessarily any 2-factor consists of a collection of cycles which partition the vertex set. A

[1]Research supported by the Australian Research Council grant A69701550.

2-factorization of a complete graph K_n is a collection of edge-disjoint 2-factors of K_n whose union is K_n. It is clear that the degree of K_n must be even for a 2-factorization to exist, and hence n must be odd. The number of 2-factors in a 2-factorization of K_n is $(n-1)/2$, or half the degree.

In this paper we investigate the possible numbers of 6-cycles which a 2-factorization of K_n may possess. First work in this area was on the number of 3-cycles or triangles which a 2-factorization of K_n may contain ([2]). This dealt with K_n when $n \equiv 1$ or 3 (mod 6); the case $n \equiv 5$ (mod 6) remains open. Then the easier case of 4-cycles was dealt with, for odd order complete graphs in [3] and for even order complete graphs minus a 1-factor in [1]. Further work in this area includes that on numbers of 8-cycles; see [4].

We now introduce some notation. Let $Q(n)$ denote the set of all non-negative integers k such that there exists a 2-factorization of K_n (n odd) containing precisely k 6-cycles. Also, following notation in [3], we let FC(n) be as follows (FC for forecast!):

Order n	FC(n)
$12k+1$	$\{0,1,\ldots,6k(2k-1)\}$
$12k+3$	$\{0,1,\ldots,(6k+1)2k\}$
$12k+5$	$\{0,1,\ldots,(6k+2)2k\}$
$12k+7$	$\{0,1,\ldots,(6k+3)2k\}$
$12k+9$	$\{0,1,\ldots,(6k+4)(2k+1)\}$
$12k+11$	$\{0,1,\ldots,(6k+5)(2k+1)\}$

In other words, FC(n) is the forecast possible numbers of 6-cycles in a 2-factorization of K_n. We also adapt this notation in an obvious way: FC(G) will denote the forecast numbers of 6-cycles possible in a 2-factorization of the simple graph G, and $Q(G)$ will denote the actual number of 6-cycles which can be obtained in a 2-factorization of G.

Our aim in this paper is to show that $Q(n) =$ FC(n) for all odd positive integers n, with no exceptions.

Some of the following constructions are complicated by the fact that when the complete bipartite graph $K_{6,6}$ is 2-factored, as is well-known, it is *not* possible to have all three 2-factors consisting of two 6-cycles. In other words, although we can take 2-factorizations of $K_{6,6}$ containing 0, 2 or even 4 6-cycles, we cannot obtain the maximum expected number of 6-cycles, namely 6. That is, $6 \notin Q(K_{6,6})$, although $6 \in$ FC($K_{6,6}$). Owing to this fact, in our constructions below, we generally work modulo 24 rather than modulo 12, and we use 2-factorizations of $K_{12,12}$ rather than of $K_{6,6}$. However, in parts we avoid this difficulty by using the fact that $Q(K_{6,6,6})$

contains 18 and $Q(K_{4,4,4})$ contains 8; see Sections 4 and 5 below.

2 The constructions

The following construction is used in the next section.

Construction I

Let the odd order complete graph K_n have $n = 12(2k+1)+\epsilon$ where $\epsilon \in \{1,3,5,7,9,11\}$. Let $Z = \{z_1, z_2, \ldots, z_\epsilon\}$. Let $(Q, \circ)$ be any idempotent commutative quasigroup of odd order $2k+1$ (for instance, take addition modulo $2k+1$ and relabel symbols so $x \circ x = x$). Take the vertex set X of K_n to be $Z \cup (Q \times J)$ where $J = \{1, 2, \ldots, 12\}$.

We take a collection of 2-factors F of K_n formed from the following:

(i) On the set $Z \cup \{(i,j) \mid 1 \le j \le 12\}$ of order $12+\epsilon$, take $6+(\epsilon-1)/2$ 2-factors $F_i = \{f_{i,1}, f_{i,2}, \ldots, f_{i,6}, \ldots, f_{i,(11+\epsilon)/2}\}$, for some fixed i with $1 \le i \le 2k+1$.

(ii) For each pair a, b with $a < b$ and $a \circ b = i$, on the set $\{(a,j) \mid 1 \le j \le 12\} \cup \{(b,j) \mid 1 \le j \le 12\}$, take a 2-factorization of $K_{12,12}$; this has six 2-factors, say $F_{a,b} = \{f_{ab,1}, f_{ab,2}, \ldots, f_{ab,6}\}$.

Then the set of final 2-factors, F, contains $6(2k+1)+(\epsilon-1)/2$ 2-factors as follows:

$$\{f_{i,m}, f_{ab,m} \mid a \circ b = i,\ a < b\} \quad \text{for } 1 \le i \le 2k+1 \text{ and } 1 \le m \le 6.$$

(This makes $6(2k+1)$ 2-factors.) Also, from $\{f_{i,7}, \ldots, f_{i,(\epsilon+11)/2}\}$, we obtain a further $(\epsilon-1)/2$ 2-factors as follows:

For $2 \le i \le 2k+1$, when $\epsilon > 1$ we ensure that the 2-factors $\{f_{i,7}, \ldots, f_{i,(11+\epsilon)/2}\}$ in F_i each contain a sub-2-factorization ζ on $Z = \{z_1, z_2, \ldots, z_\epsilon\}$ of order ϵ, and containing either 0 or $\max \mathrm{FC}(12+\epsilon)$ 6-cycles.

Let $f'_{i,t} = f_{i,t} \setminus \zeta$, for $2 \le i \le 2k+1$ and $7 \le t \le 6+(\epsilon-1)/2$; that is, $f'_{i,t}$ is the 2-factor $f_{i,t}$ with the sub-2-factorization ζ on Z removed.

Then the final $(\epsilon-1)/2$ 2-factors are:

$$\{f_{1,t}, f'_{i,t} \mid 2 \le i \le 2k+1\} \quad \text{for } t \text{ with } 7 \le t \le (11+\epsilon)/2.$$

As a consequence of the above construction and the fact that $\{0, 6, 12, 18, 24\} \subseteq \mathrm{FC}(K_{12,12})$, we have the following result. Here $m * X$ denotes the set of integers $\{mx \mid x \in X\}$.

THEOREM 2.1 *There is a 2-factorization of K_n, where $n = 24k+12+\epsilon$, and $\epsilon \in \{1,3,5,7,9,11\}$, containing*

$$\binom{2k+1}{2} * \{0,6,12,18,24\} + Q(12+\epsilon) + 2k * \{0,\ max FC(12+\epsilon)\} \text{ 6-cycles.}$$

COROLLARY 2.2 *If $Q(12+\epsilon) = FC(12+\epsilon)$ for $\epsilon \in \{1,3,5,7,9,11\}$, then $Q(n) = FC(n)$ for $n = 24k+12+\epsilon$.*

The next construction is used in Section 4.

Construction II

Let the odd order complete graph K_n have $n = 12(2k) + \epsilon$ where $\epsilon \in \{1,3,5,7,11\}$. Let $Z = \{z_1, z_2, \ldots, z_\epsilon\}$. Let $(Q, \circ)$ be any idempotent commutative quasigroup of order $|Q| = 2k$ with holes of size 2; such a quasigroup exists for all $2k \geq 6$ (see for example [5], page 19, Theorem 1.5.5). In particular, let $Q = \{1, 2, \ldots, 2k\}$ where the holes of size 2 are $\{1,2\}, \{3,4\}, \ldots, \{2k-1, 2k\}$. Let the vertex set X of K_n be $Z \cup (Q \times J)$ where $J = \{1, 2, \ldots, 12\}$.

The construction involves the following:

(i) On the set $Z \cup \{(2i-1, j), (2i, j) \mid 1 \leq j \leq 12\}$ of order $12 + \epsilon$, for each $1 \leq i \leq k$, take $12 + \frac{\epsilon - 1}{2}$ 2-factors $F_i = \{f_{i,1}, f_{i,2}, \ldots, f_{1,12}, \ldots, f_{i,(23+\epsilon)/2}\}$, for $1 \leq i \leq k$.

(ii) For *each* pair a, b in different holes in Q, with $a \circ b = 2i - 1$, on the set $P = \{(a,j) \mid 1 \leq j \leq 12\} \cup \{(b,j) \mid 1 \leq j \leq 12\}$ we place a 2-factorization of $K_{12,12}$ with 2-factors $F_{a,b} = \{f_{ab,1}, \ldots, f_{ab,6}\}$.

   Also for each pair a, b in different holes in Q with $a \circ b = 2i$, on the same set P we place a 2-factorization of $K_{12,12}$ with 2-factors $F'_{a,b} = \{f'_{ab,1}, f'_{ab,2}, \ldots, f'_{ab,6}\}$.

Then the set of final 2-factors, F, contains $12k + (\epsilon - 1)/2$ 2-factors as follows:

$$\{f_{i,m},\ f_{ab,m} \mid a \circ b = 2i - 1\} \quad \text{for } 1 \leq i \leq k, \ \text{ and } 1 \leq m \leq 6,$$

$$\{f_{i,m+6},\ f'_{ab,m} \mid a \circ b = 2i\} \quad \text{for } 1 \leq i \leq k, \ \text{ and } 1 \leq m \leq 6,$$

(making $2 \times 6 \times k$ 2-factors or $12k$ 2-factors). Then, from $\{f_{i,13}, \ldots, f_{i,(23+\epsilon)/2}\}$, we obtain a further $(\epsilon - 1)/2$ 2-factors as follows:

For $2 \leq i \leq k$, when $\epsilon > 1$, we ensure that the 2-factors $\{f_{i,13}, \ldots, f_{i,(23+\epsilon)/2}\}$ in F_i each contain a sub-2-factorization ζ on $Z = \{z_1, z_2, \ldots, z_\epsilon\}$ of order ϵ, and containing either 0 or maxFC$(24+\epsilon)$ 6-cycles.

Let $f'_{i,t} = f_{i,t} \setminus \zeta$ for $2 \leq i \leq k$ and $13 \leq t \leq (23+\epsilon)/2$; that is, $f'_{i,t}$ is the 2-factor $f_{i,t}$ with the sub-2-factorization ζ on Z removed.

Then the final $(\epsilon-1)/2$ 2-factors are: $\{f_{1,t}, f'_{i,t} \mid 2 \leq i \leq k\}$ for t with $13 \leq t \leq (23+\epsilon)/2$.

As a consequence of the above construction, and the fact that $\{0, 6, 12, 18, 24\} \subseteq$FC$(K_{12,12})$, we have the next result.

THEOREM 2.3 *There is a 2-factorization of K_n, where $n = 24k+\epsilon$ and $\epsilon \in \{1,3,5,7,11\}$, $k \geq 3$, containing*

$$4\binom{k}{2} * \{0,6,12,18,24\} + Q(24+\epsilon) + k * \{0, maxFC(24+\epsilon)\} \quad \text{6-cycles.}$$

COROLLARY 2.4 *If $Q(24+\epsilon) = FC(24+\epsilon)$ for $\epsilon \in \{1,3,5,7,11\}$, then $Q(n) = FC(n)$ for $n = 24k+\epsilon$, $k \geq 3$.*

We give one more construction here which is useful for some *ad hoc* cases; it reduces the number of "small" cases which have to be found by computer. First we need a simple lemma.

LEMMA 2.5 $Q(K_{2,2,2}) = \{1,2\}$.

Proof Let $K_{2,2,2}$ have vertex set $\{\{1,2\},\{3,4\},\{5,6\}\}$. Then $\{(1,3,2,5,4,6);(1,4,2,6,3,5)\}$ shows that $2 \in Q(K_{2,2,2})$, and $\{(1,3,5),(2,4,6);(1,4,5,2,3,6)\}$ shows that $1 \in Q(K_{2,2,2})$. It is also clear from the latter 2-factorization that it is not possible to have 0 in $Q(K_{2,2,2})$. □

Construction III

Let K_n have order $n = 2(6k+2)+\epsilon$, where $\epsilon = 1$ or 3. Let the vertex set be

$$\{\infty_i \mid 1 \leq i \leq \epsilon\} \cup \{(i,j) \mid 1 \leq i \leq 6k+2;\ j = 1,2\}.$$

On the set $\{i \mid 1 \leq i \leq 6k+2\}$ we take a *punctured* Kirkman triple system (of order $6k+3$, with one point deleted). This has $3k+1$ parallel classes, and each parallel class contains $2k$ blocks of size 3 and one block of size 2. For each of these parallel classes, in turn, we obtain two final 2-factors of

K_n (making $6k+2$), and if $\epsilon = 3$ we have one further 2-factor. These arise as follows:

Suppose one of the $3k+1$ parallel classes is $\{1\,2,\ 3\,4\,5,\ 6\,7\,8,\ \ldots, 6k\ 6k+1\ 6k+2\}$. Then on $\{\infty_i \mid 1 \le i \le \epsilon\} \cup \{(1,1),(1,2),(2,1),(2,2)\}$ we place a 2-factorization of K_5 (if $\epsilon = 1$) or of K_7 (if $\epsilon = 3$), containing, in the latter case, the triangle $(\infty_1, \infty_2, \infty_3)$. Also on $\{\{(3i,1),(3i,2)\},\{(3i+1,1),(3i+1,2)\},\{(3i+2,1),(3i+2,2)\}\}$, for $1 \le i \le 2k$, we place a 2-factorization of $K_{2,2,2}$. The resulting 2-factorization of K_n contains

$$(3k+1)*\{0\} + 2k(3k+1)*\{1,2\} = \{2k(3k+1), 2k(3k+1)+1, \ldots, 4k(3k+1)\}$$

6-cycles. (Note that the maximum number of 6-cycles possible for order $12k+4+\epsilon$, when $\epsilon = 1$ or 3, is $2k$ 6-cycles per 2-factor, so $2k(6k+2)$ when $\epsilon = 1$, or $2k(6k+3)$ when $\epsilon = 3$.)

COROLLARY 2.6 *(i)* $\{8,9,\ldots,16\} \subseteq Q(17)$; *(ii)* $\{8,9,\ldots,16\} \subseteq Q(19)$; *(iii)* $\{28,29,\ldots,56\} \subseteq Q(29)$; *(iv)* $\{28,29,\ldots,56\} \subseteq Q(31)$.

Proof These all follow from Construction III above, with $\epsilon = 1$ (in (i) and (ii)), $k = 1$ (in (i) and (iii)), $\epsilon = 3$ (in (iii) and (iv)), and $k = 2$ (in (ii) and (iv)). Note also that the 2-factorization of K_{17} contains a sub-2-factorization of K_5, and the 2-factorization of K_{19} contains a sub-2-factorization of K_7. □

In the case of order 9 (mod 24) we use a fairly similar construction; details are given in Section 5 below, since this is the only time it is used.

3 Cases $n \equiv$ 13, 15, 17, 19, 21, 23 (mod 24)

$n \equiv 13 \pmod{24}$

We start with a crucial example.

EXAMPLE 3.1 $Q(K_{12,12}) \supseteq \{0, 6, 12, 18, 24\}$.

Trivially $0 \in Q(K_{12,12})$; for instance, we can take a 2-factorization of $K_{12,12}$ into all 4-cycles.

$6 \in Q(K_{12,12})$:

(0, 12, 1, 13, 2, 14), (3, 15, 4, 16, 5, 17, 6, 18, 7, 19, 8, 20, 9, 21, 10, 22, 11, 23);
(0, 13, 3, 12, 2, 15), (1, 14, 4, 17, 7, 16, 6, 19, 5, 18, 8, 21, 11, 20, 10, 23, 9, 22);
(0, 16, 1, 15, 5, 20), (2, 17, 3, 14, 6, 12, 9, 13, 10, 18, 11, 19, 4, 21, 7, 22, 8, 23);
(0, 17, 1, 18, 2, 19), (3, 16, 8, 12, 10, 14, 9, 15, 11, 13, 4, 20, 7, 23, 5, 21, 6, 22);
(0, 18, 3, 19, 1, 21), (2, 20, 6, 23, 4, 12, 11, 17, 9, 16, 10, 15, 8, 13, 7, 14, 5, 22);
(5, 12, 7, 15, 6, 13), (0, 22, 4, 18, 9, 19, 10, 17, 8, 14, 11, 16, 2, 21, 3, 20, 1, 23).

$12 \in Q(K_{12,12})$:

(0, 12, 1, 13, 2, 14), (3, 15, 4, 16, 5, 17), (6, 18, 7, 19, 8, 20), (9, 21, 10, 22, 11, 23);
(0, 13, 3, 12, 2, 15), (1, 14, 4, 17, 6, 16), (5, 18, 8, 21, 11, 19), (7, 22, 9, 20, 10, 23);
(0, 16, 2, 17, 1, 18), (3, 14, 5, 12, 7, 20), (4, 21, 6, 22, 8, 23), (9, 13, 11, 15, 10, 19);
(0, 17, 7, 13, 4, 12, 6, 14, 8, 15, 9, 16, 10, 18, 11, 20, 1, 19, 2, 21, 3, 22, 5, 23);
(0, 19, 3, 16, 7, 15, 1, 21, 5, 20, 4, 18, 9, 12, 8, 17, 11, 14, 10, 13, 6, 23, 2, 22);
(0, 20, 2, 18, 3, 23, 1, 22, 4, 19, 6, 15, 5, 13, 8, 16, 11, 12, 10, 17, 9, 14, 7, 21).

$18 \in Q(K_{12,12})$:

(0, 12, 1, 13, 2, 14), (3, 15, 4, 16, 5, 17), (6, 18, 7, 19, 8, 20), (9, 21, 10, 22, 11, 23);
(0, 13, 3, 12, 2, 15), (1, 14, 4, 17, 6, 16), (5, 18, 8, 21, 11, 19), (7, 22, 9, 20, 10, 23);
(0, 16, 2, 17, 1, 18), (3, 14, 5, 12, 7, 20), (4, 21, 6, 22, 8, 23), (9, 13, 11, 15, 10, 19);
(0, 17, 7, 13, 4, 19), (1, 15, 5, 21, 3, 22), (2, 20, 11, 12, 6, 23), (8, 14, 9, 18, 10, 16);
(0, 20, 1, 19, 2, 21), (3, 16, 7, 14, 6, 15, 9, 12, 10, 13, 8, 17, 11, 18, 4, 22, 5, 23);
(9, 16, 11, 14, 10, 17), (0, 22, 2, 18, 3, 19, 6, 13, 5, 20, 4, 12, 8, 15, 7, 21, 1, 23).

$24 \in Q(K_{12,12})$:

(0, 12, 1, 13, 2, 14), (3, 15, 4, 16, 5, 17), (6, 18, 7, 19, 8, 20), (9, 21, 10, 22, 11, 23);
(0, 13, 3, 12, 2, 15), (1, 14, 4, 17, 6, 16), (5, 18, 8, 21, 11, 19), (7, 22, 9, 20, 10, 23);
(0, 16, 2, 17, 1, 18), (3, 14, 5, 12, 7, 20), (4, 21, 6, 22, 8, 23), (9, 13, 11, 15, 10, 19);
(0, 17, 7, 13, 4, 19), (1, 15, 6, 14, 11, 20), (2, 21, 5, 22, 3, 23), (8, 12, 9, 18, 10, 16);
(0, 20, 5, 23, 1, 21), (2, 19, 3, 18, 4, 22), (6, 12, 10, 14, 8, 13), (7, 15, 9, 17, 11, 16);
(0, 22, 1, 19, 6, 23), (2, 18, 11, 12, 4, 20), (3, 16, 9, 14, 7, 21), (5, 13, 10, 17, 8, 15).

□

EXAMPLE 3.2 $Q(13) = \mathrm{FC}(13) = \{0, 1, \ldots, 6\}$.

$0 \in Q(13)$: a hamilton decomposition of K_{13}, for example, shows this.

$1 \in Q(13)$:

(0, 1, 2, 3, 4, 5), (6, 7, 8, 9, 10, 11, 12); (0, 2, 4, 1, 3, 5, 6, 8), (7, 10, 12, 9, 11);
(0, 3, 6, 1, 5, 9, 2, 10), (4, 7, 12, 8, 11); (0, 4, 6, 2, 7, 9, 1, 11), (3, 10, 8, 5, 12);
(1, 7, 0, 9, 6, 10, 4, 12), (2, 5, 11, 3, 8); (1, 8, 4, 9, 3, 7, 5, 10), (0, 6, 11, 2, 12).

$2 \in Q(13)$:

(0, 1, 2, 3, 4, 5), (6, 7, 8, 9, 10, 11, 12); (0, 2, 4, 1, 3, 6), (5, 7, 9, 11, 8, 10, 12);
(0, 3, 5, 1, 6, 8, 2, 10), (4, 9, 12, 7, 11); (0, 4, 6, 2, 7, 10, 5, 9), (1, 8, 12, 3, 11);
(0, 7, 1, 9, 3, 10, 6, 11), (2, 5, 8, 4, 12); (0, 8, 3, 7, 4, 10, 1, 12), (2, 9, 6, 5, 11).

$3 \in Q(13)$:
$(0,1,2,3,4,5),(6,7,8,9,10,11,12)$, $(0,2,4,1,3,6),(5,7,9,11,8,10,12)$;
$(0,3,5,1,6,8),(2,9,12,4,10,7,11)$; $(0,4,6,9,1,7,2,10),(3,11,5,8,12)$;
$(1,8,3,7,4,11,6,10),(0,9,5,2,12)$; $(2,6,5,10,3,9,4,8),(0,7,12,1,11)$.

$4 \in Q(13)$:
$(0,1,2,3,4,5),(6,7,8,9,10,11,12)$; $(0,2,4,1,3,6),(5,7,9,11,8,10,12)$;
$(0,3,5,1,6,8),(2,9,12,4,10,7,11)$; $(0,4,6,9,1,7),(2,10,5,11,3,8,12)$;
$(0,9,3,12,1,10,6,11),(2,5,8,4,7)$; $(1,8,2,6,5,9,4,11),(0,10,3,7,12)$.

$5 \in Q(13)$:
$(0,1,2,3,4,5),(6,7,8,9,10,11,12)$; $(0,2,4,1,3,6),(5,7,9,11,8,10,12)$;
$(0,3,5,1,6,8),(2,9,12,4,10,7,11)$; $(0,4,6,9,1,7),(2,10,3,11,5,8,12)$;
$(0,9,3,8,4,11),(1,10,6,5,2,7,12)$; $(0,10,5,9,4,7,3,12),(1,8,2,6,11)$.

$6 \in Q(13)$:
$(0,1,2,3,4,5),(6,7,8,9,10,11,12)$; $(0,2,4,1,3,6),(5,7,9,11,8,10,12)$;
$(0,3,5,1,6,8),(2,9,12,4,10,7,11)$; $(0,4,6,9,1,7),(2,10,3,11,5,8,12)$;
$(0,9,5,2,6,10),(1,8,3,12,7,4,11)$; $(2,7,3,9,4,8),(0,11,6,5,10,1,12)$.

LEMMA 3.3 $Q(24k{+}13) = FC(24k{+}13) = \{0,1,2,\ldots,6(2k{+}1)(4k{+}1)\}$, *for all integers* $k \geq 0$.

Proof Here $n = 12(2k+1)+1$; we use Construction 1 with $\epsilon = 1$. Then Theorem 2.1 gives $Q(24k+13) = \mathrm{FC}(24k+13)$ as required. □

$n \equiv 15 \pmod{24}$

Subsequent examples, marked with an asterisk, have been placed in an Appendix, available from
`http://www.maths.uq.edu.au/~ejb/JCMCCappendix.html`
or by email from `ejb@maths.uq.edu.au` .

EXAMPLE* 3.4 $Q(15) = \mathrm{FC}(15) = \{0,1,\ldots,14\}$, and each 2-factorization contains a K_3.

LEMMA 3.5 $Q(24k{+}15) = FC(24k{+}15) = \{0,1,2,\ldots,(12k{+}7)(4k{+}2)\}$, *for all integers* $k \geq 0$.

Proof Here $n = 12(2k+1)+3$; we use Construction I with $\epsilon = 3$. Noting that $Q(K_{12,12})$ contains $\{0,6,12,18,24\}$, $Q(15) = \mathrm{FC}(15)$ and 2-factors $f_{i,7}$ (for here $t = 7$) of K_{15} for $2 \leq i \leq 2k+1$ satisfy $f_{i,7} = f'_{i,7} \cup \zeta$ where ζ is K_3

on $Z = \{z_1, z_2, z_3\}$, then Theorem 2.1 gives $Q(24k+15) = \mathrm{FC}(24k+15)$ as required. □

$n \equiv 17 \pmod{24}$

EXAMPLE* 3.6 $Q(17) = \mathrm{FC}(17) = \{0, 1, \ldots, 16\}$, and two 2-factors contain a 2-factorization of K_5 on five fixed vertices.
The Appendix shows $\{0, 1, \ldots, 7\} \subseteq Q(17)$ while the other values follow from Corollary 2.6.

LEMMA 3.7 $Q(24k+17) = FC(24k+17) = \{0, 1, 2, \ldots, (12k+8)(4k+2)\}$.

Proof Here $n = 12(2k+1)+5$; we use Construction I with $\epsilon = 5$. Noting that $Q(K_{12,12})$ contains $\{0, 6, 12, 18, 24\}$, $Q(17) = \mathrm{FC}(17)$ and 2-factors $f_{i,7}$ and $f_{i,8}$ (for here $t = 7$ and 8) of K_{17} for $2 \le i \le 2k+1$ satisfy $f_{i,7} = f'_{i,7} \cup \zeta_1$ and $f_{i,8} = f'_{i,8} \cup \zeta_2$ where $\zeta_1 \cup \zeta_2$ is K_5 on $Z = \{z_1, z_2, z_3, z_4, z_5\}$, and ζ_1, ζ_2 are two edge-disjoint 5-cycles on Z, then Theorem 2.1 gives $Q(24k+17) = \mathrm{FC}(24k+17)$ as required. □

$n \equiv 19 \pmod{24}$

EXAMPLE* 3.8 $Q(19) = \mathrm{FC}(19) = \{0, 1, \ldots, 18\}$, and three 2-factors contain a 2-factorization of K_7 on seven fixed vertices.
The Appendix shows $\{0, 1, \ldots, 7, 17, 18\} \subseteq Q(19)$, while $\{8, 9, \ldots, 16\} \subseteq Q(19)$ from Corollary 2.6.

LEMMA 3.9 $Q(24k+19) = FC(24k+19) = \{0, 1, 2, \ldots, (12k+9)(4k+2)\}$, *for all integers* $k \ge 0$.

Proof Here $n = 12(2k+1)+7$; we use Construction I with $\epsilon = 7$. We have $Q(19) = \mathrm{FC}(19)$, and 2-factors $f_{i,t}$, $t = 7, 8, 9$, of K_{19} for $2 \le i \le 2k+1$ satisfy $f_{i,t} = f'_{i,t} \cup \zeta_t$ where $\zeta_7 \cup \zeta_8 \cup \zeta_9$ is K_7 on $Z = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$, and ζ_t are a 2-factorization of K_7 on Z. Then Theorem 2.1 gives $Q(24k+19) = \mathrm{FC}(24k+19)$ as required. □

$n \equiv 21 \pmod{24}$

EXAMPLE* 3.10 $Q(21) = \mathrm{FC}(21) = \{0, 1, \ldots, 30\}$, and four 2-factors contain a 2-factorization of K_9 on nine fixed vertices. The Appendix gives

$\{25, 26, 27, 28, 29, 30\} \subseteq Q(K_{21})$. For the other values we use $K_{6,6}$, noting that $\{0, 2, 4\} \subseteq Q(K_{6,6})$, but $6 \notin Q(K_{6,6})$.

Let the vertex set of $K_{6,6}$ be $\{\{0, 1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10, 11\}\}$.
Then $0 \in Q(K_{6,6})$: $(0, 6, 1, 7, 2, 8, 3, 9, 4, 10, 5, 11)$; $(0, 7, 3, 6, 4, 8, 5, 9, 1, 11, 2, 10)$; $(0, 8, 1, 10, 3, 11, 4, 7, 5, 6, 2, 9)$.

Also $2 \in Q(K_{6,6})$: $(0, 6, 1, 7, 2, 8), (3, 9, 4, 10, 5, 11)$; $(0, 7, 3, 6, 4, 8, 5, 9, 1, 10, 2, 11)$; $(0, 9, 2, 6, 5, 7, 4, 11, 1, 8, 3, 10)$.

And $4 \in Q(K_{6,6})$: $(0, 6, 1, 7, 2, 8), (3, 9, 4, 10, 5, 11)$; $(0, 7, 3, 6, 4, 11), (1, 8, 5, 9, 2, 10)$; $(0, 9, 1, 11, 2, 6, 5, 7, 4, 8, 3, 10)$.

Now let the vertex set of K_{21} be $\{\infty_1, \infty_2, \infty_3\} \cup \{(i, j) \mid 1 \leq i \leq 3,\ 1 \leq j \leq 6\}$. On $\{\infty_i \mid 1 \leq i \leq 3\} \cup \{(i, j) \mid 1 \leq j \leq 6\}$ we place a 2-factorization of K_9, with one 2-factor containing the triangle $(\infty_1, \infty_2, \infty_3)$. Then on $\{\{(i+1, j) \mid 1 \leq j \leq 6\}, \{(i+2, j) \mid 1 \leq j \leq 6\}\}$ (addition modulo 3) we place a 2-factorization of $K_{6,6}$. Since $Q(K_9) = \{0, 1, 2, 3, 4\}$, and $Q(K_{6,6}) \supseteq \{0, 2, 4\}$, it follows that

$$Q(K_{21}) \supseteq 3 * \{0, 1, 2, 3, 4\} + 3 * \{0, 2, 4\} = \{0, 1, \ldots, 24\}.$$

□

LEMMA 3.11 $Q(24k+21) = FC(24k+21) = \{0, 1, 2, \ldots, (12k+10)(4k+3)\}$.

Proof Here $n = 12(2k+1) + 9$; we use Construction I with $\epsilon = 9$. We have $Q(21) = \mathrm{FC}(21)$, and 2-factors $f_{i,t}$, $t = 7, 8, 9, 10$, of K_{21} for $2 \leq i \leq 2k+1$ satisfy $f_{i,t} = f'_{i,t} \cup \zeta_t$ where $\zeta_7 \cup \zeta_8 \cup \zeta_9 \cup \zeta_{10}$ is K_9 on $Z = \{z_1, z_2, \ldots, z_9\}$, and ζ_t are a 2-factorization of K_9 on Z. Then Theorem 2.1 gives $Q(24k+21) = \mathrm{FC}(24k+21)$ as required. □

$n \equiv 23 \pmod{24}$

EXAMPLE* 3.12 $Q(23) = \mathrm{FC}(23) = \{0, 1, \ldots, 33\}$, and five 2-factors contain a 2-factorization of K_{11} on eleven fixed vertices. To obtain all but the top six values, we use a construction extremely similar to Example 3.10 above, using $K_{6,6}$, but we have five "infinity" elements rather than three. And we use a 2-factorization of K_{11} with a sub-2-factorization of K_5 in it (see the Appendix for this). The result is that $Q(K_{23})$ contains

$$3 * Q(K_{11}) + 3 * \{0, 2, 4\} = \{0, 1, \ldots, 27\}.$$

The values $\{28, 29, 30, 31, 32, 33\}$ are in the Appendix. □

LEMMA 3.13 $Q(24k+23) = FC(24k+23) = \{0, 1, 2, \ldots, (12k+11)(4k+3)\}$ *for all integers* $k \geq 0$.

Proof Here $n = 12(2k+1) + 11$; we use Construction I with $\epsilon = 11$. We have $Q(23) = \mathrm{FC}(23)$, and 2-factors $f_{i,t}$, $7 \leq t \leq 11$, of K_{23} for $2 \leq i \leq 2k+1$ satisfy $f_{i,t} = f'_{i,t} \cup \zeta_t$ where $\bigcup_{t=7}^{11} \zeta_t$ is K_{11} on $Z = \{z_1, z_2, \ldots, z_{11}\}$, and ζ_t are a 2-factorization of K_{11} on Z. Then Theorem 2.1 gives $Q(24k+23) = \mathrm{FC}(24k+23)$ as required. □

4 Cases $n \equiv$ 1, 3, 5, 7, 11 (mod 24)

Before we deal with separate orders mod 24, the following lemma shows that $\mathrm{FC}(v) = Q(v)$ for $v = 49$, 51, 53, 55 and 59.

LEMMA 4.1 $FC(v) = Q(v)$ *for* $v = 49$, 51, 53, 55 *and* 59.

Proof Take the vertex set $\{\infty_i \mid 1 \leq i \leq \epsilon\} \cup \{(i,j) \mid 1 \leq i \leq 8,\ 1 \leq j \leq 6\}$ where $\epsilon = 1, 3, 5, 7$ or 11 according as the order is 49, 51, 53, 55 or 59. On the set $\{1, 2, \ldots, 8\}$ we take a pairwise balanced design $\mathrm{PB}(8; \{2,3,3\}; 1\}$ with four parallel classes: 12, 345, 678; 37, 148, 256; 46, 157, 238; 58, 136, 247. (This may also be regarded as a punctured affine plane of order 3.)

We have (see the Appendix) $Q(K_{6,6,6}) \supseteq \{0, 6, 12, 18\}$, and also

(i) $Q(13) = \mathrm{FC}(13) = \{0, 1, \ldots, 6\}$ (see Example 3.2);

(ii) $Q(15) = \mathrm{FC}(15) = \{0, 1, \ldots, 14\}$, with a K_3 in each 2-factorization (see the Appendix);

(iii) $Q(17) = \mathrm{FC}(17) = \{0, 1, \ldots, 16\}$, with two 2-factors in each 2-factorization containing a sub-2-factorization of K_5 on five fixed vertices (see the Appendix and Corollary 2.6);

(iv) $Q(19) = \mathrm{FC}(19) = \{0, 1, \ldots, 18\}$, with three 2-factors in each 2-factorization containing a sub-2-factorization of K_7 on seven fixed vertices (see the Appendix and Corollary 2.6);

(v) $Q(23) = \mathrm{FC}(23) = \{0, 1, \ldots, 33\}$, with five 2-factors in each 2-factorization containing a sub-2-factorization of K_{11} on eleven fixed vertices (see the Appendix and Example 3.12).

Now for *each* of the four parallel classes of the pairwise balanced design above, we obtain six 2-factors as follows. For the parallel class 12, 345, 678: on $\{\infty_i \mid 1 \le i \le \epsilon\} \cup \{(1,j),(2,j) \mid 1 \le j \le 6\}$ we take six of the 2-factors of K_{13}, K_{15}, K_{17}, K_{19} or K_{23} (with a sub-2-factorization on $\{\infty_i \mid 1 \le i \le \epsilon\}$ for $\epsilon = 3,5,7,11$ respectively). Also on $\{\{(a,j) \mid 1 \le j \le 6\}, \{(a+1,j) \mid 1 \le j \le 6\}, \{(a+2,j) \mid 1 \le j \le 6\}\}$, for $a = 3$, and again for $a = 6$, we place a 2-factorization of $K_{6,6,6}$, which has six 2-factors also. We do likewise for the other three parallel classes. This yields 24 2-factors. For orders 51, 53, 55 or 59, when $\epsilon = 3$, 5, 7 or 11 respectively, we have a further one, two, three or five 2-factors. (For K_{15} has seven 2-factors altogether, K_{17} has eight, K_{19} has nine and K_{23} has eleven.) So saving one 2-factor from K_{15} on the sets

$$\{\infty_i \mid 1 \le i \le 3\} \cup \{(i,j) \mid 1 \le j \le 6\}$$

for each $i \in \{1,2\};\{3,7\};\{4,6\};\{5,8\}$, containing a K_3 on $\{\infty_1,\infty_2,\infty_3\}$, which is taken *once* only, we obtain one further 2-factor for K_{51}.

Similarly, saving two 2-factors from K_{17} on $\{\infty_i \mid 1 \le i \le 5\} \cup \{(i,j) \mid 1 \le j \le 6\}$ for each $i \in \{1,2\};\{3,7\};\{4,6\};\{5,8\}$, with a sub-2-factorization of K_5 on $\{\infty_i \mid 1 \le i \le 5\}$ in each of the pairs of 2-factors (taken once only), we can obtain two further 2-factors of K_{53}.

Similarly we can save three 2-factors from the 2-factorization of K_{19}, and five from the 2-factorization of K_{23}.

It follows that the total number of possible 6-cycles is

$$4 * Q(12+\epsilon) + (2 \times 4) * \{0,6,12,18\}$$

for order $48+\epsilon$, $\epsilon = 1$, 3, 5, 7 or 11. This equals FC$(48+\epsilon)$, as required. □

$n \equiv 1 \pmod{24}$

We need the following examples.

EXAMPLE 4.2 $Q(25) = \text{FC}(25) = \{0,1,\ldots,36\}$.
Let K_{25} have vertex set $\{\infty\} \cup \{(i,j) \mid 1 \le i \le 3,\ 1 \le j \le 8\}$. For $a = 1,2,3$, on $\{\infty\} \cup \{(a,j) \mid 1 \le j \le 8\}$, place a 2-factorization of K_9, and on $\{\{(a+1,j) \mid 1 \le j \le 8\}, \{(a+2,j) \mid 1 \le j \le 8\}\}$, place a 2-factorization of $K_{8,8}$, where $a+1$, $a+2$ are taken mod 3. This yields 3×4 or 12 2-factors, as required. Moreover, since $Q(9) = \{0,1,2,3,4\}$ and $Q(K_{8,8}) \supseteq \{0,4,8\}$, the 12 2-factors contain

$$3 * \{0,1,2,3,4\} + 3 * \{0,4,8\} = \{0,1,2,\ldots,36\}$$

6-cycles. Hence $Q(25) = \text{FC}(25)$, as required. □

LEMMA 4.3 *$Q(24k+1) = FC(24k+1) = \{0, 1, 2, \ldots, 12k(4k-1)\}$ for all integers $k \geq 1$.*

Proof Here $n = 12(2k) + 1$; we use Construction II with $\epsilon = 1$. We have $Q(25) = \text{FC}(25)$ from Example 4.2. Then Corollary 2.4, together with Example 4.2 and order 49 from Lemma 4.1, gives $Q(24k+1) = \text{FC}(24k+1)$ as required. □

$n \equiv 3 \pmod{24}$

EXAMPLE 4.4 $Q(27) = \text{FC}(27) = \{0, 1, 2, \ldots, 52\}$, and each 2-factorization contains a K_3.

There are 13 2-factors altogether. Let the vertex set be $\{(i, j) \mid 1 \leq i \leq 9,\ 1 \leq j \leq 3\}$. On $\{(i, 3) \mid 1 \leq i \leq 9\}$ we take a 2-factorization of K_9 (with four 2-factors, and a K_3 in each: this is easy to find, and one possibility appears in Example 5.2 in the next section). On the graph $K_{9,9} \setminus F$, with vertex set

$$\{\{(i, 1) \mid 1 \leq i \leq 9\},\ \{(i, 2) \mid 1 \leq i \leq 9\}\},$$

where F is the 1-factor $\{\{(i, 1), (i, 2)\} \mid 1 \leq i \leq 9\}$, we take a 2-factorization which also has four 2-factors. This exists with no 6-cycles and also with 12 6-cycles:

$K_{9,9} \setminus F$ into four 2-factors, each with three 6-cycles:

$$((1,1),(2,2),(3,1),(1,2),(2,1),(3,2)),$$
$$((4,1),(5,2),(6,1),(4,2),(5,1),(6,2)),$$
$$((7,1),(8,2),(9,1),(7,2),(8,1),(9,2));$$

$$((1,1),(4,2),(7,1),(1,2),(4,1),(7,2)),$$
$$((2,1),(5,2),(8,1),(2,2),(5,1),(8,2)),$$
$$((3,1),(6,2),(9,1),(3,2),(6,1),(9,2));$$

$$((1,1),(5,2),(9,1),(1,2),(5,1),(9,2)),$$
$$((2,1),(6,2),(7,1),(2,2),(6,1),(7,2)),$$
$$((3,1),(4,2),(8,1),(3,2),(4,1),(8,2));$$

$$((1,1),(6,2),(8,1),(1,2),(6,1),(8,2)),$$
$$((2,1),(4,2),(9,1),(2,2),(4,1),(9,2)),$$
$$((3,1),(5,2),(7,1),(3,2),(5,1),(7,2)).$$

$K_{9,9} \setminus F$ into four 2-factors, with no 6-cycles (each 2-factor is an 18-cycle):

$$((1,1),(2,2),(5,1),(6,2),(9,1),(7,2),(4,1),(5,2),(8,1),(9,2),(3,1),(1,2),(7,1),(8,2),(2,1),(3,2),(6,1),(4,2));$$

$$((1,1),(3,2),(9,1),(8,2),(5,1),(4,2),(7,1),(9,2),(6,1),(5,2),(2,1),(1,2),(4,1),(6,2),(3,1),(2,2),(8,1),(7,2));$$

$$((1,1),(9,2),(2,1),(6,2),(8,1),(3,2),(5,1),(1,2),(6,1),(7,2),(3,1),(4,2),(9,1),(5,2),(7,1),(2,2),(4,1),(8,2));$$

$$((1,1),(5,2),(3,1),(8,2),(6,1),(2,2),(9,1),(1,2),(8,1),(4,2),(2,1),(7,2),(5,1),(9,2),(4,1),(3,2),(7,1),(6,2)).$$

Now we have $Q(9) = \{0,1,2,3,4\}$, and $Q(K_{9,9} \setminus F)$ contains $\{0,12\}$. We repeat the above twice more, using $\{(i,2) \mid 1 \le i \le 9\}$, and then $\{(i,3) \mid 1 \le i \le 9\}$. We have left the three 1-factors from the three lots of $K_{9,9}$. These form one further 2-factor, either with no 6-cycles, as in

$$\{((m,1),(m,2),(m,3)) \mid 1 \le m \le 9\},$$

which has nine triangles, or with four 6-cycles and one triangle, as in

$$\begin{aligned}&\{((1,1),(1,2),(2,3),(2,1),(2,2),(3,3)),\\&((3,1),(3,2),(4,3),(4,1),(4,2),(5,3)),\\&((5,1),(5,2),(6,3),(6,1),(6,2),(7,3)),\\&((7,1),(7,2),(8,3),(8,1),(8,2),(9,3)),\quad ((9,1),(9,2),(1,3))\}.\end{aligned}$$

In the former case the three 1-factors are

$$\{\{(m,1),(m,2)\} \mid 1 \le m \le 9\},\ \{\{(m,2),(m,3)\} \mid 1 \le m \le 9\},$$

$$\{\{(m,1),(m,3)\} \mid 1 \le m \le 9\}.$$

In the latter case the three 1-factors are taken as:

$$\{\{(m,1),(m,2)\} \mid 1 \le m \le 9\},\quad \{\{(m,2),(m+1,3)\} \mid 1 \le m \le 9\},$$

and

$$\begin{aligned}&\{\{(1,1),(3,3)\},\quad \{(2,1),(2,3)\},\quad \{(3,1),(5,3)\},\quad \{(4,1),(4,3)\},\\&\{(5,1),(7,3)\},\quad \{(6,1),(6,3)\},\quad \{(7,1),(9,3)\},\quad \{(8,1),(8,3)\},\\&\{(9,1),(1,3)\}\}.\end{aligned}$$

In any case, the total number of 6-cycles that may be obtained in a 2-factorization of K_{27} in this manner is

$$3 * \{0,1,2,3,4\} + 3 * \{0,12\} + \{0,4\} = \{0,1,2,\ldots,52\} = \mathrm{FC}(27),$$

as required. □

LEMMA 4.5 $Q(24k+3) = FC(24k+3) = \{0,1,2,\ldots,(12k+1)4k\}$ *for all integers* $k \geq 1$.

Proof Here $n = 12(2k)+3$; we use Construction II with $\epsilon = 3$. We have $Q(27) = \mathrm{FC}(27)$ from Example 4.4. Then Corollary 2.4, together with Example 4.4 and order 51 from Lemma 4.1, gives $Q(24k+3) = \mathrm{FC}(24k+3)$ as required. □

$n \equiv 5 \pmod{24}$

EXAMPLE 4.6 $Q(29) = \mathrm{FC}(29) = \{0,1,2,\ldots,56\}$, and two 2-factors in each 2-factorization contain a sub-2-factorization of K_5 on five fixed vertices.

Construction III (and Corollary 2.6) show that $\{28,29,\ldots,56\} \subseteq Q(29)$. The next construction yields the lower numbers of 6-cycles.

Let K_{29} have vertex set $\{\infty_i \mid 1 \leq i \leq 5\} \cup \{(i,j) \mid 1 \leq i \leq 3,\ 1 \leq j \leq 8\}$. We have $Q(K_{8,8}) \supseteq \{0,4,8\}$ (see the Appendix). We also have a special case of K_{13} with two of the six 2-factors containing a sub-2-factorization of K_5, together with either a cycle of length 8, or two cycles of length 4. So this special 2-factorization of K_{13} contains $0+0+4*\{0,1\} = \{0,1,2,3,4\}$ 6-cycles.

Now we take a 2-factorization of K_{13} on $\{\infty_i \mid 1 \leq i \leq 5\} \cup \{(1,j) \mid 1 \leq j \leq 8\}$ (with of course a sub-2-factorization of K_5 on $\{\infty_i \mid 1 \leq i \leq 5\}$), and a 2-factorization of $K_{8,8}$ on $\{\{(2,j) \mid 1 \leq j \leq 8\}, \{(3,j) \mid 1 \leq j \leq 8\}\}$, which has four 2-factors.

We save the two special 2-factors of K_{13}, and place the other four 2-factors with those of $K_{8,8}$.

Repeat this three times altogether: we may then obtain a total of $3 \times 4 + 2 = 14$ 2-factors, as required. (The two extra come from the two 2-factors in each K_{13} with the sub-2-factorization of K_5, which is included just once.) We thus obtain

$$3 * \{0,1,2,3,4\} + 3 * \{0,4,8\} = \{0,1,2,\ldots,36\}$$

6-cycles. This completes the result $Q(29) = \text{FC}(29)$. □

LEMMA 4.7 $Q(24k+5) = FC(24k+5)$ *for all integers* $k \geq 1$.

Proof Here $n = 12(2k) + 5$; we use Construction II with $\epsilon = 5$. We have $Q(29) = \text{FC}(29)$ from Example 4.6. Then Corollary 2.4, together with Example 4.6 and order 53 from Lemma 4.1, gives $Q(24k+5) = \text{FC}(24k+5)$ as required. □

$n \equiv 7 \pmod{24}$

EXAMPLE 4.8 $Q(31) = \text{FC}(31) = \{0, 1, \ldots, 60\}$, and three 2-factors in each 2-factorization contain a sub-2-factorization of K_7 on seven fixed vertices. Corollary 2.6 shows $\{28, 29, \ldots, 56\} \subseteq Q(31)$, and the Appendix gives $\{57, 58, 59, 60\} \subseteq Q(31)$.

We now show that $\{0, 1, \ldots, 54\} \subseteq Q(31)$, which will complete the verification that $Q(31) = \text{FC}(31)$.

Let the vertex set be $X \cup A \cup B \cup C$ where $X = \{x_i \mid 1 \leq i \leq 7\}$, $A = \{(1,j) \mid 1 \leq j \leq 8\}$, $B = \{(2,j) \mid 1 \leq j \leq 8\}$, and $C = \{(3,j) \mid 1 \leq j \leq 8\}$. Then a decomposition is taken as follows:
(i) K_{15} on $X \cup A$, with $K_{8,8}$ on $\{B, C\}$;
(ii) $K_{15} \setminus K_7$ on $X \cup B$, with $K_{8,8}$ on $\{A, C\}$;
(iii) $K_{15} \setminus K_7$ on $X \cup C$, with $K_{8,8}$ on $\{A, B\}$.

Note that $K_{8,8}$ contains four 2-factors, and these may contain $\{0, 1, \ldots, 8\}$ 6-cycles. Also K_{15} contains seven 2-factors, and we have a 2-factorization with a sub-2-factorization of K_7, so with three of the seven 2-factors containing 2-factors of X. Thus the K_{15} or $K_{15} \setminus K_7$ contains $\{0, 1, \ldots, 8\}$ 6-cycles (This is 0, 1 or 2 in each of the four 2-factors having no edge in X).

Let the various 2-factors in the parts of the decomposition be as follows:

(i) $\{F_{Ai} \mid 1 \leq i \leq 7\}$ for K_{15} on $X \cup A$; $\{F_{BCi} \mid 1 \leq i \leq 4\}$ for $K_{8,8}$ on $\{B, C\}$.

(ii) $\{F_{Bi} = F'_{Bi} \cup F''_{Bi} \mid 1 \leq i \leq 3\} \cup \{F_{Bi} \mid 4 \leq i \leq 7\}$, for K_{15} on $X \cup B$, where F'_{Bi} is on B and F''_{Bi} is on X; $\{F_{ACi} \mid 1 \leq i \leq 4\}$ for $K_{8,8}$ on $\{A, C\}$.

(iii) $\{F_{Ci} = F'_{Ci} \cup F''_{Ci} \mid 1 \leq i \leq 3\} \cup \{F_{Ci} \mid 4 \leq i \leq 7\}$, for K_{15} on $X \cup C$, where F'_{Ci} is on C and F''_{Ci} is on X; $\{F_{ABi} \mid 1 \leq i \leq 4\}$ for $K_{8,8}$ on $\{A, B\}$.

Then the final fifteen 2-factors for K_{31} are:

$$\{F_{Ai} \cup F'_{Bi} \cup F'_{Ci} \mid 1 \le i \le 3\}, \quad \{F_{A(i+3)} \cup F_{BCi} \mid 1 \le i \le 4\},$$
$$\{F_{B(i+3)} \cup F_{ACi} \mid 1 \le i \le 4\}, \quad \{F_{C(i+3)} \cup F_{ABi} \mid 1 \le i \le 4\},$$

and these may each contain (respectively) the following numbers of 6-cycles:

$$\{0,1,2\}, \quad \{0,1,2\} + \{0,1,2\}, \quad \{0,1,2\} + \{0,1,2\}, \quad \{0,1,2\} + \{0,1,2\}.$$

Hence $3 * \{0,1,2\} + 12 * \{0,1,2,3,4\} \subseteq Q(31)$, or $\{0,1,2,\ldots,54\} \subseteq Q(31)$. This completes the case of order 31. □

LEMMA 4.9 *$Q(24k+7) = FC(24k+7)$ for all integers $k \ge 0$.*

Proof Here $n = 12(2k) + 7$; we use Construction II with $\epsilon = 7$. We have $Q(31) = \text{FC}(31)$ from Example 4.8. Then Corollary 2.4, together with Examples 4.8 and order 55 from Lemma 4.1, gives $Q(24k+7) = \text{FC}(24k+7)$ as required. □

$n \equiv 11 \pmod{24}$

EXAMPLE* 4.10 $Q(11) = \text{FC}(11) = \{0,1,\ldots,5\}$.

EXAMPLE 4.11 $Q(35) = \text{FC}(35)$, and five 2-factors in each 2-factorization contain a sub-2-factorization of K_{11} on eleven fixed vertices. We use an *ad hoc* construction here. Take the vertex set $\{\infty_1, \infty_2\} \cup A \cup B \cup C$ where $A = \{a_i \mid 0 \le i \le 10\}$, $B = \{b_i \mid 0 \le i \le 10\}$ and $C = \{c_i \mid 0 \le i \le 10\}$.

First, a decomposition of K_{35} is given by:

(i) K_{11} on vertex set A, with $K_{12,12}$ on vertex set $B \cup \{\infty_1\}$, $C \cup \{\infty_2\}$, and

(ii) K_{11} on vertex set B, with $K_{12,12} \setminus \{\infty_1, \infty_2\}$ on vertex set $A \cup \{\infty_2\}$, $C \cup \{\infty_1\}$, and

(iii) K_{11} on vertex set C, with $K_{12,12} \setminus \{\infty_1, \infty_2\}$ on vertex set $A \cup \{\infty_1\}$, $B \cup \{\infty_2\}$.

From Example 3.1, we have $\{0,6,12,18,24\} \subseteq Q(K_{12,12})$. And a 2-factorization of $K_{12,12}$ contains six 2-factors, while a 2-factorization of K_{11} contains five 2-factors. From (i), we have five 2-factors of K_{11} on vertex set A, together with five (of a possible six) 2-factors of $K_{12,12}$ with vertex set

$\{B \cup \{\infty_1\}, C \cup \{\infty_2\}\}$. From (ii), we have five 2-factors of K_{11} on vertex set B, together with five 2-factors of $K_{12,12}$ with vertex set $\{A \cup \{\infty_2\}, C \cup \{\infty_1\}\}$ (leaving out the 2-factor containing the edge $\{\infty_1, \infty_2\}$). Similarly, from (iii), we have five 2-factors of K_{11} on vertex set C, together with five 2-factors of $K_{12,12}$ with vertex set $\{A \cup \{\infty_1\}, B \cup \{\infty_2\}\}$ (leaving out the 2-factor containing the edge $\{\infty_1, \infty_2\}$). This makes a total of 15 2-factors of K_{35}, containing

$$3 * Q(11) + 3 * \{0, 4, 8, 12, 16, 20\} = \{0, 1, 2, \ldots, 75\} \quad \text{6-cycles.}$$

There remain two more 2-factors, to be formed from a 2-factor on $\{B \cup \{\infty_1\}, C \cup \{\infty_2\}\}$, and 2-factors with $\{\infty_1, \infty_2\}$ removed, on $\{A \cup \{\infty_2\}, C \cup \{\infty_1\}\}$ and on $\{A \cup \{\infty_1\}, B \cup \{\infty_2\}\}$.

Note that we may choose the 2-factor left from (i) above, as well as the paths (2-factors minus edge $\{\infty_1, \infty_2\}$) left from (ii) and (iii). For five or ten 6-cycles in the last two 2-factors of K_{35}, we start with the 4-cycles (with edge $\{\infty_1, \infty_2\}$ only once):

$$\begin{array}{ll} (i) & (\infty_1, \infty_2, b_0, c_0),\ (b_i, c_i, b_{i+1}, c_{i+1}),\ i = 1, 3, 5, 7, 9; \\ (ii) & (\infty_1, \infty_2, c_0, a_0),\ (a_i, c_i, a_{i+1}, c_{i+1}),\ i = 1, 3, 5, 7, 9; \\ (iii) & (\infty_1, \infty_2, a_0, b_0),\ (a_i, b_i, a_{i+1}, b_{i+1}),\ i = 1, 3, 5, 7, 9. \end{array}$$

These reassemble into the last two 2-factors for K_{35}: we have K_5 on $\{a_0, b_0, c_0, \infty_1, \infty_2\}$ and five lots of $K_{2,2,2}$ on $\{\{a_i, a_{i+1}\}, \{b_i, b_{i+1}\}, \{c_i, c_{i+1}\}\}$, $i = 1, 3, 5, 7, 9$. Since $K_{2,2,2}$ has a 2-factorization with either one 6-cycle and two triangles, or else two 6-cycles (see Lemma 2.5), this gives 5 or 10 6-cycles in these 2-factors of K_{35}.

For no 6-cycles at all in these last two 2-factors, we start with a different 2-factor in case (ii), namely

$$\begin{array}{ll} (ii)' & \{(\infty_1, \infty_2, c_0, a_0), (a_1, c_3, a_2, c_4), (a_3, c_5, a_4, c_6), \\ & \ (a_5, c_7, a_6, c_8), (a_7, c_9, a_8, c_{10}), (a_9, c_1, a_{10}, c_2)\}. \end{array}$$

Then (i), $(ii)'$ and (iii) re-form to give two 2-factors of K_{35} having cycles of lengths 5, 10, 10, 10 (so *no* 6-cycles):

$$\begin{array}{l} \{(a_0, b_0, c_0, \infty_2, \infty_1),\ (a_1, c_3, b_3, a_3, c_5, b_6, c_6, a_4, b_4, c_4), \\ \ (a_2, b_1, c_2, a_{10}, b_{10}, c_9, b_9, a_9, c_1, b_2),\ (a_5, b_5, a_6, c_7, b_7, a_7, c_{10}, a_8, b_8, c_8)\} \end{array}$$

and

$$\begin{array}{l} \{(a_0, c_0, \infty_1, b_0, \infty_2),\ (a_1, b_1, c_1, a_{10}, b_9, c_{10}, b_{10}, a_9, c_2, b_2), \\ \ (a_2, c_3, b_4, a_3, c_6, b_5, c_5, a_4, b_3, c_4),\ (a_5, c_7, b_8, a_7, c_9, a_8, b_7, c_8, a_6, b_6)\}. \end{array}$$

Hence $Q(35) = \{0, 1, \ldots, 75\} + \{0, 5, 10\} = \{0, 1, \ldots, 85\} = \text{FC}(35)$. □

LEMMA 4.12 *$Q(24k+11) = FC(24k+11)$ for all integers $k \geq 0$.*

Proof Here $n = 12(2k)+11$; we use Construction II with $\epsilon = 11$. We have $Q(11) = \text{FC}(11)$ from Example 4.10, and $Q(35) = \text{FC}(35)$ from Example 4.11. Then Corollary 2.4, together with Example 4.11 and order 59 from Example 4.1, gives $Q(24k+11) = \text{FC}(24k+11)$ as required. □

5 The case $n \equiv 9 \pmod{24}$

Here we have a different construction which requires fewer big examples. First we consider the tripartite graph $K_{4,4,4}$.

EXAMPLE 5.1 $Q(K_{4,4,4}) \supseteq \{0,4,8\}$.
Let the vertex set be $\{\{1,2,3,4\},\{5,6,7,8\},\{9,10,11,12\}\}$.
$0 \in Q(K_{4,4,4})$:
(1,5,2,6),(3,11,4,12),(7,9,8,10); (1,7,2,8),(3,9,4,10),(5,11,6,12);
(1,9,2,10),(3,5,4,6),(7,11,8,12); (1,11,2,12),(3,7,4,8),(5,9,6,10).

$4 \in Q(K_{4,4,4})$:
(1,5,9,2,6,10),(3,7,11,4,8,12); (1,8,11,2,5,12),(3,6,9,4,7,10);
(1,7,9),(2,8,10),(3,5,11),(4,6,12); (1,6,11),(2,7,12),(3,8,9),(4,5,10).

$8 \in Q(K_{4,4,4})$:
(1,5,9,2,6,10),(3,7,11,4,8,12); (1,8,11,2,5,12),(3,6,9,4,7,10);
(1,9,7,2,12,6),(3,8,10,4,5,11); (1,7,12,4,6,11),(2,8,9,3,5,10).

□

EXAMPLE 5.2 $Q(9) = \text{FC}(9) = \{0,1,2,3,4\}$.
Let the vertex set of K_9 be $\{0,1,2,\ldots,8\}$.
(i) $0 \in Q(9)$: take a Kirkman triple system of order 9.
(ii) $1 \in Q(9)$: take the 2-factorization
$(0,1,2,3,4,5)(6,7,8)$; $(0,2,4,1,6,3,7,5,8)$; $(0,3,1,5,6,4,8,2,7)$;
$(0,4,7,1,8,3,5,2,6)$.
(iii) $2 \in Q(9)$: take the 2-factorization
$(0,1,2,3,4,5)(6,7,8)$; $(0,2,6,3,5,7)(1,4,8)$; $(0,3,8)(1,5,6)(2,4,7)$;
$(0,4,6)(1,3,7)(2,5,8)$.
(iv) $3 \in Q(9)$: take the 2-factorization
$(0,1,2,3,4,5)(6,7,8)$; $(0,2,4,6,5,7)(1,3,8)$; $(3,5,1,6,2,7)(0,4,8)$;
$(0,3,6)(1,4,7)(2,5,8)$.
(v) $4 \in Q(9)$: take the 2-factorization
$(0,1,2,3,4,5)(6,7,8)$; $(0,2,4,6,1,7)(3,5,8)$; $(1,3,7,2,6,5)(0,4,8)$;

$(1,4,7,5,2,8)(0,3,6)$. □

Construction IV

Take K_n with $n = 24k+9$ on the vertex set $\{\infty\} \cup \{(i,j) \mid 1 \le i \le 6k+2,\ 1 \le j \le 4\}$. Take a Kirkman triple system on the set $\{1,2,3,\ldots,6k+3\}$, and suppose one of its $3k+1$ parallel classes is $\{1,2,6k+3\},\{3,4,5\},\{6,7,8\},\ldots,\{6k,6k+1,6k+2\}$. Now delete the point $6k+3$. Each parallel class will now contain one block of size 2 and $2k$ of size 3. For each such punctured parallel class we do the following:
On the set $\{\infty\} \cup \{(i,j) \mid i = 1,2,\ j = 1,2,3,4\}$, where $\{1,2\}$ is the block of size 2 in the (punctured) parallel class, we take a 2-factorization of K_9, which contains four 2-factors.
Now for each block $\{i_1,i_2,i_3\}$ in the rest of the parallel class, on the sets $\{(i_1,j) \mid j = 1,2,3,4\} \cup \{(i_2,j) \mid j = 1,2,3,4\} \cup \{(i_3,j) \mid j = 1,2,3,4\}$, we place a 2-factorization of $K_{4,4,4}$, which contains four 2-factors.
Doing this for each of the $3k+1$ (punctured) parallel classes yields $4(3k+1)$ 2-factors altogether. Moreover, these 2-factors may contain $Q(9) + 2k * Q(K_{4,4,4})$ or $\{0,1,2,\ldots,16k+4\}$ 6-cycles, so we obtain $Q(24k+9) = \{0,1,2,\ldots,(3k+1)(16k+4)\} = \mathrm{FC}(24k+9)$.

We record this result as follows.

THEOREM 5.3 $Q(24k+9) = FC(24k+9) = \{0,1,2,\ldots,4(4k+1)(3k+1)\}$. □

6 Conclusion

We have now shown:

THEOREM 6.1 *There exists a 2-factorization of K_n, n odd, containing x 6-cycles, where $0 \le x \le FC(n)$, and $FC(n)$ is given in the following table.*

Order n	*$FC(n)$*
$12k+1$	$\{0,1,\ldots,6k(2k-1)\}$
$12k+3$	$\{0,1,\ldots,(6k+1)2k\}$
$12k+5$	$\{0,1,\ldots,(6k+2)2k\}$
$12k+7$	$\{0,1,\ldots,(6k+3)2k\}$
$12k+9$	$\{0,1,\ldots,(6k+4)(2k+1)\}$
$12k+11$	$\{0,1,\ldots,(6k+5)(2k+1)\}$

In other words, $FC(n) = Q(n)$ for all odd n.

The next problem in this area is the case of K_n minus a 1-factor when n is even; the possible number of 6-cycles awaits counting!

Appendix

Please see `http://www.maths.uq.edu.au/~ejb/JCMCCappendix.html` or email a request to `ejb@maths.uq.edu.au` .

References

[1] P. Adams, E.J. Billington, I.J. Dejter and C.C. Lindner, *The number of 4-cycles in 2-factorizations of K_{2n} minus a 1-factor*, Discrete Math. **220** (2000), 1–11.

[2] I.J. Dejter, F. Franek, E. Mendelsohn and A. Rosa, *Triangles in 2-factorizations*, J. Graph Theory **26** (1997), 83–94.

[3] I.J. Dejter, C.C. Lindner and A. Rosa, *The number of 4-cycles in 2-factorizations of K_n*, J. Combin. Math. Combin. Computing **28** (1998), 101–112.

[4] S. Kucukcifci, *The number of 8-cycles in 2-factorizations of K_n*, Australas. J. Combinatorics (to appear).

[5] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, Boca Raton (1997).