Halving Families of Sets

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Abstract

We give a necessary and sufficient condition of Hall's type for a family of sets of even cardinality to be decomposable into two subfamilies having a common system of distinct representatives. An application of this result to partitions of Steiner Triple Systems into small configurations is presented.

1 Introduction

A family of sets $\mathcal{A} = \{A_1, A_2, ..., A_{2n}\}$ is *halvable* if it is possible to split $\mathcal{A}$ into two subfamilies of cardinality n each having a common system of distinct representatives. Thus $\mathcal{A}$ is halvable if there exists a family $\mathcal{B} = \{B_1, B_2, ..., B_{2n}\}$ so that $B_i \subset A_i, |B_i| = 1, i = 1, ..., 2n$, and for each $i, 1 \leq i \leq 2n$, there exist exactly one $j, 1 \leq j \leq 2n, j \neq i$, so that $B_i = B_j$. Further, a family $\mathcal{A} = \{A_1, ..., A_{2n-1}\}$ of an odd cardinality is halvable if one can halve the family $\mathcal{C} = \{C_1, ..., C_{2n}\}$, where $C_i = A_i \cup \{x\}$ for $i = 1, ..., 2n-1$ and $C_{2n} = \{x\}$, x being a new element.

The problem of characterizing halvable families of sets can be seen as a "dual" problem to characterizing pairs of families of sets having a common system of distinct representatives. For various generalizations and modification of the latter problem see [6] and [5].

A family $\mathcal{A} = \{A_i; i \in I\}$ of subsets of a set S is often represented by a

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bipartite graph G where one part is formed by the index set I and the other by S, and ix is an edge of G if $x \in A_i$. In the language of graph theory, $\mathcal{A}$ is halvable if there exists a factor F of G so that the degree $\deg_F(v) = 2$ for all $v \in I$, and $\deg_F(v) = 0$ or 1 for all $v \in S$. It follows from a result in [2] that a decision problem whether G contains the factor F can be decided in a polynomial time. However, as far as we know, no characterization of graphs possessing such a factor has been given so far.

As the main result of this paper we give a necessary and sufficient condition of Hall's type for a family of sets to be halvable. Further, it will be shown that the condition cannot be relieved. A deficient version of the result is provided as well.

In the second part of the paper we show how it is possible to apply the above result to partitioning Steiner triple systems into some small configurations.

2 Halving a family of sets

In this section we provide a necessary and sufficient condition for a family of sets to be halvable.

Let $J_1, ..., J_t$ be sets forming a partition P of $\{1, ..., 2n\}$ so that $|J_i|$ is odd for $i = 1, ..., t-1$, and there are no restrictions on J_t, we admit also $J_t = \emptyset$. Then the family of sets $\mathcal{A}^* = \{A_1^*, ..., A_t^*\}$, where $A_i^* = \bigcup_{j \in J_i} A_j, i = 1, ..., t$, will be called a reduction of the family $\mathcal{A} = \{A_1, ..., A_{2n}\}$ associated with the partition P.

2.1 Characterization of halvable families

Theorem 2.1 *A family $\mathcal{A} = \{A_1, ..., A_{2n}\}$ is halvable if and only if for any reduction $\mathcal{A}^* = \{A_1^*, ..., A_t^*\}$ of $\mathcal{A}$ it holds*

$$|\bigcup_{i=1}^{t-1} A_i^*|_2 \geq \frac{t-1-|J_t|}{2}. \qquad (1)$$

Proof. First we show the necessary part of the statement. Set $N = \{1, ..., 2n\}$. Suppose that $\mathcal{A}$ is halvable, and $\mathcal{B} = \{B_1, ..., B_{2n}\}$ is a transversal of $\mathcal{A}$. That means, $|B_i| = 1, B_i \subset A_i, i \in N$, and for each $i \in N$ there exists exactly one $j \in N, i \neq j$, so that $B_i = B_j$. Let $\mathcal{A}^* = \{A_1^*, ..., A_t^*\}$ be a reduction of $\mathcal{A}$ associated with a partition P of $\{1, ..., 2n\}$ into t parts $J_1, ..., J_t$. To show that $\mathcal{A}$ satisfies (1) it suffices to show that $\mathcal{B}^* = \{B_1^*, ..., B_t^*\}$, a reduction of $\mathcal{B}$ associated with the same partition P, satisfies (1) since $|\bigcup_{i=1}^{t-1} A_i^*|_2 \geq |\bigcup_{i=1}^{t-1} B_i^*|_2$. As $|J_i|$, $i = 1, ..., t-1$, is odd there is at least one $j \in J_i$ so that the mate B_l of B_j is off B_i^*, i.e., $B_l = B_j$ and $l \notin J_i$. Hence, there are at least $t-1$ sets B_j with their mates belonging to a different set of $\mathcal{B}^*$ than B_j. Let $B_j = \{x\}$. If the mate of B_j belongs to $B_s^*, s \leq t-1$, then $x \in (\bigcup_{i=1}^{t-1} B_i^*)_2$. Since at most $|J_t|$ of these $B_j's$ have their mates in B_t^*, $2|\bigcup_{i=1}^{t-1} B_i^*|_2 + |J_t| \geq t-1$, and (1) follows.

Given a family $\mathcal{A} = \{A_1, ..., A_{2n}\}$. Set $U = \bigcup_{i=1}^{2n} A_i$. To prove the sufficiency of (1) consider a bipartite graph $H = (N, U; E)$, where $i \in N, x \in U$ are joined by an edge, i.e., $ix \in E$ if $x \in A_i$. Construct a new graph G by substituting each vertex $x \in U$ by a set $V_x = \{x_1, ..., x_4\}$, where the vertex $x_j, j = 1, 2$, is joined with a vertex $i \in N$ iff $ix \in E$, $x_j x_m, j = 1, 2, m = 3, 4$ and $x_3 x_4$ are edges of G, that is, for the subgraph of G induced by V_x, it is $[V_x] = K_4 - x_1 x_2$.

It is easy to check that $\mathcal{A}$ can be halved iff G has a 1-factor. Suppose for the sake of contradiction that (1) is satisfied for any reduction of $\mathcal{A}$ but G does not have a 1 -factor. By Tutte's theorem there is a cutset S so that the number of odd components of the graph $G - S$ is bigger than $|S|$. We construct a cutset $S' \subset S$ as follows. If, for some x, only one of the two vertices x_1 and x_2 is in S then we remove the vertex from S. Further, if there are in S vertices x_3 and/or x_4 for some $x \in U$, we remove them from S. Denote the obtained subset of S by S'. It is easy to check from the definition of the graph G that S' is also a cutset of G, and the number of odd components of the graph $G - S'$ is bigger than $|S'|$. Moreover, from the construction of S', if C is an odd component of $G - S'$, then $|V(C) \cap N|$ is an odd number. Let $C_1, ..., C_{t-1}$ be all odd components of G. Consider a reduction of $\mathcal{A}, \mathcal{A}^* = \{A_1^*, ..., A_t^*\}$, $A_i^* = \bigcup_{j \in J_i} A_j$, where $J_i = V(C_i) \cap N$ for $i = 1, ..., t-2$, $J_{t-1} = (V(C_{t-1}) \cap N) \cup \{j, j \in N, j$ belongs to an even component of $G - S'\}$, and finally, $j \in J_t$ if $j \in S' \cap N$. Clearly, $|J_i|$ is odd for $i = 1, ..., t-1$. Moreover, if $x \in (\bigcup_{i=1}^{t-1} A_i^*)_2$ then both x_1 and x_2 are in

S', thus

$$|\bigcup_{i=1}^{t-1} A_i^*|_2 \leq \frac{|S' \cap (V(G) - N)|}{2}. \qquad (2)$$

However,

$$t - 1 > |S'| = |S' \cap N| + |S' \cap (V(G) - N)| = |J_t| + |S' \cap (V(G) - N)|,$$

i.e.,

$$\frac{|S' \cap (V(G) - N)|}{2} < \frac{t - 1 - |J_t|}{2}. \qquad (3)$$

Combining (2) and (3) we arrive at contradiction as we assumed that any reduction of $\mathcal{A}$ satisfies (1). □

2.2 Independence of (1) for distinct reductions

To verify Theorem 1 it is not necessary to verify the condition (1) for those reductions $\mathcal{A}^*$ where $t - 1 - |J_t| \leq 0$ as in this case (1) is trivially satisfied. However, it is not possible to simplify Theorem 1 any further as (1) is independent for all other reductions. To demonstrate this we will give an example where (1) fails for only one reduction given in advance.

Example. Let $J_1, ..., J_t$ form a fixed partition P of $\{1, ..., 2n\}$ so that $|J_i|$ is odd, $i = 1, ..., t-1$, and $t - 1 - |J_t| > 0$. We construct a family $\mathcal{A} = \{A_1, ..., A_{2n}\}$ which satisfies (1) for all reductions but the reduction $\mathcal{A}^* = \{A_1^*, ..., A_t^*\}$ associated with the partition P.

Let $A'_i, i \in \{1, ..., 2n\} - J_t$, be sets so that (a) if i, j belong to the same set of P then $A'_i = A'_j$; (b) if i, j belong to different sets of P then $A'_i \cap A'_j = \emptyset$; (c) $|A'_i| = 2n$. Now we are ready to construct the family $\mathcal{A} = \{A_1, ..., A_{2n}\}$. Let K be a set so that $K \cap \bigcup A'_i = \emptyset$ and $|K| = \frac{t-1-|J_t|}{2} - 1$. Then, for $i \notin J_t$, $A_i = A'_i \cup K$, and, for $i \in J_t$, $A_i = \bigcup_{j \notin J_t} A_j$.

Clearly, $\mathcal{A}^*$ does not satisfy (1) as $\left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 = |K| < \frac{t-1-|J_t|}{2}$. It is not difficult to check that (1) is satisfied for all other reductions $\mathcal{A}^{**}$of $\mathcal{A}$. To see this it suffices to note that $|A_i \cup A_j|_2 \geq 2n$ for i, j from the same part of P and for $i \notin J_t, j \in J_t$. Hence, for any partition P' of $\{1, ..., 2n\}$ into t' sets $J'_1, ..., J'_{t'}$ where the above mentioned i and j belong to different sets J'_s, $s \leq t' - 1$ we have $\left|\bigcup_{s=1}^{t'-1} A_s^{**}\right|_2 \geq |A_i \cup A_j|_2 \geq 2n \geq \frac{t'-1-|J_{t'}|}{2}$ as the number t' of parts of any partition of $\{1, ..., 2n\}$ is at most $2n$. Further, if P' is obtained from P by adding another element of $\{1, ..., 2n\}$ to J_t, then the value of right-hand side of (1) is decreased by at least 1, as well as for a partition P' obtained by merging sets of P but in both cases the left-hand part of (1) is at least $|K|$, thus (1) is satisfied. □

2.3 Deficiency

Given a family of sets $\mathcal{A} = \{A_1, ..., A_{2n}\}$. If $\mathcal{A}$ is not halvable then it is desirable to know what is the largest subfamily of $\mathcal{A}$ which can be halved. For a reduction $\mathcal{A}^* = \{A_1^*, ..., A_t^*\}$ of $\mathcal{A}$, define $def(\mathcal{A}^*) = -\left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 + \frac{t-1-|J_t|}{2}$, and call this number the deficiency of $\mathcal{A}^*$. An answer to the above mentioned question is given by:

Theorem 2.2 *Let $\mathcal{A} = \{A_1, ..., A_{2n}\}$ be a family of sets. Then there exist a halvable subfamily $\mathcal{A}', |\mathcal{A}'| = 2n - 2k$, of $\mathcal{A}$ if and only if $k \geq def(\mathcal{A}^*)$ for all reductions $\mathcal{A}^*$ of $\mathcal{A}$.*

Proof. Consider a family $\mathcal{B} = \{B_1, ..., B_{2n}\}$, $B_i = A_i \cup K, i = 1, ..., 2n$, where K is a set of k new elements, i.e., $|K| = k, K \cap (\bigcup_{i=1}^{2n} A_i) = \emptyset$. It is obvious that there is a halvable subfamily $\mathcal{A}'$ of $\mathcal{A}$, $|\mathcal{A}'| = 2n - 2k$, if and only if $\mathcal{B}$ is halvable. Thus we need to show that $\mathcal{B}$ is halvable if and only if $k \geq d \doteq \max def(\mathcal{A}^*)$, where the maximum runs over all reductions $\mathcal{A}^*$ of $\mathcal{A}$.

Let $k \geq d$. Consider a reduction $\mathcal{B}^* = \{B_1^*, ..., B_t^*\}$ of $\mathcal{B}$ associated with a partition P of $\{1, ..., 2n\}$, and $\mathcal{A}^* = \{A_1^*, ..., A_t^*\}$ be a reduction of $\mathcal{A}$ associated with the same partition P. For $t = 2$ there is nothing to prove since in this case $t - 1 - |J_t| \leq 0$ as $|J_t| > 0$. For $t \geq 3$,

$$\left|\bigcup_{i=1}^{t-1} B_i^*\right|_2 = \left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 + k \geq \left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 + def(\mathcal{A}^*)$$

$$= \left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 - \left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 + \frac{t-1-|J_t|}{2} = \frac{t-1-|J_t|}{2}.$$

Thus $\mathcal{B}$ is halvable. On the other hand, let $\mathcal{B}$ be halvable. Then, for any reductions $\mathcal{A}^*, \mathcal{B}^*$ associated with a partition P of $\{1, ..., 2n\}$ into t parts

$J_1, ..., J_t, t > 2$ (again, the case $t = 2$ is trivial, as in this case $def(\mathcal{A}^*) \leq 0$), it is

$$\left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 + k = \left|\bigcup_{i=1}^{t-1} B_i^*\right|_2 \geq \frac{t-1-|J_t|}{2}$$

$$= \left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 - \left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 + \frac{t-1-|J_t|}{2} = \left|\bigcup_{i=1}^{t-1} A_i^*\right|_2 + def(\mathcal{A}^*),$$

hence $k \geq def(\mathcal{A}^*)$ which in turn implies $k \geq d$. □

3 Partitioning Steiner Triple Systems

Steiner triple system of order $v, STS(v)$, is a pair $(V, \mathcal{B})$, where V is a v-set and $\mathcal{B}$ is a collection of 3-subsets of V , called triples, such that every 2-subset of V is contained in exactly one triple of $\mathcal{B}$. It is well known that an $STS(v)$ exists iff $v \equiv 1, 3(\text{mod} 6)$. If in the definition of an STS we replace "exactly" with "at most" we have a partial triple system. In this paper we use the term "k-configuration" to describe a partial triple system with k triples. A Steiner triple system $S = (V, \mathcal{B})$ is said to be decomposable into copies of a k-configuration T if either the set $\mathcal{B}$ of triples, or a set of triples $\mathcal{B}$ from which less than k triples have been deleted, is decomposable into copies of T (this definition abuses slightly the standard language because, when $k > 0$, the word packing would be more appropriate).

The interest in decomposing Steiner triple systems into configurations has been triggered by a conjecture due to Füredi. For the formulation and partial results on the conjecture see [7] and [4]. As a matter of fact a question along this line has already been considered by Brown, Erdös and Sós [9]and subsequently by Ruzsa and Szemerédi in their celebrated paper [8].

We deal in this paper only with 2-configurations. There are two of them. A 2-configuration called the bow-tie comprising two intersecting triples and a 2-configuration comprising two disjoint triples called 2 -parallel configuration, shortly 2-PC. The interested reader can find in [4] also results dealing with decompositions of Steiner triple systems into all five 3-configurations as well as k-configurations comprising k triples intersecting at the same point or k pairwise disjoint triples. It has been shown:

Theorem 3.1 *[4](i) Any $STS(v)$ can be decomposed into bow-ties. (ii) Any $STS(v)$, $v \neq 7, 9$, can be decomposed into 2-PC.*

In fact there are in [4] two different proofs of the part (i). Another proof of (i) can be found in [1]. However, the argument employed to prove (ii) differs from the main idea in all of these three proofs of (i). As an application of the result on halving families we show that it allows to prove (i) and (ii) as a consequence of the same result. This also suggests that finding a sufficient condition for splitting a family of sets into three subfamilies having a common system of distinct representatives might be a possible way how to tackle Füredi's conjecture.

Proof of Theorem 3. Let $S = (V, \mathcal{B})$ be an $STS(v)$ and C be a 2-configuration. Then $|\mathcal{B}| = \frac{v(v-1)}{6} \doteq n$. Denote $T_1, ..., T_n$ the triples of $\mathcal{B}$. Clearly, the total number of 2–configurations in S is $\binom{n}{2}$. Let m be the number of those 2-configuration of S which are isomorphic to the configuration C. (A surprising result by Grannel, Griggs and Mendelhson [3] says that if C is a k-configuration where $k \leq 3$, then the number m does not depend on the choice of Steiner triple system S but only on C.) Denote the 2-configurations of triples of S isomorphic to C by $C_1, ..., C_m$. Consider a family $\mathcal{A}(S, C) = \{A_1, ..., A_n\}$ of subsets of $\{1, ..., m\}$, where $j \in A_i$ iff the triple T_i is in C_j. Clearly, each number from $\{1, ..., m\}$ occurs in exactly 2 sets of $\mathcal{A}$, and $|A_i| = |A_j|$ for $1 \leq i, j \leq n$. It is easy to see that the Steiner triple system S can be decomposed into 2-configuration C if and only if the family $\mathcal{A}(S, C)$ can be partitioned into 3 subfamilies $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ so that $|\mathcal{A}_1| = |\mathcal{A}_2|, |\mathcal{A}_3| \leq 1$ such that the subfamilies $\mathcal{A}_1, \mathcal{A}_2$ have a common system of distinct representatives. Indeed, if c is an element of a common system of distinct representatives chosen from sets $T_i \in \mathcal{A}_i, i = 1, 2$, then the triples of S corresponding to T_i's make up a copy of the configuration C. On the other hand, let triples T_1, T_2 form a copy of C in a decomposition of S into C. Then we construct a partition of $\mathcal{A}(S, C)$ into $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ so that the sets of $\mathcal{A}(S, C)$ corresponding to T_i's belong to different subfamilies $\mathcal{A}_1, \mathcal{A}_2$ and a triple of S, if any, which is not in any copy of C, is put into $\mathcal{A}_3$.

To simplify the proof, in the case that $|\mathcal{B}| = \frac{v(v-1)}{6}$ is odd we remove from $\mathcal{A}(S, C)$ an arbitrary set (=triple). By $\mathcal{A}'(S, C)$ and $\mathcal{B}'$ we will mean either the original $\mathcal{A}(S, C)$ and $\mathcal{B}$ or $\mathcal{A}(S, C)$ and $\mathcal{B}$ with the triple removed, respectively. Thus we need to show that $\mathcal{A}'(S, C)$ is halvable. By Theorem 1, $\mathcal{A}'(S, C)$ is halvable if any reduction of $\mathcal{A}'(S, C)$ satisfies the condition (1). Consider a reduction of $\mathcal{A}'(S, C\}$, $\mathcal{A}^* = \{A_1^*, ..., A_t^*\}$, associated with a partition P of $\{1, ..., 2n\}$ into t sets $J_1, ..., J_t$. We remind the reader that each set of $\mathcal{A}'(S, C)$ corresponds to a triple of $\mathcal{B}$ and we will sometimes abuse the language and say a triple A_i, triples from A_j^*, etc. For $s = 1, ..., t$, denote by V_s the subset of V where $x \in V$ belongs to V_s if there is a triple T in A_s^* so that $x \in T$. Thus, $|V_s| \geq 3, s = 1, ..., t-1$, and $|V_t| = 0$ or

$|V_t| \geq 3$.

We prove (1) by induction with respect to t. For $t = 2$, $|J_t|$ is odd, that is, $|J_2| > 0$, and we are done as $\frac{t-1-|J_t|}{2} = \frac{1-|J_2|}{2} \leq 0$. For $t = 3$, (1) is again trivially satisfied if $|J_3| > 0$ as then $|J_3| \geq 2$. Otherwise, for $|J_t| = 0$, it is sufficient to show that

$$|A_1^* \cup A_2^*|_2 \geq 1. \tag{4}$$

Obviously, A_1^* and A_2^* form a partition of $\mathcal{B}'$. It is not difficult to see that, for $v \geq 7$, there are triples $T_1 \in A_1^*, T_2 \in A_2^*$ of $\mathcal{B}'$ sharing a point, i.e., (4) is satisfied for C being the bow-tie; for $v \geq 13$, there are disjoint triples $T_1 \in A_1^*, T_2 \in A_2^*$, i.e., (4) is satisfied also for C being 2-PC.

Now suppose that $t > 3$. If there is an $c \in |\bigcup_{i=1}^{t-1} A_i^*|_2$ we construct a new reduction $\mathcal{A}^{**}$ of $\mathcal{A}(S, C)$ associated with a partition P' of $\{1,..,2n\}$ into $t-2$ sets, as follows: Let the sets containing c be in A_s^* and $A_r^*, 1 \leq s < r \leq t-1$, and let $q \leq t-1, s \neq q \neq r$. Then $\mathcal{A}^{**} = \{A_i^{**}, i \in I\}$, where $I = \{1, ..., t\} - \{s, r\}$, $A_i^{**} = A_i^*$ for $i \in \{1, ..., t\} - \{s, r, q\}$ and $A_q^{**} = A_r^* \cup A_s^* \cup A_q^*$. By the induction hypothesis applied to $\mathcal{A}^{**}$ we get

$$|\bigcup_{i=1}^{t-1} A_i^*|_2 \geq |\bigcup_{i \in I} A_i^{**}|_2 + 1 \geq \frac{t-3-|J_t|}{2} + 1 = \frac{t-1-|J_t|}{2}$$

and we are done.

Now let $|\bigcup_{i=1}^{t-1} A_i^*|_2 = 0$. We show that then $t - 1 - |J_t| \leq 0$, hence (1) is trivially satisfied.

First we consider the case when C is the bow-tie. Then the sets $V_1, ..., V_{t-1}$ are pairwise disjoint. As $|V_s| \geq 3, s = 1, ..., t-1$, it is $v \geq 3(t-1)$. Hence, if $\{x, y\}$ is a pair of elements of V which do not belong to the same set $V_s, s \leq t-1$, then the triple of $\mathcal{B}'$ containing both of them is in A_t^*. Thus,

$$|J_t| \geq \frac{\binom{v}{2} - \sum_{i=1}^{t-1} \binom{|V_i|}{2}}{3} - 1.$$

A routine calculus exercise gives that maximum of $\sum_{i=1}^{t-1} \binom{|V_i|}{2}$ is attained in the case when one of $V_i's, 1 \leq s \leq t-1$, has cardinality $v - 3(t-2)$ and the other $V_i's$ have cardinality 3. We get

$$|J_t| \geq \frac{\binom{v}{2} - \binom{v-3(t-2)}{2} - (t-2)\binom{3}{2}}{3} - 1 = \frac{6v(t-2) - 9(t-2)^2 - 6(t-2)}{6} - 1$$

$$= \frac{(t-2)(2v-3t+4)}{2} - 1 \geq \frac{(t-2)(3t-2)}{2} - 1 = 1.5t^2 - 4t + 1$$

as $v \geq 3(t-1)$. Thus, for the right-hand side of (1) it is $\frac{t-1-|J_t|}{2} \leq \frac{-1.5t^2+5t-2}{2} \leq 0$ for $t > 3$, and also in this case (1) is satisfied.

Suppose now that C is the 2-PC. In this case the condition $|\bigcup_{i=1}^{t-1} A_i^*|_2 = 0$ means that if T_1 and T_2 are two disjoint triples of $\mathcal{B}'$ then at least one of them is from A_t^* or there is an $s, 1 \leq s \leq t-1$, so that both T_1 and T_2 belong to A_s^*. Set $M = \max_{x \in V} |\{s, 1 \leq s \leq t-1, x \in V_s\}|$. Since we have $t > 3$, it follows that $M \geq 2$. Let $x \in V$ occurs in M of sets $V_s, s = 1, ..., t-1$. If $M > 4$, then any triple of $\mathcal{B}'$ not containing x has to belong to A_t^*. Since there are in $\mathcal{B}'$ at most $\frac{v-1}{2}$ triples containing x, it is $|J_t| \geq (\frac{v(v-1)}{6} - 1) - \frac{v-1}{2}$ and $t-1 \leq \frac{v-1}{2}$. Hence also in this case $t-1-|J_t| \leq 0$ for $v \geq 13$. If $2 \leq M \leq 3$, then $t-1 \leq M+4 \leq 7$ as there are at most 4 triples not containing x, having pairwise non-empty intersection and, at the same time, intersecting three (two) fixed triples containing x , these seven triples constitute a Fano plane. For $M = 4$ it is $t-1 = 4$ and at most 4 triples not containing x do not belong to J_t. Thus, for $2 \leq M \leq 4$, $|J_t| > \frac{v(v-1)}{6} - 1 - (\frac{v-1}{2} + 4)$ as A_t^* has to contain all triples without x with the possible exception of the four triples, and $t-1 \leq 7$. We get $t-1-|J_t| \leq -v^2 + 4v + 75 \leq 0$ for $v \leq 13$. □

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