$(2, C)$-ordered path designs $P(v, 3, 1)$

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ABSTRACT. Let C be the underlying graph of a configuration of l blocks in a path design of order v and block size 3, $(V, \mathcal{B})$. We say that $(V, \mathcal{B})$ is (l, C)-ordered if it is possible to order its blocks in such a way that each set of l consecutive blocks has the same underlying graph C. In this paper we completely solve the problem of the existence of a $(2, C)$-ordered path design $P(v, 3, 1)$ for any configuration having two blocks.

1 Introduction

Let K_v be the complete undirected graph on v vertices and let G be a subgraph of K_v with no isolated vertices. A *G-design* of K_v is a pair $(V, \mathcal{B})$, where V is the vertex set of K_v and $\mathcal{B}$ is an edge-disjoint decomposition of K_v into copies of the graph G. Usually we say that B is a block of the G-design if $B \in \mathcal{B}$, and $\mathcal{B}$ is called the block-set.

A *balanced G-design* [4, 6] is a G-design in which each vertex belongs to the same number of copies of G. A K_k-design is well-known as a bal-

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anced incomplete block design of order v and block-size k; this is of course balanced in the G-design sense.

A *path design* $P(v,k,1)$ [4] is a P_k-design of K_v, where P_k is a simple path with $k-1$ edges and k vertices, written as $[a_1,a_2,\dots,a_k] = \{\{a_1,a_2\},\{a_2,a_3\},\dots,\{a_{k-1},a_k\}\}$.

Clearly a path design $P(v,2,1)$ $(V,\mathcal{B})$ exists for every $v \geq 2$ and it is always balanced. Hung and N.S. Mendelsohn [6] proved that a balanced path design $P(v,2h+1,1)$ $(h \geq 1)$ exists if and only if $v \equiv 1 \pmod{4h}$, and a balanced path design $P(v,2h,1)$ $(h \geq 2)$ exists if and only if $v \equiv 1 \pmod{2h-1}$. Tarsi [7] proved that the necessary conditions for the existence of a $P(v,k,1)$, namely $v \geq k$ (if $v > 1$) and $v(v-1) \equiv 0 \pmod{2(k-1)}$, are also sufficient.

A *configuration* on p points and l blocks in a G-design $(V,\mathcal{B})$ is a pair $(P,\mathcal{L})$, where $P \subseteq V$, $|P| = p$, $\mathcal{L} \subseteq \mathcal{B}$, $|\mathcal{L}| = l$ and $x \in P$ if and only if x is a vertex of at least one block of $\mathcal{L}$.

The *underlying graph* of a configuration $(P,\mathcal{L})$ is the subgraph C of K_v having P as vertex-set and such that e is an edge of C if and only if e is an edge of some block of $\mathcal{L}$. An *ordered G-design* is a G-design $(V,\mathcal{B})$ which has the blocks in $\mathcal{B}$ ordered by a $1-1$ mapping: $f\colon \mathcal{B} \to \{1,2,\dots,|\mathcal{B}|\}$.

From now on we suppose that the blocks of an ordered G-design are always written in an ordered fashion, i.e. $\mathcal{B} = \{B_1,B_2,\dots,B_n\}$ means that $f(B_i) = i$ for $i = 1,2,\dots,n$.

Definition 1. *We say that an ordered G-design is* (l,C)-ordered *if each set of l consecutive blocks has the same underlying graph C.*

Example 1: Let G be a path P_3 and $V = \mathbb{Z}_5$. Let C be the underlying graph of the configuration $(P,\mathcal{L})$ where $P = \{0,1,2,3\}$ and $\mathcal{L} = \{[0,1,2],[0,3,1]\}$. Put $\mathcal{B}_1 = \{[2,0,4],[0,1,3],[1,4,3],[1,2,4],[2,3,0]\}$ and $\mathcal{B}_2 = \{[0,1,4],[1,3,4],[1,2,4],[2,3,0],[4,0,2]\}$. It is easy to see that $(V,\mathcal{B}_1)$ is not $(2,C)$-ordered, but if we order the blocks of $\mathcal{B}_1$ in the following way $\mathcal{B}_1 = \{[0,1,3],[1,4,3],[1,2,4],[2,0,4],[2,3,0]\}$ then $(V,\mathcal{B}_1)$ is $(2,C)$-ordered. It is also easy to check that it is impossible to order the blocks of $\mathcal{B}_2$ in such a way that $(V,\mathcal{B}_2)$ is $(2,C)$-ordered.

The idea and the motivation for ordering the blocks of a G-design are given by Colbourn and Johnstone [2]; they investigated how to order the blocks of a simple twofold triple system in such a way that the minimal change property holds, that is, so that each two consecutive blocks share exactly two vertices. In our terminology this problem is equivalent to constructing a $(2,C)$-ordered twofold triple system where C is the underlying multigraph of the configuration $(\{0,1,2,3\},\{\{0,1,2\},\{0,1,3\}\})$.

Recently many papers have dealt with the problem of decomposing a Steiner triple system into a given small configuration. The seminal paper on this topic is by Horak and Rosa [5]; see also the very interesting survey [3] by Grannell and Griggs for more results and references. Clearly an (l, C)-ordered G-design will be C-decomposable (in the sense of [5]) if and only if l divides the number of blocks in the design.

Let C be the underlying graph of a configuration of two blocks in a Steiner triple system. Theorem 5.1 of [5] proves that any Steiner triple system of order v is exactly $(2, C)$-decomposable (when the number of blocks in the triple system is even), except for the cases $v = 7$ or 9 when the two blocks of the configuration are disjoint. The same proof of Horak and Rosa's theorem gives the following stronger result.

Theorem 1. [5] *Let C be the underlying graph of a configuration of two blocks in a Steiner triple system. Any Steiner triple system of order v can be $(2, C)$-ordered except when $v = 7, 9$ if the two blocks of the configuration are disjoint.*

In this paper we consider the existence problem for a $(2, C)$-ordered design in the case when G has three nonisolated vertices.

The above theorem gives a complete answer to this problem for $G = K_3$, so only the case $G = P_3$ remains. For $l = 2$ there are just the following six configurations (we write only the block set, the point set P being straightforward):

$$\mathcal{L}_1 = \{[0,1,2],[0,3,1]\}; \quad \mathcal{L}_2 = \{[0,1,2],[3,0,4]\};$$
$$\mathcal{L}_3 = \{[0,1,2],[0,3,4]\}; \quad \mathcal{L}_4 = \{[0,1,2],[3,4,5]\};$$
$$\mathcal{L}_5 = \{[0,1,2],[3,1,4]\}; \quad \mathcal{L}_6 = \{[0,1,2],[0,3,2]\}.$$

For $i = 1, 2, \ldots, 6$, let the underlying graph of $\mathcal{L}_i$ be denoted by C_i. It is easy to see that for $i = 5$ or 6 there is no $(2, C_i)$-ordered path design $P(v, 3, 1)$. We study the remaining four cases in the next section.

Note that the most interesting case is the first one; in fact a $(2, C_1)$-ordered path design $P(v, 3, 1)$ gives a path design $P(v, 3, 1)$ with a minimal change property (see [2]).

2 Main results

We base our constructions on a slight variation of the well-known difference method. Let $B = [x, y, z]$ be a block in a path design $P(v, 3, 1)$ based either on $\mathbb{Z}_v$ or on $\mathbb{Z}_v \cup \{\infty\}$. For $\alpha \in \mathbb{Z}_v$, let $B + \alpha$ be the path $[x+\alpha, y+\alpha, z+\alpha]$ where we suppose that the sum is taken modulo v if $x \in \mathbb{Z}_v$, or is $\infty + \alpha = \infty$ if $x = \infty$. Let $D = \{d_1, d_2, \ldots, d_{2h}\}$ be a set of differences in $\mathbb{Z}_v$ such that $\frac{v}{2} \notin D$ if v is even. Let $\Gamma = \{E_1, E_2, \ldots, E_h\}$ be an ordered set of blocks.

Definition 2. *We say that Γ is a* $(2, C)$-ordered set of base blocks using the differences in D *if the following conditions are satisfied:*

1) *Each difference in D appears in exactly one block of* Γ.

2) *The blocks of* Γ *are* $(2, C)$*-ordered, that is, each pair of consecutive blocks of* Γ *gives the same underlying graph* C.

3) *There is an element* $\alpha \in \mathbb{Z}_v$ *such that* $\gcd(\alpha, v) = 1$ *and* C *is the underlying graph of the configuration* $\{E_h, E_1 + \alpha\}$.

Remark 1: It is easy to check that a $(2, C)$-ordered set of base blocks Γ gives the following set of $(2, C)$-ordered blocks $\mathcal{F} = \{F_\sigma \mid \sigma = 1, 2, \ldots, hv\}$ where $F_\sigma = E_{\sigma - jh} + j\alpha$, and where j is the element of $\mathbb{Z}_v$ such that $\sigma - jh \in \{1, 2, \ldots, h\}$.

Henceforth the notation F_i will denote blocks defined in the sense of this remark.

Note that if $v \equiv 1 \pmod 4$ then from the existence of a $(2, C)$-ordered set of base blocks Γ using the differences in $D = \{1, 2, \ldots, \frac{v-1}{2}\}$, the existence of a $(2, C)$-ordered balanced path design $P(v, 3, 1)$ having $\mathcal{F}$ as block set follows.

Example 2: Let $v = 13$, $\alpha = 3$, $E_1 = [0, 1, 6]$, $E_2 = [0, 6, 4]$ and $E_3 = [3, 0, 4]$. Then $\Gamma = \{E_1, E_2, E_3\}$ is a $(2, C_1)$-ordered set of base blocks and the set $\mathcal{F} = \{[0,1,6], [0,6,4], [3,0,4], [3,4,9], [3,9,7], [6,3,7], [6,7,12], [6,12,10], [9,6,10], [9,10,2], [9,2,0], [12,9,0], [12,0,5], [12,5,3], [2,12,3], [2,3,8], [2,8,6], [5,2,6], [5,6,11], [5,11,9], [8,5,9], [8,9,1], [8,1,12], [11,8,12], [11,12,4], [11,4,2], [1,11,2], [1,2,7], [1,7,5], [4,1,5], [4,5,10], [4,10,8], [7,4,8], [7,8,0], [7,0,11], [10,7,11], [10,11,3], [10,3,1], [0,10,1]\}$ is the block set of a $(2, C_1)$-ordered $P(13, 3, 1)$.

Theorem 2. *For each* $v \equiv 1 \pmod 4$ *there is a* $(2, C_1)$*-ordered balanced path design* $P(v, 3, 1)$.

Proof: For $v = 5, 13$ the theorem is proved by Examples 1 and 2. For the remaining v it is sufficient to construct a $(2, C_1)$-ordered set of base blocks Γ using the differences in $D = \{1, 2, \ldots, \frac{v-1}{2}\}$.

Case $v = 9$. Put $\alpha = 8$ and $\Gamma = \{[0, 1, 3], [0, 3, 8]\}$.

Case $v = 17$. Put $\alpha = 16$ and $\Gamma = \{[0, 1, 6], [0, 6, 4], [0, 4, 7], [0, 7, 16]\}$.

Case $v = 5 + 8k$, $k \geq 2$. Put $\alpha = 1 + 2k$ and $\Gamma = \{E_1, E_2, \ldots, E_{1+2k}\}$, where $E_1 = [0, 1, 3(k+1)]$, $E_2 = [0, 3(k+1), 3k+1]$, $E_{2i+1} = [0, 3k+2-i, 3k+3+i]$, $E_{2i+2} = [0, 3k+3+i, 3k+1-i]$ for $i = 1, 2, \ldots, k-1$, and $E_{2k+1} = [1+2k, 0, 2+2k]$.

Case $v = 1 + 8k$, $k \geq 3$. Put $\alpha = 8k$ and $\Gamma = \{E_1, E_2, \ldots, E_{2k}\}$, where $E_1 = [0, 1, 3k]$, $E_2 = [0, 3k, 3k-2]$, $E_{2i+1} = [0, 3k-1-i, 3k+i]$, $E_{2i+2} = [0, 3k+i, 3k-2-i]$ for $i = 1, 2, \ldots, k-2$, $E_{2k-1} = [0, 2k, 4k-1]$ and $E_{2k} = [0, 4k-1, 8k]$. □

Theorem 3. *For each $v \equiv 0 \pmod 4$ there is a $(2, C_1)$-ordered path design $P(v, 3, 1)$.*

Proof: Case $v = 4$. Put $\mathcal{B} = \{[0,1,2], [0,2,3], [0,3,1]\}$.

Case $v = 8$. Let $V = \mathbb{Z}_7 \cup \{\infty\}$. Put $\mathcal{B} = \{B_1, B_2, \dots, B_{14}\}$, where $B_1 = [\infty, 0, 3]$, $B_2 = [\infty, 3, 6]$, $B_3 = [\infty, 6, 2]$, $B_4 = [\infty, 2, 5]$, $B_5 = [\infty, 5, 1]$, $B_6 = [1, \infty, 4]$, $B_7 = [0, 4, 1]$, $B_{8+i} = [0, 1, 6] + i$ for $i \in \mathbb{Z}_7$.

Case $v = 12$. Let $V = \mathbb{Z}_{11} \cup \{\infty\}$. Define the following $(2, C_1)$-ordered set of base blocks $\pmod{11}$ using $\alpha = 10$ and $D = \{1,2,3,4\}$: $\Gamma = \{E_1, E_2\}$, where $E_1 = [0,1,3]$ and $E_2 = [0,3,10]$. Put $B_1 = [\infty, 1, 7]$, $B_2 = [\infty, 7, 2]$, $B_3 = [\infty, 2, 8]$, $B_4 = [\infty, 8, 3]$, $B_5 = [\infty, 3, 9]$, $B_6 = [\infty, 9, 4]$, $B_7 = [\infty, 4, 10]$, $B_8 = [\infty, 10, 5]$, $B_9 = [\infty, 5, 0]$, $B_{10} = [0, \infty, 6]$, $B_{11} = [0, 6, 1]$, $B_{11+i} = F_i$ for $i = 1, 2, \dots, 22$.

Case $v = 16$. Let $V = \mathbb{Z}_{15} \cup \{\infty\}$. Define the following $(2, C_1)$-ordered set of base blocks $\pmod{15}$ using $\alpha = 14$ and $D = \{1, 2, \dots, 6\}$: $\Gamma = \{E_1, E_2, E_3\}$, where $E_1 = [0,1,6]$, $E_2 = [0,6,2]$ and $E_3 = [0,2,14]$. Put $B_1 = [\infty, 0, 7]$, $B_{2(j+1)} = [\infty, 7-j, 14-j]$, $B_{2j+3} = [\infty, 14-j, 6-j]$ for $j = 0, 1, \dots, 5$, $B_{14} = [1, \infty, 8]$, $B_{15} = [0, 8, 1]$ and $B_{15+i} = F_i$ for $i = 1, 2, \dots, 45$.

Case $v = 8k$, $k \geq 3$. Let $V = \mathbb{Z}_{8k-1} \cup \{\infty\}$. Define the following $(2, C_1)$-ordered set of base blocks $\pmod{8k-1}$ using $\alpha = 8k-2$ and $D = \{1, 2, \dots, 4k-2\}$: $\Gamma = \{E_1, E_2, \dots, E_{2k-1}\}$, where $E_1 = [0, 1, 3k]$, $E_{2i} = [0, 3k+i-1, 3k-i-1]$, $E_{2i+1} = [0, 3k-i-1, 3k+i]$ for $i = 1, 2, \dots, k-2$, $E_{2k-2} = [0, 4k-2, 2k-2]$, $E_{2k-1} = [0, 2k-2, 8k-2]$. Put $B_1 = [\infty, 0, 4k-1]$, $B_{2(j+1)} = [\infty, 4k-1-j, 8k-2-j]$, $B_{2j+3} = [\infty, 8k-2-j, 4k-2-j]$ for $j = 0, 1, \dots, 4k-3$, $B_{8k-2} = [1, \infty, 4k]$, $B_{8k-1} = [0, 4k, 1]$ and $B_{8k-1+i} = F_i$ for $i = 1, 2, \dots, (2k-1)(8k-1)$.

Case $v = 4 + 8k$, $k \geq 2$. Let $V = \mathbb{Z}_{3+8k} \cup \{\infty\}$. Define the following $(2, C_1)$-ordered set of base blocks $\pmod{3+8k}$ using $\alpha = 2 + 8k$ and $D = \{1, 2, \dots, 4k\}$: $\Gamma = \{E_1, E_2, \dots, E_{2k}\}$, where $E_1 = [0, 1, 1+2k]$, $E_{2i} = [0, 2k+i, 4k-i+1]$, $E_{2i+1} = [0, 4k-i+1, 2k+i+1]$ for $i = 1, 2, \dots, k-1$, $E_{2k-1} = [0, 3k, 8k+2]$. Put $B_1 = [\infty, 0, 4k+1]$, $B_{2(j+1)} = [\infty, 4k-j+1, 8k-j+2]$, $B_{2j+3} = [\infty, 8k-j+2, 4k-j]$ for $j = 0, 1, \dots, 4k-1$, $B_{8k+2} = [1, \infty, 4k+2]$, $B_{8k+3} = [0, 4k+2, 1]$ and $B_{8k+3+i} = F_i$ for $i = 1, 2, \dots, 2k(8k+3)$. □

Theorem 4. *For each $v \equiv 1 \pmod 4$ there is a $(2, C_2)$-ordered balanced path design $P(v, 3, 1)$.*

Proof: Case $v = 5$. Put $V = \mathbb{Z}_5$ and $\mathcal{B} = \{[0,1,2], [3,0,4], [2,3,1], [4,2,0], [1,4,3]\}$.

Case $v = 9$. Put $V = \mathbb{Z}_9$, $\alpha = 1$ and $\Gamma = \{[0,1,5], [2,0,3]\}$.

Case $v = 13$. Put $V = \mathbb{Z}_{13}$, $\alpha = 1$ and $\Gamma = \{[0,1,5], [11,0,3], [0,7,2]\}$.

Case $v = 5 + 8k$, $k \geq 2$. Let $V = \mathbb{Z}_v$ and put $\alpha = 1$ and $\Gamma = \{E_1, E_2, \dots, E_{1+2k}\}$, where $E_1 = [0, 1, 1 + 4k]$, $E_{2i} = [4i - 2, 0, 4i - 1]$, $E_{2i+1} = [0, 4i, 8i + 1]$ for $i = 1, 2, \dots, k - 1$, $E_{2k} = [4k - 2, 0, 4k - 1]$ and $E_{1+2k} = [0, 3 + 4k, 2]$.

Case $v = 9 + 8k$, $k \geq 1$. Let $V = \mathbb{Z}_v$ and put $\alpha = 1$ and $\Gamma = \{E_1, E_2, \dots, E_{2k+2}\}$, where $E_1 = [0, 1, 5 + 4k]$, $E_{2i} = [4k - 4i + 6, 0, 4k - 4i + 7]$, $E_{2i+1} = [0, 4k - 4i + 4, 8k - 8i + 9]$ for $i = 1, 2, \dots, k$, and $E_{2k+2} = [2, 0, 3]$. □

Theorem 5. *For each $v \equiv 0 \pmod 4$, $v \geq 8$, there is a $(2, C_2)$-ordered path design $P(v, 3, 1)$.*

Proof: It is easy to see that for $v = 4$ there is no $(2, C_2)$-ordered path design $P(4, 3, 1)$.

Case $v = 8$. Let $V = \mathbb{Z}_7 \cup \{\infty\}$. Put $\mathcal{B} = \{[0,1,3], [1,5,\infty], [5,3,2], [3,0,\infty], [0,5,4], [5,2,\infty], [2,0,6], [0,4,\infty], [4,2,1], [2,6,\infty], [6,4,3], [4,1,\infty], [1,6,5], [6,3,\infty]\}$.

Case $v = 12$. Let $V = \mathbb{Z}_{11} \cup \{\infty\}$. Define the following $(2, C_2)$-ordered set of base blocks $\pmod{11}$ using $\alpha = 2$ and $D = \{3, 4, 5\}$: $\Gamma = \{E_1, E_2\}$, where $E_1 = [0, 3, 7]$ and $E_2 = [\infty, 0, 5]$. Put $B_{i+1} = [0, 1, 3] + i$ for $i = 0, 1, \dots, 10$ and $B_{11+i} = F_i$ for $i = 1, 2, \dots, 22$.

Case $v = 16$. Let $V = \mathbb{Z}_{15} \cup \{\infty\}$. Let Γ_1 and Γ_2 be the two following $(2, C_2)$-ordered sets of base blocks $\pmod{15}$ obtained using $\alpha = 1$, $D_1 = \{1, 2, 3, 4\}$ and $\alpha = 2$, $D_2 = \{5, 6, 7\}$ respectively: $\Gamma_1 = \{E_1, E_2\}$, where $E_1 = [0, 1, 5]$ and $E_2 = [2, 0, 3]$; $\Gamma_2 = \{\overline{E}_1, \overline{E}_2\}$, where $\overline{E}_1 = [\infty, 1, 8]$ and $\overline{E}_2 = [3, 8, 14]$. Put $B_i = F_i$ and $B_{i+30} = \overline{F}_i$ for $i = 1, 2, \dots, 30$.

Case $v = 20$. Let $V = \mathbb{Z}_{19} \cup \{\infty\}$. Let Γ_1 and Γ_2 be the two following $(2, C_2)$-ordered sets of base blocks $\pmod{19}$ obtained using $\alpha = 1$, $D_1 = \{1, 2, \dots, 6\}$ and $\alpha = 2$, $D_2 = \{7, 8, 9\}$ respectively: $\Gamma_1 = \{E_1, E_2, E_3\}$, where $E_1 = [0, 1, 17]$, $E_2 = [5, 0, 6]$ and $E_3 = [0, 4, 2]$; $\Gamma_2 = \{\overline{E}_1, \overline{E}_2\}$, where $\overline{E}_1 = [\infty, 18, 8]$ and $\overline{E}_2 = [3, 10, 18]$. Put $B_i = F_i$ for $i = 1, 2, \dots, 57$ and $B_{i+57} = \overline{F}_i$ for $i = 1, 2, \dots, 38$.

Case $v = 8k$, $k \geq 3$. Let $V = \mathbb{Z}_{8k-1} \cup \{\infty\}$. Let Γ_1 and Γ_2 be the two following $(2, C_2)$-ordered sets of base blocks $\pmod{8k - 1}$ obtained using $\alpha = 1$, $D_1 = \{1, 2, \dots, 4k - 4\}$ and $\alpha = 2$, $D_2 = \{4k - 3, 4k - 2, 4k - 1\}$ respectively: $\Gamma_1 = \{E_1, E_2, \dots, E_{2k-2}\}$, where $E_1 = [0, 1, 4k - 3]$, $E_{2i} = [2k - 2i, 0, 2k + 2i - 3]$, $E_{2i+1} = [0, 2k - 2i - 1, 4k - 3]$ for $i = 1, 2, \dots, k - 2$ and $E_{2k-2} = [2, 0, 4k - 5]$; $\Gamma_2 = \{\overline{E}_1, \overline{E}_2\}$, where $\overline{E}_1 = [\infty, 1, 4k]$ and $\overline{E}_2 = [3, 4k, 8k - 2]$. Put $B_i = F_i$ for $i = 1, 2, \dots, (8k - 1)(2k - 2)$ and $B_{i+(8k-1)(2k-2)} = \overline{F}_i$ for $i = 1, 2, \dots, 2(8k - 1)$.

Case $v = 4 + 8k$, $k \geq 3$. Let $V = \mathbb{Z}_{3+8k} \cup \{\infty\}$. Let Γ_1 and Γ_2 be the two following $(2, C_2)$-ordered sets of base blocks $\pmod{8k + 3}$ obtained using $\alpha = 1$, $D_1 = \{1, 2, \dots, 4k - 2\}$ and $\alpha = 2$, $D_2 = \{4k - 1, 4k, 4k + 1\}$

respectively: $\Gamma_1 = \{E_1, E_2, \ldots, E_{2k-1}\}$, where $E_1 = [0, 1, 8k+1]$, $E_{2i} = [4i+1, 0, 4i+2]$, $E_{2i+1} = [0, 4i+3, 8i+7]$ for $i = 1, 2, \ldots, k-2$, $E_{2k-2} = [4k-3, 0, 4k-2]$ and $E_{2k-1} = [0, 4, 2]$; $\Gamma_2 = \{\overline{E}_1, \overline{E}_2\}$, where $\overline{E}_1 = [\infty, 2+8k, 4k]$ and $\overline{E}_2 = [3, 2+4k, 8k+2]$. Put $B_i = F_i$ for $i = 1, 2, \ldots, (8k+3)(2k-1)$ and $B_{i+(8k+3)(2k-1)} = \overline{F}_i$ for $i = 1, 2, \ldots, 2(8k+3)$. □

Theorem 6. *For each $v \equiv 1 \pmod 4$, $v \geq 9$, there is a $(2, C_3)$-ordered balanced path design $P(v, 3, 1)$.*

Proof: Let $V = \mathbb{Z}_v$. It is easy to see that for $v = 5$ there is no $(2, C_3)$-ordered path design $P(5, 3, 1)$.

Case $v = 9$. Put $\alpha = 1$ and $\Gamma = \{[0, 3, 1], [1, 5, 6]\}$.

Case $v = 13$. Put $\alpha = 1$ and $\Gamma = \{[0, 7, 2], [2, 5, 1], [1, 12, 11]\}$.

Case $v = 17$. Put $\alpha = 1$ and $\Gamma = \{[0, 9, 2], [2, 8, 3], [1, 5, 2], [15, 16, 1]\}$.

Case $v = 21$. Put $\alpha = 1$ and $\Gamma = \{[0, 11, 2], [2, 10, 3], [3, 9, 4], [1, 5, 3], [18, 19, 1]\}$.

Case $v = 29$. Put $\alpha = 1$ and $\Gamma = \{[0, 15, 2], [2, 14, 3], [3, 13, 4], [4, 12, 5], [3, 9, 4], [1, 5, 3], [26, 27, 1]\}$.

Case $v = 13 + 8k$, $k \geq 3$. Put $\alpha = 1$ and $\Gamma = \{E_1, E_2, \ldots, E_{3+2k}\}$, where $E_1 = [0, 7+4k, 2]$, $E_{i+1} = [i+1, 4k-i+7, i+2]$ for $i = 1, 2, \ldots, k+1$, $E_{k+3} = [k+1, 3k+3, k+2]$, $E_{k+3+i} = [k-i+1, 3k-3i+3, k-i+2]$ for $i = 1, 2, \ldots, k-2$, $E_{2k+2} = [1, 5, 3]$ and $E_{2k+3} = [10+8k, 11+8k, 1]$.

Case $v = 9 + 8k$, $k \geq 2$. Put $\alpha = 1$ and $\Gamma = \{E_1, E_2, \ldots, E_{2k+2}\}$, where $E_1 = [0, 5+4k, 2]$, $E_{i+1} = [i+1, 4k-i+5, i+2]$ for $i = 1, 2, \ldots, k$, $E_{k+2} = [k, 3k+2, k+1]$, $E_{k+2+i} = [k-i, 3k-3i+2, k-i+1]$ for $i = 1, 2, \ldots, k-1$ and $E_{2k+2} = [8k+7, 8k+8, 1]$. □

Theorem 7. *For each $v \equiv 0 \pmod 4$, $v \geq 8$, there is a $(2, C_3)$-ordered path design $P(v, 3, 1)$.*

Proof: It is easy to see that for $v = 4$ there is no $(2, C_3)$-ordered path design $P(4, 3, 1)$.

Case $v = 8$. Let $V = \mathbb{Z}_7 \cup \{\infty\}$. Put $\mathcal{B} = \{[0, 1, 3], [3, 4, 6], [6, 0, 2], [2, 3, 5], [5, 6, 1][1, 2, 4], [4, 5, 0], [\infty, 3, 0], [\infty, 4, 1], [\infty, 5, 2], [\infty, 6, 3], [\infty, 0, 4], [\infty, 1, 5], [\infty, 2, 6]\}$.

Case $v = 12$. Let $V = \mathbb{Z}_{11} \cup \{\infty\}$. Define the following $(2, C_3)$-ordered set of base blocks $\pmod{11}$ using $\alpha = 1$ and $D = \{1, 2, 3, 4\}$: $\Gamma = \{E_1, E_2\}$, where $E_1 = [0, 4, 1]$ and $E_2 = [1, 10, 9]$. Put $B_i = F_i$ for $i = 1, 2, \ldots, 22$ and $B_{23+i} = [\infty, 5, 0] + i$ for $i = 0, 1, \ldots, 10$.

Case $v = 16$. Let $V = \mathbb{Z}_{15} \cup \{\infty\}$. Define the following $(2, C_3)$-ordered set of base blocks $\pmod{15}$ using $\alpha = 1$ and $D = \{1, 2, \ldots, 6\}$: $\Gamma = \{E_1, E_2, E_3\}$, where $E_1 = [0, 6, 1]$, $E_2 = [1, 5, 2]$ and $E_3 = [13, 14, 1]$. Put $B_i = F_i$ for $i = 1, 2, \ldots, 45$ and $B_{46+i} = [\infty, 7, 0] + i$ for $i = 0, 1, \ldots, 14$.

Case $v = 20$. Let $V = \mathbb{Z}_{19} \cup \{\infty\}$. Define the following $(2, C_3)$-ordered set of base blocks (mod 19) using $\alpha = 1$ and $D = \{1, 2, \ldots, 8\}$: $\Gamma = \{E_1, E_2, E_3, E_4\}$, where $E_1 = [0, 8, 1]$, $E_2 = [1, 7, 2]$, $E_3 = [3, 5, 1]$ and $E_4 = [1, 17, 16]$. Put $B_i = F_i$ for $i = 1, 2, \ldots, 76$ and $B_{77+i} = [\infty, 9, 0] + i$ for $i = 0, 1, \ldots, 18$.

Case $v = 28$. Let $V = \mathbb{Z}_{27} \cup \{\infty\}$. Define the following $(2, C_3)$-ordered set of base blocks (mod 27) using $\alpha = 1$ and $D = \{1, 2, \ldots, 12\}$: $\Gamma = \{E_1, E_2, \ldots, E_6\}$, where $E_1 = [0, 12, 1]$, $E_2 = [1, 11, 2]$, $E_3 = [2, 10, 3]$, $E_4 = [3, 9, 4]$, $E_5 = [1, 5, 3]$ and $E_6 = [24, 25, 1]$. Put $B_i = F_i$ for $i = 1, 2, \ldots, 162$ and $B_{163+i} = [\infty, 13, 0] + i$ for $i = 0, 1, \ldots, 26$.

Case $v = 8 + 8k$, $k \geq 2$. Let $V = \mathbb{Z}_{8k+7} \cup \{\infty\}$. Define the following $(2, C_3)$-ordered set of base blocks (mod $8k + 7$) using $\alpha = 1$ and $D = \{1, 2, \ldots, 4k+2\}$: $\Gamma = \{E_1, E_2, \ldots, E_{2k+1}\}$, where $E_i = [i-1, 4k-i+3, i]$ for $i = 1, 2, \ldots, k+1$, $E_{k+i+1} = [k-i, 3k-3i+2, k-i+1]$ for $i = 1, 2, \ldots, k-1$ and $E_{2k+1} = [8k+5, 8k+6, 1]$. Put $B_i = F_i$ for $i = 1, 2, \ldots, (2k+1)(8k+7)$ and $B_{(2k+1)(8k+7)+1+i} = [\infty, 4k+3, 0] + i$ for $i = 0, 1, \ldots, 6 + 8k$.

Case $v = 4 + 8k$, $k \geq 4$. Let $V = \mathbb{Z}_{3+8k} \cup \{\infty\}$. Define the following $(2, C_3)$-ordered set of base blocks (mod $8k + 2$) using $\alpha = 1$ and $D = \{1, 2, \ldots, 4k\}$: $\Gamma = \{E_1, E_2, \ldots, E_{2k}\}$, where $E_i = [i-1, 4k-i+1, i]$ for $i = 1, 2, \ldots, k+1$, $E_{k+i+1} = [k-i, 3k-3i, k-i+1]$ for $i = 1, 2, \ldots, k-3$, $E_{2k-1} = [1, 5, 3]$ and $E_{2k} = [8k, 8k+1, 1]$. Put $B_i = F_i$ for $i = 1, 2, \ldots, 2k(8k+3)$ and $B_{2k(8k+3)+1+i} = [\infty, 4k+1, 0] + i$ for $i = 0, 1, \ldots, 3 + 8k$. □

Theorem 8. *For each $v \equiv 1 \pmod 4$, $v \geq 9$, there is a $(2, C_4)$-ordered balanced path design $P(v, 3, 1)$.*

Proof: It is easy to see that for $v = 5$ there is no $(2, C_4)$-ordered path design $P(5, 3, 1)$.

Let $v = 1 + 4k$, $k \geq 2$. Put $\Gamma = \{E_1, E_2, \ldots, E_k\}$, where $E_1 = [0, 2, 1]$, $E_i = [2i-1, 4i-1, 2i]$ for $i = 2, 3, \ldots, k$, and either $\alpha = 8$ if $k = 2$, or $\alpha = 1$ if $k \geq 3$. □

Theorem 9. *For each $v \equiv 0 \pmod 4$, $v \geq 8$, there is a $(2, C_4)$-ordered path design $P(v, 3, 1)$.*

Proof: It is easy to see that for $v = 4$ there is no $(2, C_4)$-ordered path design $P(4, 3, 1)$.

Case $v = 8$. Let $V = \mathbb{Z}_8$ and $\mathcal{B} = \{[6,4,3], [0,1,2], [7,4,5], [0,2,6], [1,3,5], [4,0,6], [5,1,7], [2,3,0], [6,5,7], [2,4,1], [6,7,3], [2,5,0], [1,6,3], [0,7,2]\}$.

Case $v = 4k$, $k \geq 3$. Let $V = \mathbb{Z}_{4k-1} \cup \{\infty\}$. Define the following $(2, C_4)$-ordered set of base blocks (mod $4k - 1$) using $\alpha = 1$ and the differences in $D = \{1, 2, \ldots, 2k-2\}$: $\Gamma = \{E_1, E_2, \ldots, E_{k-1}\}$, where $E_1 = [0, 2, 1]$, $E_i = [2i-1, 4i-1, 2i]$ for $i = 2, 3, \ldots, k-1$, and $E_k = [\infty, 0, 2k-1]$. Put $B_i = F_i$ for $i = 1, 2, \ldots, k(4k-1)$. □

Remark 2: The $P(v, 3, 1)$ given in Theorems 2, 4, 6, 8, 9 are $(2, C)$-ordered in a cyclic way, that is, the configuration given by the first and last block also has C as underlying graph.

3 Concluding remarks

The next step in the study of the existence of (l, C)-ordered G-designs could be either to increase the number of vertices of G or to increase l. The latter seems to be more interesting, in particular when $G = K_3$ and $\mathcal{L} = \{\{1,2,3\},\{1,4,5\},\{2,4,6\}\}$. In fact Colbourn [1] pointed out that the solution of this problem has an interesting application to RAID (redundant arrays of independent disks) disk arrays.

References

[1] C.J. Colbourn, Open problems, Lecture given at University of Catania, December 1999.

[2] M.J. Colbourn and J.K. Johnstone, Twofold Triple Systems with a Minimal Change Property, *Ars Combinatoria* **18** (1983), 151–160.

[3] M.J. Grannell and T.S. Griggs, Configurations on Steiner triple systems, Combinatorial designs and their applications, Research Notes in Mathematics Series (F.C. Holroy, K.L. Quinn, C. Rowley and B.S. Webb, editors), Chapman & Hall CRC, Boca Raton 1999.

[4] P. Hell and A. Rosa, Graph decompositions, handcuffed prisoners, and balanced P-designs, *Discrete Math.* **2** (1972), 229–252.

[5] P. Horak and A. Rosa, Decomposing Steiner triple systems into small configurations, *Ars Combinatoria* **26** (1988), 91–105.

[6] S.H.Y. Hung and N.S. Mendelsohn, Handcuffed designs, *Aequationes Math.* **18** (1974), 256–266.

[7] M. Tarsi, Decomposition of a complete multigraph into simple paths: nonbalanced handcuffed designs, *J. Combin. Theory Ser. A* **34** (1983), 60–70.