Fire Control on Graphs

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Abstract

Let $G = (V, E)$ be a connected undirected graph. Suppose a fire breaks out at a vertex of G and spreads to all its unprotected neighbours in each time interval. Also, one vertex can be protected in each time interval. We are interested in the number of vertices that can be "saved", that is, which will never be burned. An algorithm is presented to find the optimal solution in the 2-dimension grid graphs and 3-dimension cubic graphs. We also determined the upper and lower bounds of the maximum number o f vertices that can be saved on the large product graphs. The problem of containing the fire with one firefighter or more is also considered.

1 Introduction

We shall use the graph-theoretic notation of [1], so that a graph has vertex set $V(G)$, edge set $E(G)$, and $\epsilon(G)$ edges. If the vertices u and v are connected in G, the **distance** between u and v in G, denoted by $d(u, v)$, is the length of a shortest path between u and v. Let P_n denote a path containing **n** vertices (length $n - 1$). A subset S of $V(G)$ is called an independent set of G if no two vertices of S are adjacent in G. A weighted graph is a graph with a real positive number associated to each vertex of G. A maximum weighted independent set is a set of independent vertices of maximum weight.

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Imagine a graph as an apartment building with vertices representing the individual apartments of the building and an edge between two vertices if and only if the two apartments are adjacent. Imagine also, that at time zero, a **fire** breaks out in an apartment of the building. The fire will spread at each time interval to all the adjacent non-protected vertices of each burning vertex. If a firefighter is present, he can protect one apartment per time interval. In this paper, a vertex where a firefighter was present will be referred to as a **protected vertex**, and any vertices that are not burned at the end are considered **saved vertices**. In most of this paper, we will consider this fire fighting problem under the condition of using one firefighter to defend the graph. That is one vertex can be saved at each time interval.

Let $MVS(G, v)$ denote the maximum number of vertices saved in a given graph G in which the fire starts at a given vertex v, and let S be the set of vertices that have been protected in the graph G.

This problem was proposed by Bert L. Hartnell [5]. It may also be known as virus control on a network. There are two different aspects of this problem. From the firefighters' point of view, the object is to save the maximum number of vertices in the graph with a given number of firefighters, by building a fire wall or barrier (see [7]). From an architects' point of view, the object is to design a graph such that firefighter can defend the most vertices in the network given a random subset of vertices where the fires break out. In the other words, they try to find graphs in which the firefighters can defend the graph most efficiently if fires break out at random vertices. More precisely, Bert Hartnell et al (see [2], [4], [6] [3], [8], and [5]) studied the following problem. Whenever a vertex is attacked, all vertices within distance 2 are also destroyed indirectly. They were interested in designing a connected graph such that when a random subset of the vertices are attacked the expected number of vertices that are destroyed is minimized.

In general, it is difficult to find the value of $MVS(G, v)$. G. MacGillivray and P. Wang [7] show that following decision problem is NP-complete even when restricted to bipartite graphs.

FIRE
INSTANCE: A graph G with a special vertex v, and a positive integer k.
QUESTION: Can at least k vertices be saved? That is, is there a sequence at most $|V(G)|$ $s_1, s_2, \ldots$ vertices of G such that if vertex s_i is protected at time I then at least k vertices are saved when the process ends.

Theorem 1.1. *FIRE is NP-complete for bipartite graphs.*

Since it is NP-complete, we shall focus our attention on finding solutions to special graphs. In [7], G. MacGillivray and P. Wang developed two algorithms to find the optimal solution of stopping a fire in a tree, where the fire breaks out at the root of the tree. Furthermore, they converted this problem into a problem of finding the maximum weighted independent set, which can solved by using integer linear programming. In the next section, we shall concentrate on explaining strategies that will save the maximum number of vertices in the product graphs, P_n, $P_n \times P_n$, and $P_n \times P_n \times P_n$. In section 3, we shall discuss this problem in the large product graphs. Finally, we shall also consider the possibility of containing the fire with a number of firefighters.

2 Optimal Fire Fighting Strategies in the Product Graphs

It is obvious that if a fire were to break out at one of the end vertices of P_n, it would take only one firefighter to protect the rest of the n vertices; $MVS(P_n, v) = n - 1$. If the fire were to break out at any of the other vertices in P_n, $MVS(P_n, v) = n - 2$.

Algorithm 2.1.

Given: $P_n \times P_n$ a $n \times n$ grid. A fire breaks out at a point (a, b), where a is the row index and b is the column index, $1 \leq a, b \leq \lceil n/2 \rceil$ and $a \geq b$.

Protect: The following vertices are protected in the order given: $(a, b+1)$, $(a-1, b+1)$,$(a+1, b+2)$, $(a-2, b+2)$, $(a+2, b+3)$, $(a-3, b+3)$, ..., $(1, b+a-1)$, $(2a-2, b+a-1)$, $(2a-1, b+a)$, ..., $(n, b+a)$.

An example of the fire control on a 7×7 grid, with the fire starting at point $(4, 4)$, is given in Figure 1. The total number of vertices saved is 12. Clearly, $MVS(P_n \times P_n, (a, b)) \geq n(n-b) - (a-1)(n-a)$, if $a \geq b$. We think that this is the optimal solution but unable to prove it in general.

The problem becomes even harder if one wants to find an optimal solution for a rectangular grid instead of a square grid. For example, one has to protect two disconnected regions in the optimal solution for a 5×11 grid where the fire starts at $(1, 6)$. There is a need to have a computer program that we can use to explore the optimal solutions in small product graphs.

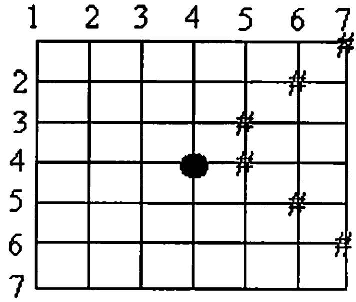

Figure 1: $MVS(P_7 \times P_7) = 12$

In the following, we shall describe a back tracking program that can be used to find optimal solution in grids.

Let g be a graph of dimensions n (the number of rows) by m (the number of columns), and let x be the row coordinate and y the column coordinate of where the fire starts. Let *best_count* be the maximum number of vertices burned, let best be the grid that has the minimum number of vertices burned, and let old be the grid that will change as the fire spreads when the procedure $FIND\_BEST()$ is called recursively. Initially, best-count = m*n, that is, we assume all vertices burned.

Algorithm 2.2.

FIND_BEST(g)

  If (*max_burned*(g) $\geq$ *best_count*) then *exit*;

  For $i = 1$ to $m \times n$ do

    If (the vertex i in the grid g has value 1), then *old* = g;

      Save the vertex i in old by giving it the value 0;

      Burn all the vertices (give them a negative value) in old, that a adjacent to the vertices that have been burned in previous time intervals

      If old is not all burned (that is there are still vertices that can be burned) then

        $FIND\_BEST$(*old*);

      Else if (*max_burned*(*old*) < *best_count*) then

        best_count = *max_burned*(*old*); *best* = *old*;

  end{For}

END

The Algorithm 2.2 is basically a back tracking algorithm that has a recursive procedure to find all possible solutions and keep the one that can save most vertices. To make the program more efficient, it exits the back tracking procedure when the number of vertices burned exceed the previous lowest level. This program have been used to find optimal solutions in small grids (n and $m \leq 10$). We also made some changes to Algorithm 2.2 so that it can be used in three-dimension product graphs, $P_l \times P_m \times P_n$. As one might expect, it would not be an efficient way to find an optimal solution for large product graphs. Instead, we will find a solution first and then run our program only on a small part of $P_l \times P_m \times P_n$ to verify that it is an optimal solution. This idea is illustrated in the following example. Let (x, y, z) be a vertex of $P_l \times P_m \times P_n$ on level x, row y and column z. We will focus our attention on the cases where l and m are relatively small and n is large.

First, since n is relatively large compared to l and m, one vertex from each column must be protected to provide a possible optimal solution in $P_l \times P_m \times P_n$. It follows that we have to protect at least lm vertices. A fire wall in a graph is defined as a series of protected vertices that will separate the cube into a section containing the burned vertices of the graph and a section containing the saved vertices of the graph. Secondly, we start to build this fire wall from the vertex which has distance lm from the vertex where the fire breaks out and build it toward the fire. This is because the fire will spread to all unprotected neighbouring vertices at each time interval

Let us consider $P_3 \times P_3 \times P_6$ and the fire breaks out at the point $(1, 1, 1)$, as shown in Figure 2. By using the above method, a fire wall contains these 9 protected vertices: level 3: (3, 3, 6), (3, 2, 5), (3, 1, 4), level 2: (2, 3, 6), (2, 2, 5), (2, 1, 4), and level 1: (1, 3, 5), (1, 2, 4), (1, 1, 3). It saves 21 vertices among 54 vertices. Our program proves that this is an optimal solution. This also provides an optimal solution for $n \geq 6$.

Theorem 2.1. *$MVS(P_3 \times P_3 \times P_n, (1,1,1)) = 9n - 33$ where $n \geq 6$.*

Proof. Assume that a solution exists for a $P_3 \times P_3 \times P_n$, $n \geq 6$, that is better than the above solution. Then it is a solution containing a series of protected vertices that saves more than 21 vertices and allows less then 33 vertices burned in $P_3 \times P_3 \times P_6$, a contradiction with the fact that one can save at most 21 vertices in $P_3 \times P_3 \times P_6$. □

The optimal solution in $P_l \times P_m \times P_n$ is rather different than the one in $P_n \times P_n$. The one obtained from Algorithm 2.1 protects one vertex from

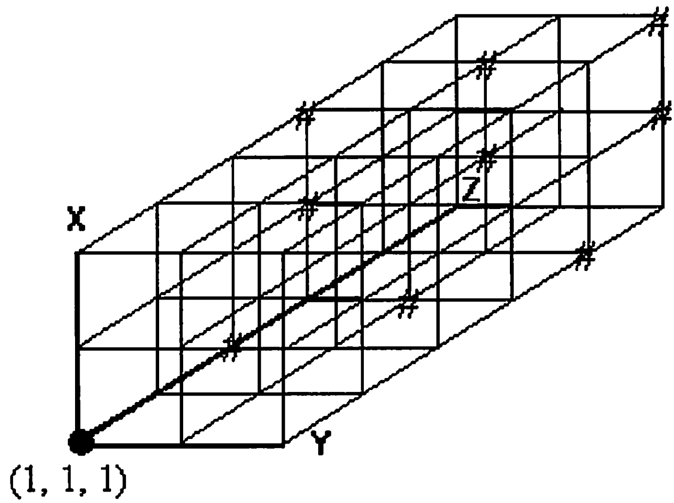

Figure 2: $P_3 \times P_3 \times P_6$

each distance. For example, the protected vertices in the solution in Figure 1 has a distance sequence — 1, 2, 3, 4, 5, 6. The strategy is rather aggressive and always protects a neighbouring vertex from those vertices to where the fires just spread. On the other hand, firefighters may want to build a fire wall away from the fire as in the case of fighting with a forest fire. This is exactly what occurred in $P_l \times P_m \times P_n$. For example, the protected vertices in the optimal solution in Figure 2 has a distance sequence — 2, 4, 4, 5, 6, 6, 7, 8, 9.

The above strategy of building a fire wall leads to an lower bound of $MVS(P_n \times P_n \times P_p, (1,1,1))$. Let $k = n^2 - 2(n-1) + 1$. We describe the strategy of protecting the vertices needed to obtain the following lower bound, by determining which vertices per level need to be protected. The following vertices form the fire wall.

Level $n : (n,n,k), (n,n-1,k-1), \cdots, (n, n-(n-1), k-(n-1))$;
Level $n-1 : (n-1,n,k), (n-1,n-1,k-1), \cdots, (n-1, n-(n-1), k-(n-1))$;
Level $n-2 : (n-2,n,k-1), (n-2,n-1,k-2), \cdots, (n-2, n-(n-1), k-(n-1))$;
...
Level $1 : (1,n,k-(n-2)), (1,n-1,k-(n-1)), \cdots, (1, n-(n-1), (k-(n-2)-(n-1)))$.

Note that the lower bound in Theorem 2.2 is sharp because of Theorem 2.1.

Theorem 2.2. *$MVS(P_n \times P_n \times P_p, (1,1,1)) \geq n^2(n+1)/2 + n(n-1)(n-2)/2$, if $p \geq n^2 - 2(n-1) + 1$ and $n \geq 3$.*

Proof. The number of vertices that can be saved in a $P_n \times P_n \times P_p$ graph with $p = n^2 - 2(n-1) + 1$ is equal to the following. We will consider the results in terms of levels. In each of the levels n and level $n-1$, $1+2+3+\cdots+n$ vertices are saved. In each of the levels $n-2$ down to 1, $1+2+3+\ldots+n$ vertices plus an additional number of vertices are saved. This additional number of vertices is equal to $n+2n+3n+\cdots+(n-2)n = n(n-1)(n-2)/2$. Thus, the total number of vertices saved is equal to $n^2(n+1)/2 + n(n-1)(n-2)/2$. Therefore, if $p \geq n^2 - 2(n-1) + 1$ then the total number of vertices saved is greater than or equal to $n^2(n+1)/2 + n(n-1)(n-2)/2$. □

3 Asymptotic Results

As the number of vertices in the product graphs becomes very large, the number of vertices that can be saved is affected. Let us examine the upper and lower bounds of the ratio of the number of vertices

Let $R(G, v) = \frac{\text{the number of vertices can be saved}}{\text{the total number of vertices in G}}$.

Theorem 3.1.

$$R(P_n, v) = \begin{cases} 1 - \frac{1}{n} & \text{if } v \text{ is one the end vertices of } P_n; \\ 1 - \frac{2}{n} & \text{otherwise.} \end{cases} \quad (1)$$

Proof. Note that the total number of vertices in P_n equals n and one can save either $n-1$ or $n-2$ vertices depends whether the fire breaks out at the ends or not. □

Note that we are mainly interested in asymptotic values of $R(G)$. This result implies that one can save almost all vertices of P_n for large n.

Theorem 3.2. *$\frac{1}{4} + \epsilon \leq R(P_n \times P_n, v) \leq 1$, where $\epsilon = \frac{1}{2n}$ if n is even and $\epsilon = \frac{-1}{4n^2}$ if n is odd.*

Proof. If the fire breaks out at (1,1), then the solution obtained from Algorithm 2.1 will protect every vertex on the second column and will save $n^2 - n$ vertices. We shall prove in Theorem 3.3 that this is indeed the optimal solution. If the fire breaks at $(\lceil n/2 \rceil, \lceil n/2 \rceil)$ then the solution obtained from Algorithm 2.1 will give the lower bound of $R(P_n \times P_n)$. □

Before we proceed, several notations need to be clarified. We use distance in the common sense, that is the length of the shortest path between a pair of vertices. We call a path a fire-path if it starts at the vertex where the fire breaks out and ends at a protected vertex or burned vertex, and all intermediate vertices on the path are burned vertices. Furthermore, we define a fire-distance $d_f(u)$ where u is a protected vertex or burned vertex to be the length of the shortest fire-path from the fire to u. Note that $d_f(u)$ changes according to different vertices being protected. For example, $d_f((1,4)) = 3$ if $(1,4)$ is the only protected vertex in $P_7 \times P_7$, but $d_f((1,4)) = 5$ if both $(1,2)$ and $(1,4)$ are protected. Clearly, $d_f(u) \geq d(v,u)$ where the fire breaks out at v. Furthermore, $d_f(u) = d(v,u) + 2k$ where $k \geq 1$ in $P_n \times P_n$. We call a protected vertex u an interior-boundary vertex if $d_f(u) = d(v,u)$ where the fire breaks out at v. In the above example, $(1,2)$ is an interior-boundary vertex when both $(1,2)$ and $(1,4)$ are protected. Now, we are ready to prove the upper bound in Theorem 3.2 is sharp.

Observation 3.1. *The number of interior-boundary vertices is less or equal to maximum distance from the fire to all interior-boundary vertices.*

Proof. Since one can protect one vertex at each time interval, the number of interior-boundary vertices is less or equal to maximum distance from the fire to all interior-boundary vertices. □

Theorem 3.3. $MVS(P_n \times P_n, (1,1)) = n^2 - n$.

Proof. We shall prove an equivalent statement that there is at least one vertex of distance t from $(1,1)$ catching the fire at time t for $1 \leq t \leq n$ by induction. For $t = 1$, the fire will spread to at least one of the two vertices $(1,2)$ and $(2,1)$. Assume that there is at least one vertex say (x,y) where $x + y - 2 = t - 1$ at distance $t - 1$ from the fire caught the fire at time $t - 1$. Let P be the shortest fire-path from $(1,1)$ to (x,y). At rime t, if there is one vertex catch the fire then we done otherwise $(x+1,y)$ and $(x,y+1)$ have to be protected, and one vertex on each row has to be protected from P to each vertex (x',y') at distance t with $x' > x+1$, and one vertex on each column from P to each vertex (x',y') at distance t with $y' > y+1$ has to be protected. Since $d_f((x,y)) \geq d((1,1),(x,y))$, over all there is at

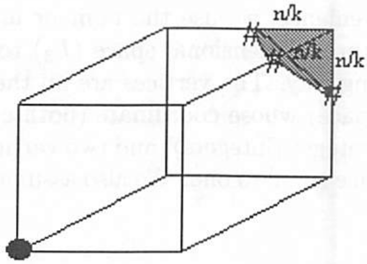

Figure 3: $P_n \times P_n \times P_n$

least $x + y - 2 + 2 = t + 1$ vertices that have been protected in time t, a contradiction with the fact that at most t vertices can be protected in time t. Therefore, there is at least one vertex catching the fire at each time interval t for $1 \leq t \leq n$. This implies that the fire started at $(1, 1)$ spreads to at least n vertices before it ends. □

Conjecture 3.1.

$$\lim_{n \to \infty} R(P_n \times P_n \times P_n, v) = 0.$$

Note that one would have the best chance to save a portion of n^3 vertices if the fire breaks out at a corner vertex, $(0, 0, 0)$ and suppose all the protected vertices are interior-boundary vertices, and the furthest away corner of this cube (see Figure 3) can be saved. Suppose a fraction (n/k, where k is a positive integer) of the three edges of the cube in the corner can be saved. It follows that $[n/k + (n/k - 1) + (n/k - 2) +, \cdots, +2 + 1] = 1/2(n^2/k^2 - n/k)$ vertices have to be protected in order to save this corner. On the other hand, the distance from the fire to the protected vertices is less than $3n$, which, in turn, is less that $1/2(n^2/k^2 - n/k)$ for n large with respect to k, a contradiction with Observation 3.1.

4 Containing a Fire

In this section, we examine whether we can contain the fire in infinite graphs. That is, we must build a fire barrier that isolates the fire from all directions. Since we consider the problem of containing the fire as quick as possible, the protected vertices must be adjacent to at least a burned vertex. We call those protected vertices boundary vertices if they are adjacent to both burned vertices and saved vertices that aren't protected. Clearly, one of the vertices protected in the last time interval must be a boundary vertex. For the convenience, we use the number line (L_1), the Cartesian Plane (L_2) and the three-dimensional space (L_3) to represent the infinite graphs in the following way. The vertices are all the points in the line (in the plane or in the space) whose coordinate (both coordinates or all three coordinates) is (are) integer (integers), and two vertices are adjacent if they have geometric distance equal to one. We also assume the fire always breaks out at origin.

Theorem 4.1. *It is possible to contain the fire with one firefighter in L_1.*

Proof. It is obvious one can protect two vertices at 1 and -2 to contain the fire. □

Theorem 4.2. *It is impossible to contain the fire with one firefighter in L2.*

Proof. Assume that one can contain the fire in L_2. There exists a fire barrier that isolates the burning vertices. Let B be the set of vertices on interior-boundary and let the vertex $(\bar{x}, \bar{y})$ be an interior-boundary vertex with the maximum distance from the fire, that is $d((0,0),(\bar{x},\bar{y})) =$ max $\{\ d((0,0),(x,y))|$ where (x,y) is any interior-boundary vertex $\}$. We may assume that both $\bar{x}$ and $\bar{y}$ are positive integers and $\bar{x} \geq \bar{y}$. Otherwise, we can rotate the x-axis and y-axis to place the vertex in the first quadrant. It follows that $d((0,0),(\bar{x},\bar{y})) = \bar{x} + \bar{y}$. Since $(\bar{x},\bar{y})$ is the furthest away interior-boundary vertex, one must protect two vertices on each column $x = c$ where $0 \leq c \leq \bar{x}$. That is, there are at least $2\bar{x}$ vertices that have to be protected. At least one vertex on left of the origin has to be protected. Therefore, $|B| \geq 2\bar{x} + 1 > \bar{x} + \bar{y}$ since $\bar{x} \geq \bar{y}$. By Observation 3.1., $d((0,0),(\bar{x},\bar{y})) = \bar{x} + \bar{y} \leq |B|$, a contradiction. Therefore, it is not possible to isolate the fire in L_2. □

In practice, there are more than one firefighter in any fire department. The natural question is how many firefighters does it need to isolate the fire in $L2$. That is, two or more vertices in the graph can be protected simultaneously in each time interval. The solution below is similar to the one obtained by B. Hartnell, S. Finbow and K. Scmeisser However, the fire can be contained within eight time intervals instead eleven time interval in our solution. Figure 4 shows how to build the fire barrier by two firefighters. We are able to show that this is the best solution in terms of the number of time intervals.

Lemma 4.1. *If (x,y) is the vertex that has been protected during the last time interval in the best solution, then $y \geq 2$.*

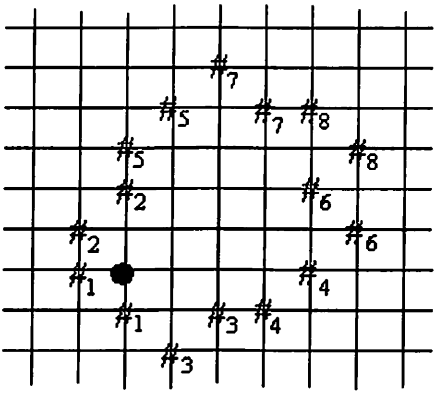

Figure 4: Containing a fire in L_2 with minimum number of time intervals

Proof. Clearly, we can protect at most $2d_f((x,y))$ vertices. If $y = 0$, then one needs to protect at least two vertices (one above the x-axis and one below the x-axis) in same column with $x < d_f((x,y))$. This implies $2d_f((x,y)) + 1$ interior-boundary vertices have to be protected, a contradiction. If $y = 1$, then one has to protect the vertices $(-1,0),(0,1)$ and $(0,-1)$ in order to use less than $2d_f((x,y))+1$ interior-boundary vertices to contain the fire. This is impossible because all three vertices have distance 1 from the fire and one can protect at most two vertices at distance 1. □

Theorem 4.3. *One can not contain the fire with two firefighters in less than* 8 *time intervals in* L_2.

Proof. We shall prove this in two steps. Let t be the minimum number of time intervals needed to contain the fire by two firefighters.

Case 1. $1 \le t \le 4$.

Note that if the fire is contained in t time intervals then all the vertices at distance t must be saved. There are $4t$ vertices at distance t. It is easy to see that two vertices at distance i can save at most $t + 2 - i$ vertices at distance t. It follows that the total number of vertices that can be saved at distance t is $(t+1)+t+(t-1)+(t-2) < 4t$ for $t = 4$. Therefore, it is impossible to contain the fire in four time intervals.

Case 2. $5 \le t \le 7$.

The same argument is used for $t = 5, 6$ and 7. For simplicity, we shall illustrate the case when $t = 7$. Since $t = 7$, the vertices protected in the last time interval, say (x, y), should have $d_f((x, y)) = 7$. This plus the facts that $y \leq x$ and $y \geq 2$ leads to at least one vertex from $\{(2,2), (3,2), (4,2), (5,2), (3,3), (4,3)\}$ has to be protected in the last time interval. It is impossible to protect any vertex from $\{(2,2), (3,3), (4,2)\}$ in time interval seven since the fire-distance has to be the distance plus $2k$ for $k \geq 1$. If the vertex $(3,2)$ is protected in time interval 7, then the shortest fire-path P has to pass either $(3,3)$ or $(4,2)$, say $(4,2)$. Thus, $(3,1)$ has to be protected to force P pass $(4,2)$ and back to $(3,2)$. This implies that P is on x-axis, turns at $(4,0)$ and passes through $(4,1)$ and $(4,2)$. One needs to protect at least two vertices on the left of y-axis, two vertices on each $x = 0, x = 1, x = 2$ and $x = 4$, three vertices on $x = 3$, and three vertices on the right of P, a contradiction with the fact that at most fourteen vertices can be protected in seven time intervals. It follows that either $(5,2)$ or $(4,3)$ is one of the vertices that has been protected in the last time interval.

(i) $(5,2)$ is one of the vertices that has been protected in the last time interval.

Suppose three vertices with $x < 0$ are protected. It follows that exactly two vertices (one in the upper boundary and one in the lower boundary) will be protected on each column from $x = 0$ to $x = 4$. Among the three vertices with $x < 0$, one vertex with $x < 0$ and $y = 0$ has to be protected and it is within distance 3 from the origin. The other two protected vertices must be on the column $x = -1$. Otherwise, the fire will penetrate the containing boundary on column $x = -1$. Moreover, these two vertices must have $|y| \leq 2$. Hence, they must be within distance 3 from the origin. Since these two vertices have $|y| \leq 2$, the two vertices on the column $x = 0$ must have $|y| \leq 3$. Again, they are within distance 3 from the origin. To contain the fire and connect to the vertex $(5,2)$, the first three vertices in the lower boundary can be no further than distance 3 from the origin. These vertices are $(1,-2), (2,-1)$ and $(0,3)$. So far, there are eight vertices protected within distance 3 from the origin. But one can protect at most six vertices within distance 3, a contradiction. Thus, there must be two protected vertices with $x < 0$. Without loss of generality, we may assume they are $(0,-1), (-1,0), (-1,1)$ and $(0,2)$. The two vertices on the column $x = 1$ can be no further away than $(1,3)$ and $(1,-2)$. The vertex in the lower boundary on the column $x = 2$ must have $y \geq -2$. Otherwise, one cannot contain the fire from spreading with three vertices from the lower boundary between $-2 \leq x \leq 1$(four rows to cover). It follows that the next two vertices in the lower boundary must be within distance 4. Again, nine vertices have to be protected within distance 4, a contradiction.

(ii) $(4,3)$ is one of the vertices that has been protected in the last time interval.

Since nine vertices with $x \geq 0$, have to be protected, $2 \leq |$ the number of vertices with $x < 0$ that has to be protected $| \leq 5$. We shall proceed with four different cases.

(a) Two vertices with $x < 0$ are protected.

We may assume that $(-1,0)$ and $(-1,1)$ are the two protected vertices with $x < 0$. Then $(0,-1)$ and $(0,2)$ must be the two vertices protected in the $x = 0$ column. It follows that one has to protect at least one vertex in the upper boundary on the columns $x = 1, 2$, and 3. This implies that there are at most six vertices available to be used in the lower boundary between $(0,-1)$ and $(4,3)$. The protected vertex in the lower boundary on the column $x = 1$ must have $y \geq -2$. It cannot have $y \leq -1$ due to the fact that there are five vertices within distance 2. Hence, $(1,-2)$ must be protected. Consequently, either $(2,-1), (3,-1)$ and $(4,0)$ or $(2,-2), (3,-1)$ and $(4,0)$ are protected and exactly 6 vertices are needed in the lower boundary between $(0,-1)$ and $(4,3)$. Clearly, the vertex that has to be protected in the upper boundary on the column $x = 1$ is $(1,3)$. There are nine vertices protected within distance 4, a contradiction.

(b) Three vertices with $x < 0$ are protected.

This similar to the previous case, where there are at most five vertices available to be used in the lower boundary between the y-axis and $(4,3)$. This implies that two vertices on the columns $x = 1$ and 2 must have $0 \geq y \geq -2$ and $1 \geq y \geq -1$ respectively. These vertices are also within distance 3 from the origin and the five vertices with $x \leq 0$ are within distance 3, a contradiction.

(c) Four vertices with $x < 0$ are protected.

There are at most four vertices available to be used in the lower boundary between the y-axis and $(4,3)$. This implies that the three vertices on the columns $x = 1, 2$ and 3 must have $0 \geq y \geq -1, 1 \geq y \geq 0$ and $2 \geq y \geq 1$ respectively. If the vertex in the lower boundary of column $x = 1$ has $y = 0$, then $(0,-1), (-1,-1), (-2,0), (-2,1), (-1,2)$ and $(0,4)$ must be the six protected vertices on the left of $(1,0)$, and $(2,0)$ and $(3,1)$ must be the next two protected vertices on the right of $(1,0)$. Clearly, these nine protected vertices are all within distance 4. This is impossible. Hence, the

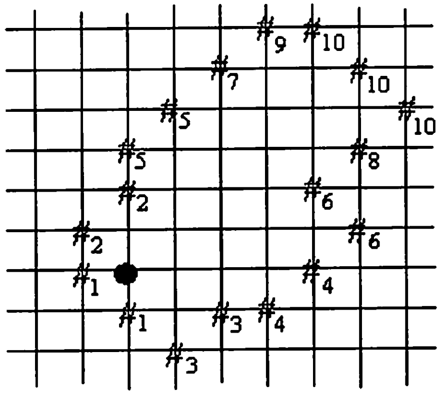

Figure 5: Containing a fire in L_2

vertex in the lower boundary of column $x = 1$ must be $(1, -1)$. It follows that $(2, 0)$ and $(3, 1)$ must be the next two protected vertices on its right. These three vertices in addition to the six vertices with $x \leq 0$ are within distance 4, a contradiction.

(d) Five vertices with $x < 0$ are protected.

Similar to the previous case, $(1, 0), (2, 1), (3, 2)$ are the three protected vertices between y-axis and $(4, 3)$. The vertex in the upper boundary of column $x = 3$ must be $(3, 4)$ and the vertex in the upper boundary of column $x = 2$ must be $(2, 4)$ because in the case of $(2, 5)$ we can change the x-coordinate with the y-coordinate and use Case (i). It follows that the vertex in the upper boundary of column $x = 2$ must be $(2, 5)$. Otherwise, there are eleven vertices within distance 5. Since $(1, 0)$ is the vertex in the lower boundary of column $x = 1, (0, -1)$ must be the vertex in the lower boundary of column $x = 0$. It follows that $(-1, 0)$ cannot be protected. This implies that the vertex in the upper boundary of column $x = -1$ must have $y \leq 3$ and the vertex in the upper boundary of column $x = 0$ must be $y = 4$. Hence, all fourteen protected vertices except $(3, 3), (4, 3), (3, 4), (2, 4)$ and $(1, 5)$ are within distance 4, a contradiction. This completes the proof. □

This is not a unique way to build a fire wall. In fact, it is not necessary to have two firefighters working full time. This may prolong the process of containing the fire and will eventually lead to more damage. In Figure 5, only one fire firefighter is used at time intervals 7 and 8. This makes the problem of determining the number of firefighters needed to contain the fire in the three-dimensional space a much harder problem. It implies that two firefighters could start to contain the fire at a certain $z - level$, for example $z = 0$, and then one (or more) of the firefighters could move to another $z - level$ and come back sometime later. The following theorem gives an upper bound of the number of firefighters needed to contain a fire in a n-regular graph.

Theorem 4.4. *One can contain a fire in two time intervals by $n - 1$ firefighters in a graph with regular degree n.*

Proof. One can first protect $n-1$ neighbours of the fire and then the fire will spread to the unprotected neighbor vertex. On can protect $n-1$ neighbours of the unprotected vertex in the second time interval. □

Corollary 4.1. *The number of fire fighters to contain a fire in L_3 is less or equal to five.*

As we know, we cannot contain the fire in a L_2 with one fire fighter. Therefore, two firefighters are needed to contain the fire in L_3. We believe that five firefighters are needed to contain the fire in L_3.

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