On Covering of Pairs by Quintuples

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Let V be a finite set of order v. A (v, k, λ) covering design of index λ and block size k is a collection of k-element subsets, called blocks, such that every 2-subset of V occurs in at least λ blocks. The covering problem is to determine the minimum number of blocks, $\alpha(v, k, \lambda)$, in a covering design. It is well known that $\alpha(v, k, \lambda) \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdot \lambda \right\rceil \right\rceil = \phi(v, k, \lambda)$, where $\lceil x \rceil$ is the smallest integer satisfying $x \leq \lceil x \rceil$. In this paper we determine the value $\alpha(v, 5, \lambda)$, with few possible exceptions, for $\lambda = 3$, $v \equiv 2 \pmod 4$ and $\lambda = 9, 10$, $v \geq 5$ and $\lambda \geq 11, v \equiv 2 \pmod 4$.

1. Introduction

A (v, k, λ) *covering design* (or respectively *packing design*) of order v, block size k and index λ is a collection β of k-element subsets, called blocks, of a v-set V such that every 2-subset of V occurs in at least (at most) λ blocks.

Let $\alpha(v, k, \lambda)$ denote the minimum number of blocks in a (v, k, λ) covering design; and $\sigma(v, k, \lambda)$ denote the maximum number of blocks in a (v, k, λ) packing design. A (v, k, λ) covering design with

$|\beta| = \alpha(v, k, \lambda)$ is called a *minimum covering design.* Similarly, a (v, k, λ) packing design with $|\beta| = \sigma(v, k, \lambda)$ will be called a *maximum packing design.* It is well known that [43]

$$\alpha(v, k, \lambda) \geq \left\lceil \frac{v}{k}\left\lceil \frac{v-1}{k-1}\lambda \right\rceil \right\rceil = \phi(v, k, \lambda) \text{ and}$$

$$\sigma(v, k, \lambda) \leq \left[\frac{v}{k}\left[\frac{v-1}{k-1}\lambda \right] \right] = \psi(v, k, \lambda)$$

where $\lceil x \rceil$ is the smallest and $[x]$ is the largest integer satisfying $[x] \leq x \leq \lceil x \rceil$.

Hanani has sharpened this bound in some cases by proving the following result.

Theorem 1.1

(i) If $\lambda(v-1) \equiv 0 \pmod{k-1}$ and $\lambda v(v-1)/(k-1) \equiv -1 \pmod{k}$ then $\alpha(v, k, \lambda) \geq \phi(v, k, \lambda) + 1$.

(ii)If $\lambda(v-1) \equiv 0 \pmod{k-1}$ and $\lambda v(v-1)/(k-1) \equiv 1 \pmod{k}$ then $\sigma(v, k, \lambda) \leq \psi(v, k, \lambda) - 1$.

When $\alpha(v, k, \lambda) = \phi(v, k, \lambda)$ the (v, k, λ) covering design is called a *minimal covering design.* Similarly, when $\sigma(v, k, \lambda) = \psi(v, k, \lambda)$ the (v, k, λ) packing design is called an *optimal packing design.*

Many researchers have been involved in determining the covering numbers known to date (see bibliography) most notably W.H. Mills and R.C. Mullin. The following two Theorems summarize what is known about covering with $k = 5$.

Theorem 1.2 [37] Let v be an odd integer greater than 5.

(i) If $v \equiv 1 \pmod 4$ and $\lambda > 1$, then $\alpha(v,5,\lambda) = \phi(v,5,\lambda) + e$ where $e = 1$ if $\lambda(v-1) \equiv 0 \pmod 4$ and $\frac{\lambda v(v-1)}{4} \equiv -1 \pmod 5$ and $e = 0$ otherwise with the exceptions that $\alpha(9,5,2) = \phi(9,5,2)+1$, $\alpha(13,5,2) = \phi(13,5,2)+1$ and the possible exceptions of the pairs $(v, \lambda) \in \{(53,2), (73,2)\}$ and,

(ii) If $v \equiv 3 \pmod 4$ and $\lambda \geq 1$ then $\alpha(v,5,\lambda) = \phi(v,5,\lambda) + e$ where e is as in (i) with the exceptions that $\alpha(15,5,\lambda) = \phi(15,5,\lambda)+1$ for $\lambda = 1, 2$ and the possible exception of the pairs $(v, \lambda) \in \{(63,2), (83,2)\}$.

For the next theorem let S = {389,469,789,869,1189,1269,1589,1609,1669, 1729,1789,1849, 1909,1929,1969,1989,2009,2049,2069,2089,2109,2129, 2149,2169,2189,2209,2229,2269, 2289,2309}.

<u>Theorem 1.3</u> Let $v \geq 5$ be an integer and e as in Theorem 1.2. Then:

1) Let $v \equiv 17 \pmod{20}$ be a positive integer. Then $\alpha(17, 5, 1) = \phi(17, 5, 1) + 2$ [34] , and $\alpha(v,5,1) = \phi(v,5,1)$ with the possible exceptions of $v \in \{37,57,77,137,157,177,237,257,277,337,357,377,437,457,637\}$, [3, 29, 39].

2) Let $v \equiv 9 \pmod{20}$ be a positive integer. Then $\alpha(v,5,1) = \phi(v,5,1)$ for all $v \geq 2349$ and $v \in S$ with the possible exception of $v = 3149$ [3, 29, 39].

3) Let $v \equiv 13 \pmod{20}$ be a positive integer. Then $\alpha(13, 5, 1) = \phi(13, 5, 1) + 2$, [34] , and $\alpha(v,5,1) = \phi(v,5,1)+1$ with the possible exceptions of $v \in \{13,53,73,153,273\}$, [3, 29, 39].

4) Let $v \equiv 2 \pmod 4$ be a positive integer. Then $\alpha(v,5,1) = \phi(v,5,1)$, [29, 39].

5) Let $v \geq 5$ be an even integer. Then $\alpha(v,5,2) = \phi(v,5,2)$ [40].

6) Let $v \geq 5$, $v \not\equiv 2 \pmod 4$ be an integer. Then $\alpha(v,5,3) = \phi(v,5,3)+e$ [18, 19, 37].

7) $\alpha(v,5,\lambda) = \phi(v,5,\lambda)+e$ for all integers $v \geq 5$ and $\lambda \equiv 0 \pmod 4$ [7, 12, 15, 16]

8) $\alpha(v,5,5) = \phi(v,5,5)$ for all even v, $v \geq 5$ with the possible exceptions of $v \in \{24,28,56,104,124,144,164,184\}$, [4].

9) $\alpha(v,5,6) = \phi(v,5,6)$ for all even integers v, $v \geq 5$ with the possible exception of $v = 18$ [5].

10) $\alpha(v,5,7) = \phi(v,5,7)$ for all even integers v, $v \geq 5$ with the possible exceptions of $v = 22,28,142,162$, [6].

11) $\alpha(v,5,\lambda) = \phi(v,5,\lambda)$ for all $v \equiv 0 \pmod 4$, $v \geq 5$, and $11 \leq \lambda \leq 21$ with the possible exceptions of $(v, \lambda) = (44,13)\ (28,17)\ (44,17)$ [8].

The previous two theorems do not cover the cases $v \equiv 2 \pmod 4$ and $\lambda = 3$; v even, and $\lambda = 9, 10$; and $v \equiv 2 \pmod 4$ and $\lambda \geq 11$. Our goal in this paper is to deal with these cases. We shall prove the following:

<u>Theorem 1.4</u> 1) $\alpha(v,5,3) = \phi(v,5,3)$ for $v \equiv 2 \pmod 4$, $v \geq 6$ with the possible exceptions of $v \in \{18,26,122,126,138,142,146,158,162,178,186,218,226,278\}$.

2) $\alpha(v,5,\lambda) = \phi(v,5,\lambda)$ for $\lambda = 9, 10$ and even $v \geq 5$ with the possible exceptions of $(v,\lambda) = (28,9)\ (56,9)$.

3) $\alpha(v,5,\lambda) = \phi(v,5,\lambda)+e$, where e as before, and for all $v \equiv 2 \pmod 4$, $v \geq 6$, and $11 \leq \lambda \leq 19$.

2. Recursive Constructions

In order to describe our recursive constructions we require the notions of *transversal designs, group divisible designs, and balanced incomplete block* *designs*. For the definition of these combinatorial designs we refer the reader to [4]. We shall adopt the following notations: a $T[k,\lambda,m]$ stands for a transversal design with block size k, index λ and group size m. A (k,λ)-GDD of type $1^a 2^b 3^c$.... denotes a group divisible design with block size k, index λ and a groups of order 1, b groups of order 2, etc. A $B[v, k, \lambda]$ stands for a balanced incomplete block design on v points, with block size k and index λ. It is clear that if a $B[v,k,\lambda]$ exists then $\alpha(v, k, \lambda) = \lambda v(v-1)/k(k-1) = \phi(v,k,\lambda)$. In the case k=5 Hanani [26] has proved the following:

Theorem 2.1 Necessary and sufficient conditions for the existence of a $B[v,5,\lambda]$ are :

$\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v(v-1) \equiv 0 \pmod{20}$ and $(v, \lambda) \neq (15, 2)$.

The following obvious lemma is most useful to us.

Lemma 2.1 If there exist a $B[v, 5, \lambda]$ and $\alpha(v, 5, \lambda') = \phi(v, 5, \lambda')$, then $\alpha(v, 5, \lambda+\lambda') = \phi(v, 5, \lambda+\lambda')$.

Corollary 2.1 Let $v \equiv 0$ or $6 \pmod{10}$ be a positive integer. Then $\alpha(v, 5, \lambda) = \phi(v, 5, \lambda)$ for $\lambda \geq 9$ with the possible exception of $(v, \lambda) = (56,9)$.

Proof If $v \equiv 0$ or $6 \pmod{10}$ then there exists a $B[v, 5, 4]$. On the other hand for such v, $\alpha(v, 5, \lambda) = \phi(v, 5, \lambda)$ holds for $\lambda = 5, 6, 7$ with the possible exception of $(v,\lambda) = (56,5)$. Furthermore, $\alpha(56,5,13) = \phi((56,5,13)$ [8]. Now invoke Lemma 2.1 to get the result.

Let v, h and k be positive integers, $v > h$. A (v, k, λ) minimal covering design (or respectively optimal packing design) with a hole of size h is a triple (V, H, β) where V is a v-set, H is a subset of V of cardinality h, and β is a collection of k-subsets of V, called blocks, such that:

1) no 2-subset of H appears in any block;

2) every 2-subset {x, y} of V where at least one of x, y does not lie in H, appears in at least (at most) λ blocks;

3) $|\beta| = \phi(v,k,\lambda) - \phi(h,k,\lambda)$, ($|\beta| = \psi(v,k,\lambda) - \psi(h,k,\lambda)$).

We shall make use of the following

Lemma 2.2 If there exists a (v,k,λ) minimal covering design with a hole of size $h \geq k$ and $\alpha(h,k,\lambda) = \phi(h,k,\lambda)$ then $\alpha(v,k,\lambda) = \phi(v,k,\lambda)$.

In many places throughout this paper, instead of constructing a $(v,5,\lambda)$ minimal covering design we construct a $(v,5,\lambda)$ minimal covering design with a hole of size $h \geq 5$ where $\alpha(h,5,\lambda) = \phi(h,5,\lambda)$ and then apply Lemma 2.2.

The proof of the following theorem may be found in [1, 26] and references therein.

Theorem 2.2 (1) There exists a T[6,1,m] for all positive integers m, $m \neq 2, 3, 4, 6$ with the possible exceptions of $m \in \{10,14,18,22\}$.

(2) There exists a $T[6,\lambda,m]$ for all positive integers $\lambda > 1$.

By deleting (m - u) points from a group of $T[6,\lambda,m]$ and from all blocks containing them we obtain a $(\{5,6\}, \lambda)$ - GDD of type m^5u^1. Furthermore, by inflating a $(\{5,6\}, \lambda)$ - GDD of type m^5u^1 by a factor of four and replacing the blocks of size 5 and 6 by the blocks of a (5,1) - GDD of type 4^5 and 4^6 respectively we obtain a $(5,\lambda)$ - GDD of type $(4m)^5(4u)^1$. Hence, we have the following.

Theorem 2.3 Let m, u and $\lambda > 1$, $m \geq u$ be positive integers such that m, u $\equiv 0 \pmod 4$. Then there exists a $(5,\lambda)$ - GDD of type m^5u^1.

In the case $\lambda = 3$ and $v \equiv 2 \pmod 4$ we shall make use of the following theorem.

Theorem 2.4 Let m, u $\equiv 0 \pmod 4$ $m \geq u \geq 0$ and $h \equiv 2 \pmod 4$, $h \geq 0$ be integers. Further, assume we have the following :

(1) a (5,3)-GDD of type m^5u^1. (2) a (u+h, 5,3) minimal covering design.

(3) a (m+h, 5, 3) minimal covering design with a hole of size h.

(4) $\{3(m+h)^2 - 2(m+h)\}$ and $\{3h^2 - 2h\}$ are the same congruency modulo 20.

Then $\alpha(5m + u + h,5,3) = \phi(5m + u + h,5,3)$.

Proof Take a (5,3)-GDD of type m^5u^1 and adjoin a set H of h points to the groups. On the first five groups we construct a (m+h,5,3) minimal covering design with a hole of size h and on the last group we construct a (u+h,5,3) minimal covering design.

To complete the proof of Theorem 2.4, we need to show that the total number of blocks obtained by this construction is equal to $\phi(5m + u + h,5,3)$. But a (5,3)-GDD of type m^5u^1 has the following number of blocks :

$$\frac{3}{2}(2m^2 + mu). \quad \text{(I)}$$

Since $u+h \equiv 2 \pmod 4$ it follows that a (u+h, 5, 3) covering design has the following number of blocks

$$\phi(u+h, 5, 3) = \left\lceil \frac{(u+h)}{5} \left\lceil \frac{3(u+h-1)}{4} \right\rceil \right\rceil = \left\lceil \frac{(u+h)}{5} \frac{(3(u+h)-2)}{4} \right\rceil =$$

$$\frac{3(u+h)^2 - 2(u+h) + c}{20} \quad \text{(II)}$$

where c is an integer uniquely determined by u and h.

A (m+h, 5, 3) minimal covering design with a hole of size h has the following number of blocks: ϕ (m+h, 5, 3) - ϕ (h, 5, 3) =

$$\left\lceil \frac{3(m+h)^2 - 2(m+h)}{20} \right\rceil - \left\lceil \frac{3h^2 - 2h}{20} \right\rceil$$

And since $\{3(m+h)^2 - 2(m+h)\}$ and $\{3h^2 - 2h\}$ are the same congruency modulo 20, the above number is equal to $\frac{3m^2+6mh-2m}{20}$ (III)

On the other hand,

$$\phi(5m+u+h) = \frac{3(5m+u+h)^2 - 2(5m+u+h) + c}{20} \qquad \text{(IV)}$$

Where c is the same integer as in (II) since 5m+u+h and u+h are the same congruency modulo 20.

Now it is easily checked that the total number of blocks in (I), (II) and 5 times the blocks in (III) is equal to the total number of blocks in (IV).

Theorem 2.5 Let m $\equiv$ 10 or 14 (mod 20). If there exists a (5,3) - GDD of type m^n and $\alpha(m,5,3)=\phi(m,5,3)$ then $\alpha(mn,5,3) = \phi(mn,5,3)$.

The following two constructions are modifications of Theorem 2.4 and Theorem 2.18 of [13] respectively.

Theorem 2.6 If there exists a (6,λ) - GDD of type 5^n and a (20 + h,5,λ) minimal covering design with a hole of size h then there exists a (20(n-1)+4u+h,5,λ) minimal covering design with a hole of size 4u+h where $0 \leq u \leq 5$.

Theorem 2.7 If there exists a (6,λ) - GDD of type 5^n, a (20+h,5,λ) minimal covering design with a hole of size h and a (20+h,5,λ) minimal covering design then there exists a $(20n+h,5,\lambda)$ minimal covering design.

The application of the above two theorems requires the existence of a $(6,\lambda)$ - GDD of type 5^n. Our authority for this is the following lemma of Hanani [26 p. 286]

<u>Lemma 2.3</u> (1) There exists a $(6,\lambda)$ - GDD of type 5^7 for $\lambda \geq 2$.
(2) There exists a (6,10) - GDD of type 5^9.

<u>Theorem 2.8</u> Assume there exists (1) a (5,1) - RGDD of type 5^m (2) a $(5,\lambda)$ - GDD of type t^6 where $t \equiv 0 \pmod 4$ (3) a $(5t+h,5,\lambda)$ minimal covering design with a hole of size *h*. Then there exists a $(5mt+ut+h,5,\lambda)$ minimal covering design with a hole of size tu+h where $0 \leq u \leq \frac{5(m-1)}{4}$.

<u>Proof</u> Take a (5,1) - RGDD of type 5^m and inflate it by a factor of t. To each of u parallel classes we adjoin t new points and on each block of the u parallel classes construct a $(5,\lambda)$ - GDD of type t^6. On the remaining parallel classes we construct a $(5,\lambda)$ - GDD of type t^5 for each block in the parallel classes. To the groups we adjoin a set H of *h* new points and on each group we construct a $(5t+h,5,\lambda)$ minimal covering design with a hole of size *h on H*. Then it is clear that the resultant design is a $(5mt+ut+h,5,\lambda)$ minimal covering design with a hole of size ut+h.

Another notion that is used in this paper is *modified group divisible design* (MGDD). We refer the reader to [4] for the definition. A $(5,\lambda)$ - MGDD of type n^m stands for a modified group divisible desgin with block size 5, index λ, groups size n and row size m. A resolvable modified group divisible design is one the blocks of which can be partitioned into parallel classes. It is clear that a (5,1) - RMGDD of type 5^m is the same as RT[5,1,m] with one parallel class of blocks singled out, and since RT[5,1,m] is equivalent to a T[6,1,m] we have the following existence theorem:

Theorem 2.9 There exists a (5,1) - RMGDD of type 5^m for all positive integers m, $m \neq 2, 3, 4, 6$ with the possible exception of $m \in \{10,14,18,22\}$.

The following theorem is our main recursive construction and it is a generalization of Theorem 2.3 of [3].

Theorem 2.10 Let $r \equiv 0 \pmod 5$ and $q,s,t \equiv 0 \pmod 4$ be positive integers. Further, assume the following designs exist :

(1) a (r,1)- RMGDD of type r^m which is equivelant to (r-1) MOLS of order m

(2) a $(5,\lambda)$ - GDD of type t^{r+1}, $t^r q^1$ and $t^m s^1$

(3) a $(rt+h,5,\lambda)$ minimal covering design with a hole of size h.

Then there exists a $(rmt+tu+wq+h+s,5,\lambda)$ minimal covering design with a hole of size tu+wq+h+s where u and w are nonegative integers such that $0 \leq u+w \leq m-1$.

Proof Take a (r,1)-RMGDD of type r^m and inflate this design by a factor of t. To each of u parallel classes adjoin t new points and on each block of the parallel classes construct a $(5,\lambda)$-GDD of type t^{r+1}. To each of w parallel classes adjoin q new points and on each block of the parallel classes construct a $(5,\lambda)$-GDD of type $t^r q^1$. On the remaining parallel classes we construct a $(5,\lambda)$-GDD of type t^r for each block in the parallel classes.

To the parallel class of size m we adjoin s new points and construct a $(5,\lambda)$-GDD of type $t^m s^1$. Finally, to the groups we adjoin a set H of h new points and on each group construct a $(rt+h,5,\lambda)$ minimal covering design with a hole of size h on H. It is clear that the total number of points we adjoined is ut+wq+s+h and that the resultant design is a $(rmt+tu+wq+h+s,5,\lambda)$ minimal covering design with a hole of size tu+wq+h+s.

To complete the proof of Theorem 2.10 we need to show that the number of blocks obtained by this construction is equal to the number of blocks of a (rmt+tu+wq+h+s,5,λ) minimal covering design wilh a hole of size tu+wq+h+s. Notice first that a (r,1)-RMGDD of type r^m has (m-1) parallel class each parallel class has m blocks. Now following the steps of this construction we observe that

1) The u parallel classes contribute $\frac{\lambda mu\, t^2(r^2+r)}{20}$ blocks (I)

2) The w parallel classes contribute $\frac{\lambda mw}{20}(r^2t^2 + 2rtq - rt^2)$ blocks (II)

3) The remaining parallel classes contribute

$\frac{\lambda m(m-u-w-1)\, t^2 r\,(r-1)}{20}$ blocks (III)

4) The parallel class of size m contributes $\frac{\lambda r(t^2m^2 + 2tms - t^2m)}{20}$ blocks (IV)

5) Since $t \equiv 0 \pmod 4$, a (rt+h, 5,λ) minimal covering design with a hole of size h has the following number of blocks: $\frac{\lambda r^2t^2 + 2\lambda rth - \lambda rt + crt}{20}$ (V)

where c is an integer determined by λ and h. On the other hand, a (rmt+tu+wq+h+s,5,λ) minimal covering design with a hole of size tu+wq+h+s has the following number of blocks:

ϕ (rmt+tu+wq+h+s,5,λ) - ϕ (tu+wq+h+s,5,λ);

And since $q,s,t \equiv 0 \pmod 4$, this expression can be simplified to:

$$\frac{\lambda r^2m^2t^2 + 2\lambda rmt(tu+wq+h+s) - \lambda rmt + crmt}{20} \qquad \text{(VI)}$$

Now it is easily checked that the number of blocks is (I), (II), (III), (IV) and m times the blocks of (V) is equal to the number of blocks in (VI).

Again the proof of the following theorem is very similar to the proof of Theorem 2.4 of [3].

Theorem 2.11 If there exists (1) a (5,1)-RMGDD of type 5^m (2) a $(5,\lambda)$-GDD of type $4^{m-1}8^1$ (3) a $(20, 5, \lambda)$ minimal covering design (4) a $(24,5,\lambda)$ minimal covering design with a hole of size 4. Then there exists a $(20m+4u+4,5,\lambda)$ minimal covering design with a hole the size 4u+4 where $0 \leq u \leq m-1$.

Proof Take a (5,1)-RMGDD of type 5^m and inflate this design by a factor of 4. To each of u parallel classes, $0\leq u\leq m-1$, adjoin four points and on each block of these parallel classes construct a $(5,\lambda)$-GDD of type 4^6. On the remaining blocks of the parallel classes construct a $(5, \lambda)$-GDD of type 4^5. For each block of the parallel class of block size m, after inflating by 4, adjoin four new points {a,b,c,d} to the last group and then construct a $(5, \lambda)$-GDD of type $4^{m-1}8^1$. Finally on the first (m-1) groups we construct a $(20, 5, \lambda)$ minimal covering design and on the last group we construct a $(24, 5, \lambda)$ minimal covering design with a hole of size 4 such that the hole is {a,b,c,d}. It is clear that this construction yields a $(20m+4u+4, 5, \lambda)$ minimal covering design with a hole of size 4u+4.

Theorem 2.12 [3] If there exist (1) a (5,1) - RMGDD of type 5^m (2) a $(5,\lambda)$ - GDD of type $4^{m-1}s^1$ (3) a $(20+h,5,\lambda)$ minimal covering design with a hole of size h. Then there exists a $(24(m-1)+h+s,5,\lambda)$ minimal covering design with a hole of size 4(m - 1)+h+s.

It is clear that the application of the above theorems requires the existence of a $(5,\lambda)$- GDD of type $4^m s^1$. We observe that we may choose s = 0 if m $\equiv$ 1(mod 5), s=4 if m$\equiv$ 0 or 4(mod 5) and $s = \frac{4(m-1)}{3}$ if m $\equiv$ 1 (mod 3). (See [3]). We may also apply the following:

Theorem 2.13 [25] There exists a (5,1) - GDD of type $4^m 8^1$ where m ≡ 0 or 2 (mod 5), m≥ 7 with the possible exception of m = 10.

Our last recursive construction is the following:

Theorem 2.14 If there exists (1) (5,1) - MGDD of type 5^m (2) a (5,λ) - GDD of type 4^m (3) a (20+h,5,λ) minimal covering design with a hole of size h (4) $\alpha(20+h,5,\lambda) = \phi(20+h,5,\lambda)$ Then $\alpha(20m+h,5,\lambda) = \phi(20m+h,5,\lambda)$.

Proof Take a (5,1) - RMGDD and inflate this design by a factor of 4. Replace all its blocks by the blocks of a (5,λ) - GDD of type 4^5. Add h points to the groups and on the first m - 1 groups construct a (20+h,5,λ) minimal covering design with a hole of size h and on the last group construct a (20+h,5,λ) minimal covering design . Finally, on the blocks of size m construct a (5,λ) - GDD of type 4^m.

3. The Structure of Packing and Covering Designs

Let (V, β) be a (v, k, λ) packing design, for each 2-subset e = {x, y} of V define m(e) to be the number of blocks in β which contain e. Note that by the definition of a packing design we have m(e) ≤ λ for all e.The complement of (V, β), denoted by C (V, β) is defined to be the graph with the vertex set V and edges e occurring with multiplicity λ-m(e) for all e. The number of edges (counting multiplicities in C (V, β)) is given by $\lambda\binom{v}{2} - |\beta|\binom{k}{2}$. The degree of each vertex is $\lambda(v-1) - r_x(k-1)$ where r_x is the number of blocks containing x.

In a similar way we define the excess graph of a (V, β) covering design denoted by E(V, β), to be the graph with vertex set V and edges e occurring with multiplicity m(e) - λ for all e. The number of edges in E(V, β) is given by

$|\beta|\binom{k}{2}-\lambda\binom{v}{2}$; and the degree of each vertex is $r_x(k-1)-\lambda(v-1)$ where r_x is as before.

The following lemmas are easy to prove:

Lemma 3.1 Let $v \equiv 2$ or 4 (mod 5) be a positive integer greater than 5. Then the complement graph of a (v,5,4) optimal packing design consists of v-2 isolated vertices and another two vertices joined by four edges.

Lemma 3.2 The complement graph of a (v,5,1) optimal packing design, $v \equiv 2$ (mod 20), and the excess graph of a (v,5,3) minimal covering design, $v \equiv 10$ or 14 (mod 20) is a 1-factor.

Lemma 3.3 The excess graph of a (v,5,4) minimal covering design for $v \equiv 2$ or 4 (mod 20) consists of v-3 isolated vertices and other three vertices the pairs of which are joined by two edges.

Lemma 3.4 Let $v \equiv 3$ (mod 10), $v \geq 23$ be positive integer. Then the complement graph of a (v,5,2) optimal packing designs consists of v-3 isolated vertices and 3 other vertices the pairs of which are connected by 2 edges.

Theorem 3.1 If there exists

1) A $(v,5,\lambda)$ covering design with $\phi(v,5,\lambda)$ blocks.
2) A $(v,5,\lambda')$ packing design with $\psi(v,5,\lambda')$ blocks.
3) $\phi(v,5,\lambda)+\psi(v,5,\lambda')=\phi(v,5,\lambda+\lambda')$.
4) The complement graph $C(V,\beta)$ of the packing design is isomorphic to a subgraph G of the excess graph $E(V,\beta)$ of the covering design. Then there exists a $(v,5,\lambda+\lambda')$ covering design with a $\phi(v,5,\lambda+\lambda')$ blocks, that is, a $(v,5,\lambda+\lambda')$ minimal covering design.

4. Notations and a Few More Designs

In this short section we discuss the notations used through this paper and construct a few more minimal covering designs for $2 \leq \lambda \leq 7$.

A block <k k+m k+n k+j f(k)> (mod v) where f(k) = a if k is even and f(k)=b if k is odd is denoted by <0 m n j> $\cup$ {a, b} (mod v). Similarly, a block <k k+m k+n k+j f(k)> where f(k) = a if k $\equiv$ 0 (mod 4); f(k) = b if k $\equiv$ 1 (mod 4); f(k) = c if k $\equiv$ 2 (mod 4), and f(k) = d if k $\equiv$ 3 (mod 4) is denoted by <0 m n j> $\cup$ {a, b, c, d} (mod v). Note that a, b, c, d are not necessarily distinct.

In a similar way, a block <(0, k) (0, k+m) (1, k+n) (1, k+j) f(k)> mod (-, v) where f(k) = a if k is even, f(k) = b if k is odd is denoted by <(0, 0) (0, m) (1, n) (1, j)> $\cup$ {a,b} mod (-, v).

We now improve the result of Theorem 1.3.

<u>Lemma 4.1</u>

(i) α(v,5,5) = ϕ(v,5,5) for v = 24,104,124,144,164,184.

(ii) α(18,5,6) = ϕ(18,5,6)

(iii) α(273,5,1) = ϕ(273,5,1) + 1.

(iv) α(28,5,7) = ϕ(28,5,7).

<u>Proof</u> For a (24,5,5) minimal covering design proceed as follow:

1) Take a (24,5,1) optimal packing design. This design is constructed by deleting all blocks through the point 25 from the blocks of a B[25,5,1]. Assume in this design we have the block <a b c 1 22>. In this block change 22 to 24.

2) Take a (26,5,1) minimal covering design with a hole of size two. This

design is constructed by taking a B[25,5,1] then partitioning the 25 points of B[25,5,1] into quadruples and adjoining a new point, say 26, to these quadruples. Since $25 \equiv 1 \pmod 4$ there will be a pair of points, say {25, 26} which we delete. Further, we may assume that the repeated pairs in this design are exactly those missing in design (1). We also can assume that we have the following two blocks <a b c 24 25> <d e f 24 26>

In the first block change 25 to 22 and in the second change 26 to 23. In all other blocks change 25 and 26 to 24.

3) Take the (23,5,2) optimal packing design in [14]. This design has a triple, say, {21, 22, 23} the pairs of which appear in zero blocks. Furthermore this design has the following four blocks: <1 2 3 21 4>

<5 6 7 22 8> <9 10 11 23 12> <d e f 5 23>.

In the first block change 4 to 22, in the second change 8 to 23, in the third change 12 to 21, and in the fourth change 23 to 24.

4) Take a B[25,5,1] and assume that the blocks through the point 25 are

<8 13 14 21 25><12 15 16 22 25><4 17 18 23 25>

<1 2 3 19 25 ><5 6 7 20 25><9 10 11 24 25>

Then in the first block change 25 to 22, in the second change 25 to 23, in the third change 25 to 21, in the fourth change 25 to 4, in the fifth change 25 to 8, and in the sixth change 25 to 12. Then it is easy to check that the above four steps yield the blocks of a (24, 5, 5) minimal covering design.

For v = 104,144,164,184 apply Theorem 2.14 with h = 4, $\lambda = 5$ and m=5,7,8,9 and see [11] for a (5,5)-GDD of type $4^5,4^7,4^8,4^9$ and [4] for a (24,5,5) minimal covering design with a hole of size 4.

For v=124 apply Theorem 2.10 with m=r=5, h=s=t=u=4, w=0 and $\lambda = 5$.

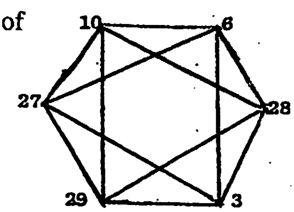
(ii) For an (18,5,6) minimal covering design proceed as follows:

1) Take an (18,5,4) packing design with $\psi(18,5,4)$ -1 blocks [17]. This design has a triple, say, {a, b, c} the pairs of which appear in zero blocks.

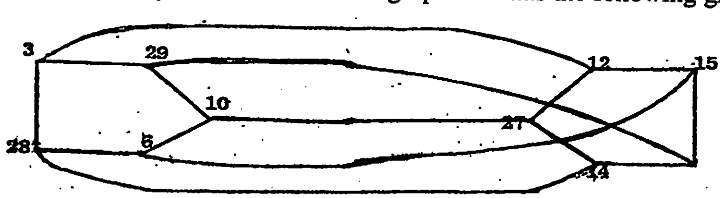
2) Take the (18,5,2) minimal covering design in [40]. It can be seen that there is a triple, say {a,b,c} the pairs of which appear in five blocks.
3) Adjoin the block <a b c> to the blocks obtained in (1) and (2). Then it is easy to check that the above three steps yield the blocks of an (18,5,6) minimal covering design.

(iii) For v = 273 apply Theorem 2.10 with r = 5, t = 4, m = 12, u = 6, h = 1, s= 8, w = 0 and $\lambda = 1$.

(iv) For a (28,5,7) minimal covering design proceed as follows:
1) Take a (27,5,4) minimal covering design. This design has a triple, say, {3,6,27} the pairs of which appear in six blocks. assume in this design we have the block <a b c 27 3 >. In this block we replace 3 by 28.
2) Take a (29,5,2) packing design with ψ (29, 5, 2)-1 blocks [14]. The complement graph of this design consists of the following 12 edges. Assume in this design we have the block < a b c 28 29 >. In this block replace 29 by 3 and in all other blocks replace 29 by 28.



3) Take the (30,5,1) minimal covering design in [29]. Close observation of this design shows that its excess graph consists of three mutually isomorphic graphs G_1, G_2 and G_3 so that the vertices of these graphs partition the point set of the design. Hence G_1 has ten vertices and 15 edges. We may permute the points of this design so that the excess graph contains the following graph.



In this design we need to replace 29 and 30 by 28 and to gain the edges of the complement graph of design (2). Notice that the edge { 28, 29 } can be dismissed since 29 is to be replace by 28. Further, the edges of the triple { 3, 6, 27 } are repeated in design (1), and the edges {27,28} and {3,28} are gained in step (1) and step (2) respectively. Also the pairs {6,10} {6,28} {3,28} {3,29} {10,29} and {10,27} are edges in the excess graph of design (3). Hence the previous three steps yield a design such that each pair appears in at least seven blocks except (10, 28) which appear in exactly six blocks. Assume in design (3) we have the following three blocks : < 1 2 3 28 30 > <4 5 6 29 30 > < 7 8 9 28 29 >, where { 1,2,...9 } are arbitrary numbers. Then in the first block replace 30 by 10, in the second replace 29 by 8 and 30 by 27 and in the third replace 29 by 12. Finally, assume in design (2) we have the block < 12 13 14 15 28 > and in design (1) we have the block < 4 5 6 14 27 >. In the first block replace 28 by 10 and in the second replace 27 by 28.

5. Covering with Index 3, v $\equiv$ 2 (mod 4)

The following construction combines other known designs to handle the case v$\equiv$ 2 (mod 20).

Lemma 5.1 (a) $\alpha(v,5,3) = \phi(v,5,3)$ for v = 22,42,62,82,102.

(b) There exists a (v,5,3) minimal covering design with a hole of size 2 for v = 42,62,82.

(c) There exists a (v,5,3) minimal covering design with a hole of size 6 for v =38,46,58,66,86,106.

Proof (a) For v = 22 the construction is as follows

1) Take a B[21,5,1] on {1, 2, . . ., 21} and assume we have the following three blocks <1 2 3 4 21> <5 6 7 8 21> <9 10 11 12 21>. In these three blocks change 4, 8 and 12 to 22.

2) Take the (v+1,5,2) optimal packing design in [14] on {1, 2, . . ., 23}. This design has a triple {21,22, 23}, say, the pairs of which appear in zero blocks while each other pair appears in two blocks. Close observation of this design shows that we may assume we have the following four blocks <1 2 3 14 23> <5 6 7 15 23>, <9 10 11 16 23>, <4 8 12 19 23> where {14, 15, 16, 19} are arbitrary numbers. In the first block change 23 to 4, in the second change 23 to 8, in the third change 23 to 12 and in the fourth change 23 to 21. In all other blocks we change 23 to 22.

For v = 42, 62, 82 the construction is as follows:

1) Take a B[v -1, 5, 1]. Assume in this design we have the blocks < a b c d v-1 > and < x y z m v-1 >. In the first block we replace d by v and in the second block we replace m by v.

2) Take a (v,5,1) optimal packing design which is equivalent to a (5,1) - GDD of type 2^u where $u = \frac{v}{2}$ [2]. Assume the groups are (2i+1, 2i+2) for $i \in Z_{((v-2)/2)}$.

3) Take a (v+1,5,1) minimal covering design with a hole of size three, say { v-1, v, v+1 }, such that the excess graph contains a 1 - factor on v - 2 vertices. We may assume that the 1- factor is (2i+1, 2i+2) for $i \in Z_{(v-4)/2}$. In this design we replace v+1 by v. Further, assume we have the blocks < a b c e v >, < x y z d v >. In the first block replace v by d and in the second replace v by m.

4) Finaly,adjoin the block < v-1 v m e d >.

To complete our construction we have to show that there exists a (v+1,5,1) minimal covering design with a hole of size three such that the excess graph contains a subgraph which is a 1 - factor on v - 2 vertices for v+1 = 43,63,83.

For v = 42 see [38], for v = 62,82 see [4].
For a (102,5,3) minimal covering design let $X = Z_2 \times Z_{40} \cup H_{22}$. On

$Z_2 \times Z_{40} \cup H_{13}$ construct a (93,5,1) minimal covering design with a hole of size 13 on H_{13} [25]. Further, take the following blocks:

<(i, 0) (i, 10) (i, 20) (i, 30) h_{14}> orbit length 10, i = 0, 1.

<(i, 0) (i, 1) (i, 3) (i, 18)> $\cup \{h_1, h_2\}$, i = 0, 1,

<(i, 0) (i, 5) (i, 11) (i, 26)> $\cup \{h_3, h_4\}$, i = 0, 1.

<(i, 0) (i, 7) (i, 17) (i, 26)> $\cup \{h_5, h_6\}$, i = 0, 1.

<(0,0) (0, 13) (1, 0) (1,39)> $\cup \{h_{15}, h_{16}\}$

<(0, 0) (0, 3) (1, 5) (1, 12)> $\cup \{h_{17}, h_{18}\}$

<(0, 0) (0, 9) (1, 10) (1, 23)> $\cup \{h_{19}, h_{20}\}$

<(0, 0) (0, 11) (1, 17) (1, 30)> $\cup \{h_{21}, h_{22}\}$

<(0, 0) (0, 4) (1, 7) (1, 38) h_7> <(0, 0) (0, 8) (1, 21) (1, 37) h_8>

<(0, 0) (0, 12) (1, 4) (1, 8) h_9> <(0, 0) (0, 16) (1, 11) (1, 31) h_{10}>

<(0, 0) (0, 5) (1, 25) (1, 33) h_{11}> <(0, 0) (0, 6) (1, 22) (1, 24) h_{12}>

<(0, 0) (0, 4) (1, 0) (1, 3) h_{13}> <(0, 0) (0, 8) (1, 1) (1, 5) h_{14}>

<(0, 0) (0, 12) (1, 2) (1, 31) h_{15}> <(0, 0) (0, 16) (0, 8) (1, 26) h_{16}>

<(0, 0) (0, 18) (1, 29) (1, 34) h_{17}> <(0, 0) (0, 1) (0, 18) (0, 24) h_{18}>

<(0, 0) (0, 7) (0, 20) (1, 28) h_{19}> <(0, 0) (0, 13) (1, 22) (1, 38) h_{20}>

<(0, 0) (0, 2) (1, 6) (1, 14) h_{21}> <(0, 0) (0, 20) (1, 7) (1, 35) h_{22}>

To complete the construction for the (102,5,3) minimal covering design we fill the hole, H_{22}, with a (22,5,3) minimal covering design.

(b) For a (v,5,3) covering design with a hole of size two, v=42, 62, 82,proceed as follows:

1) Take a B[v -1, 5, 1].

2) Take a (v,5,1) optimal packing design, which is equivalent to a (5,1) - GDD of type 2^u where $u = \frac{v}{2}$, and assume that {v-1, v} is a group, [2].

3) Take a (v+1,5,1) minimal covering design with a hole of size three, say { v-1 v v+1 }, such that the excess graph contains a 1 - factor on v - 2

vertices. We may assume that the 1- factor here covers all but the pair {v-1, v} of the complement graph of the design in (2). Replace v+1 by v, then it is easy to see that the above 3 steps yield the blocks of a (v,5,3) covering design with a hole of size two for v=42, 62, 82.

(c) See [6].

Lemma 5.2 $\alpha(v,5,3) = \phi(v,5,3)$ for $v \equiv 2 \pmod{20}$, $v \geq 22$ with the possible exceptions of v= 122,142,162.

Proof For v = 22,42,62,82,102 the result follows from Lemma 5.1.
For $v \equiv 2 \pmod{40}$ $v \geq 202$ take a (5,3)- GDD of type 40^n [11]. Adjoin two points to the groups and on all the groups except one we construct a (42,5,3) minimal covering design with a hole of size two and on that group construct a (42,5,3) minimal covering design.

For v = 262 apply Theorem 2.4 with m = 44 u = 32 and h = 10. To complete the construction of v = 262 we need to construct a (54, 5, 3) minimal covering design with a hole of size 10. For this purpose let $X = Z_2 \times Z_{22} \cup H_{10}$.
Then take the following blocks mod (_, 22)

$< (a, 0)\ (a, 1)\ (a, 3)\ (a, 5)\ (a, 13) >$ a = 0, 1

$< (0, 0)\ (0, 4)\ (0, 9)\ (0, 15)\ (1, 1) >$ $< (0, 0)\ (1, 2)\ (1, 3)\ (1, 6)\ (1, 10) >$

$< (0, 0)\ (0, 3)\ (0, 9)\ (1, 16) > \cup \{h_1, h_2\}$

$< (0, 0)\ (1, 0)\ (1, 5)\ (1, 11) > \cup \{h_1, h_2\}$

$< (0, 0)\ (0, 1)\ (1, 0)\ (1, 3) > \cup \{h_1, h_2\}$

$< (0, 0)\ (0, 3)\ (1, 7)\ (1, 20) > \cup \{h_3, h_4\}$

$< (0, 0)\ (0, 5)\ (1, 14)\ (1, 15) > \cup \{h_5, h_6\}$

$< (0, 0)\ (0, 7)\ (1, 13)\ (1, 18) > \cup \{h_7, h_8\}$

$< (0, 0)\ (0, 11)\ (1, 1)\ (1, 8) > \cup \{h_9, h_{10}\}$ $< (0, 0)\ (0, 1)\ (1, 0)\ (1, 7)\ h_3 >$

$< (0, 0)\ (0, 2)\ (1, 3)\ (1, 15)\ h_4 >$ $< (0, 0)\ (0, 4)\ (1, 16)\ (1, 18)\ h_5 >$
$< (0, 0)\ (0, 6)\ (1, 4)\ (1, 17)\ h_6 >$ $< (0, 0)\ (0, 7)\ (1, 9)\ (1, 17)\ h_7 >$
$< (0, 0)\ (0, 8)\ (1, 5)\ (1, 16)\ h_8 >$ $< (0, 0)\ (0, 8)\ (1, 4)\ (1, 20)\ h_9 >$
$< (0, 0)\ (0, 10)\ (1, 9)\ (1, 15)\ h_{10} >$

For v = 182, 222, 302, 342, apply Theorem 2.4 with (m,u,h) = (32,16,6), (40,16,6), (60,0,2), (60,40,2) respectively .

For v = 582 apply Theorem 2.8 with m=13, t=8, h=6 and u=7 and then apply Lemma 2.2 for the hole of size 62.

For all $v \equiv 22 \pmod{40}$, $v \geq 382$, $v \neq 582$ simple calculations show that v can be written as v = 40m+8u+h+s where m,u,h, and s are chosen so that

1) There exists a (5,1)-RMGDD of type 5^m;
2) $m \equiv 0,1$ or 4(mod 5), $m \neq 10$, 14; s=8 if $m \equiv 0$ or 4 (mod 5) and s=0 if $m \equiv 1$ (mod 5);
3) $0 \leq u \leq m-1$, r=5, h=2,6, t=8;
4) 8u+h+s = 22,42,62,82,102;
5) There exists a $(5,\lambda)$-GDD of type $8^m s^1$;

Now apply Theorem 2.10 with $\lambda=3$, and w=0 to get the result.

<u>Lemma 5.3</u> $\alpha(v,5,3) = \phi(v,5,3)$ for $v \equiv 10 \pmod{20}$, $v \geq 10$.

<u>Proof:</u> For v=10,30 see [6].
For $v \geq 50$ take a (5,3)-GDD of type $10^{(v/10)}$, $v \neq 230, 270, 390$ and then apply Theorem 2.5 [11].
For v = 230,390 apply Theorem 2.10 with r=5, h=6, $\lambda=3$, s=t=8, u=2, w=0, and m=5,9 respectively. It should be noted that when invoking Theorem 2.10 for v=230,390, we require a (5,3)-GDD of type 8^6 and 8^{10}, our authority is [11]
For v = 270 take a T[5,3,54] and then apply Theorem 2.5.

Lemma 5.4 $\alpha(v,5,3) = \phi(v,5,3)$ for $v \equiv 14 \pmod{20}$, $v \geq 14$.

Proof: For v= 14,34,54,74,94 see [6].

For v = 114, 134 see next table. In general, the construction in this table, and all other tables to come is as follows: Let $X = Z_{v-n} \cup H_n$ or $Z_2 \times Z_{\frac{v-n}{2}} \cup H_n$ where $H_n = \{h_1, h_2, \ldots h_n\}$ is the hole. The blocks are constructed by taking the orbit of the tabulated base blocks, mod (v–n) or $\text{mod}\ \frac{(v-n)}{2}$.

For v = 154 apply Theorem 2.5 and see [11] for a (5,3)-GDD of type 14^{11}.

For v = 254 apply Theorem 2.4 with m = 44, u = 24, and h = 10 and see Lemma 5.2 for a (54,5,3) minimal covering design with a hole of size 10.

For v= 174, 194, 274, 294, 314, 334, 354 apply Theorem 2.4 with h=6 and (m,u)= (32,8),(32,28),(52,8), (52,28), (60, 8), (60, 28), (60, 48) respectively and see [6] for a (38,5,3) and a (58,5,3) minimal covering design with a hole of size 6.

For v= 534,554,574,594 apply Theorem 2.8 with m=13, λ=3, t=8, h=2,6 and u=1, 4, 6, 9 respectively and see Lemma 5.1 for the appropriate (v,5,3) minimal covering design with a hole of size 2 and 6.

For all other values of v, simple calculations show that v can be written as v= 40m+8u+h+s where m, u, h, and s are choosen as in Lemma 5.2 with the difference that 8u+h+s=14,34,54,74,94.

Now apply Theorem 2.10 with λ=3, t=8, and w=0 to get the result.

V	Point Set	Base Blocks
114	$Z_{104} \cup H_{10}$	<0 7 52 59>$\cup \{h_9,h_{10}\}$ half orbit <0 1 3 7 12> <0 8 21 46 72><0 10 33 61 77><0 14 34 63 82> <0 15 39 57 74><0 6 24 46 54><0 1 3 9 25> <0 4 31 52 69><0 5 34 47 67><0 10 28 54 68> <0 11 41 60 72><0 15 29 50>$\cup \{h_i\}_{i=1}^{4}$ <0 11 34 49>$\cup \{h_i\}_{i=5}^{8}$ <0 1 3 26>$\cup \{h_1, h_2\}$ <0 4 9 37> $\cup \{h_3, h_4\}$ <0 7 20 39>$\cup \{h_5, h_6\}$ <0 10 27 63>$\cup \{h_7, h_8\}$ <0 12 43 59>$\cup \{h_9, h_{10}\}$.
134	$Z_{120} \cup H_{14}$	On $Z_{120} \cup H_9$ construct a (129,5,1) minimal covering design with a hole of size 9 say H_9. Further, take the following blocks <0 30 60 90 h_{10}> orbit length 30 <0 1 6 14 35> <0 7 40 56 84><0 9 26 57 67><0 18 38 68 98> <0 19 42 66 88><0 4 20 48 66><0 3 15 37 39> <0 1 3 8>$\cup \{h_1, h_2\}$ <0 4 13 23> $\cup \{h_3, h_4\}$ <0 11 49 80>$\cup \{h_5, h_6\}$ <0 11 53 68> $\cup \{h_7, h_8\}$ <0 12 37 87>$\cup \{h_9, h_{10}\}$ <0 14 41 73>$\cup \{h_{11}, h_{12}\}$ <0 17 56 77> $\cup \{h_{13}, h_{14}\}$ <0 6 35 61>$\cup \{h_i\}_{i=11}^{14}$

<u>Lemma 5.5</u> $\alpha(v,5,3) = \phi(v,5,3)$ for $v \equiv 6 \pmod{20}$, $v \geq 6$ with the possible exceptions of v=26,126,146,186,226.

<u>Proof:</u> For v=6,46,66,86 see [6].

For v≡ 6 (mod 40), v ≥ 206, take a (5,3)-GDD of type 40^n. Adjoin six points to the groups and on all but one group construct a (46,5,3) minimal covering design with a hole of size six and on that group, construct a (46,5,3) minimal covering design.

For v=106 let $X=Z_2 \times Z_{50} \cup H_6$. On $Z_2 \times Z_{50} \cup H_5$ construct a B[105,5,1] with a hole of size 5, say, $<h_1 h_2 h_3 h_4 h_5>$. Further take the following blocks mod (–, 50)

<(0, 0) (0, 25) (1, 0) (1, 25) h_6> half orbit
<(i, 0) (i, 2) (i, 10) (i, 16) (i, 29)> i = 0, 1 <(0, 0) (0, 1) (0, 4) (0, 24) (1, 2)>
<(0, 0) (1, 3) (1, 4) (1, 8) (1, 10)> <(0, 0) (0, 7) (0, 18) (0, 35) (1, 41)>
<(0, 0) (1, 5) (1, 15) (1, 27) (1, 42)> <(0, 0) (0, 1) (0, 25) (1, 19) (1, 37)>
<(0, 0) (0, 13) (1, 9) (1, 29) (1, 45)> <(0, 0) (0, 9) (0, 12) (1, 0) (1, 5)>
<(0, 0) (0, 14) (1, 7) (1, 31) (1, 40)> <(0, 0) (0, 5) (0, 20) (1, 17) (1, 26)>
<(0, 0) (0, 16) (1, 30) (1, 38) (1, 49)> <(0, 0) (1, 20) (1, 31) (1, 32) (1, 35)>
<(0, 0) (0, 4) (0, 22) (1, 23) (1, 40)> <(0, 0) (0, 2) (0, 19) (1, 13) (1, 39)>
<(0, 0) (0, 9) (0, 21) (1, 4) (1, 24)> <(0, 0) (0, 5) (1, 13) (1, 16) h_1>
<(0, 0) (0, 6) (1, 30) (1, 48) h_2> <(0, 0) (0, 7) (1, 9) (1, 34) h_3>
<(0, 0) (0, 8) (1, 22) (1, 29) h_4> <(0, 0) (0, 10) (1, 7) (1, 35) h_5>
<(0, 0) (0, 11) (1, 10) (1, 39) h_6>

For v = 166 apply Theorem 2.4 with m = 32, h = 6 and u = 0.
For v = 266,306,346,386,466 apply Theorem 2.4 with (m,u,h) = (52,0,6), (60,0,6),
(60,40,6), (64,56,10), (80,60,6) respectively and see [6] for a (74,5,3) minimal covering design with a hole of size 10.
For v = 586, 626 and 786 apply Theorem 2.8 with t = 8, $\lambda = 3$ and (m,u,h) = (13,8,2), (13, 13,2), (17,13,2) respectively.

For v = 986 apply Theorem 2.10 with m = 11, t = 16, h = 2, r = 5, s = 0, u = 6, w = 1, and q =8 and see [11] for a (5,3)-GDD of type 16^{11}.

For v = 1186,1386,1586,1786,1986 apply Theorem 2.10 with r = 10, t = 4, w = 0, h=6 and (m, u,s) = (27,23,8), (32,23,8), (37,23,8), (43,1,56), (47,23,8) respectively.

For all other values of v$\equiv$ 186 (mod 200), v$\geq$ 2186 write v = 40m + 346 then apply Theorem 2.10 with r = 5, t = 8, h = 2, u = 43, s = q = 0 and m$\equiv$ 1 (mod 5).

For all other values of v, v$\not\equiv$ 186 (mod 200), v$\neq$ 146,166,186,226 write v = 40m + 8u + h + s, then the proof is the same as Lemma 5.2 with the difference that 8u +h + s = 6,46,66,86,106.

<u>Lemma 5.6</u> α (v,5,3) = ϕ (v,5,3) for all v$\equiv$ 18 (mod 20) with the possible exceptions of v = 18,138,158,178,218,278.

<u>Proof</u> For v = 38,58,78,98 see [6].

For v = 118 let X = $Z_{112} \cup H_6$. Then the blocks are:

<0 43 56 99> $\cup \{h_5, h_6\}$ half orbit.

<0 1 3 7 28> <0 5 19 64 81> <0 8 32 73 83> <0 11 34 49 69>

<0 12 30 56 72> <0 1 3 9 19> <0 5 25 57 83> <0 11 35 72 84>

<0 13 36 66 81> <0 14 41 62 79> <0 4 22 42 66> <0 1 3 7 12>

<0 7 15 55 85> <0 14 31 67 83> <0 9 22 55> $\cup \{h_i\}_{i=1}^{4}$

<0 10 33 87>$\cup \{h_1, h_2\}$ <0 19 41 80>$\cup \{h_3, h_4\}$

<0 21 47 84> $\cup \{h_5, h_6\}$

For v = 258 apply Theorem 2.4 with m = 44, u = 28, and h = 10, see Lemma 5.2 for a(54,5,3) minimal covering design with a hole of size 10.

For v = 198,238,298,318,338,358,378,458,538,558,578,598, apply Theorem 2.4 as indicated in the table below and see [6] and Lemma 5.1 (c) for the appropriate (v,5,3) minimal covering design with a hole of size h.

v	198	238	298	318	338	358	378	458	538	558	578	598
m	32	40	52	52	60	60	64	80	100	100	100	100
u	32	32	32	52	32	52	48	52	32	52	72	92
h	6	6	6	6	6	6	10	6	6	6	6	6

For v= 618,778 apply Theorem 2.8 with r = 5, t = 8, h = 2 and (m,u) = (13,12) (17,12) respectively.

For v = 978,1778 apply Theorem 2.10 with r = 5, t = 12, w = 0 and (m,u,h,s) = (15,6,6,0), (29,2,2,12) respectively.

For v = 1178,1378,1578,1978 apply Theorem 2.10 with r = 10, t = 4, h = 6, w = 0, u = 21, s = 8 and m = 27,32,37,47 respectively.

For v≡ 178 (mod 200), v≥ 2178 write v = 40m + 338 then apply Theorem 2.10 with r = 5, t = 8, h = 2, s = w = 0, u = 42 and m≡ 1 (mod 5).

For all other values of v, v≠ 138,158,178,218,258,278 write v = 40m + 8u + h +s and then the proof is the same as Lemma 5.2 with the difference that 8u + h + s =38,58,78,98,118.

In this section we have shown the following:

<u>Theorem 5.1</u> Let v≡ 2 (mod 4), v≥ 6 be an integer. Then α (v,5,3) = ϕ (v,5,3) with the possible exceptions of v ∈ {18,26,122,126,138,142,146,158,162,178,186,218,226,278}.

From now on we make use of Theorem 2.10 in its simplest form, that is, when r = 5, w = 0 and t = 4. It should be noted that the requirement in (1) is equivalent to 4 MOLS of order m. This Theorem can be stated as follows:

Theorem 5.2 If there exists (1) a (5,1)-RMGDD of type 5^m, (2) a $(5,\lambda)$ - GDD of type $4^m s^1$, (3) a $(20+h,5,\lambda)$ minimal covering design with a hole of size h, then there exists a $(20m+4u+h+s,5,\lambda)$ minimal covering design with a hole of size 4u+h+s where $0 \leq u \leq m-1$.

6. Covering With Index 9

In this section, we distinguish the following cases.

6.1 $v \equiv 4 \pmod{20}$

Lemma 6.1 (a) $\alpha(v,5,9) = \phi(v,5,9)$ for v = 24,44,64,84.
(b) There exists a (24,5,9) minimal covering design with a hole of size 4.

Proof For a (24,5,9) minimal covering design with a hole of size 4 proceed as follows:
1) Take a (23,5,2) optimal packing design [14]. In this design each pair appears in precisely two blocks except a triple, say, {21,22,23}, the pairs of which appear in zero blocks.
2) Take two copies of a B[25,5,1]. Assume in each copy we have the block <21 22 23 24 25>. Delete this block and in all other blocks change 25 to 24.
3) Take a (24,5,5) minimal covering design with a hole of size 4 [7].

For a (24,5,9) minimal covering design, proceed as follows:
1) Take a (24,5,4) optimal packing design [17]. By Lemma 3.1 each pair of this design appears in four blocks except one pair, say, {23,24} that appears in zero blocks. Assume that this design has the following three blocks through the

point 24: < a b c 24 22 > < d e f 24 1 > < g h i 24 5 > where {a,b,c,}, {d,e,f} and {g,h,i,} are arbitrary numbers not necessarily disjoint. Then in the first block replace 22 by 23, in the second replace 1 by 23 and in the third 5 by 23.

2) Take the (24,5,5) minimal covering design in Lemma 4.1. Close observation of this design shows that the following pairs appear at least six times {23,24}, {22,24}, {5,24} {1,24}, {6,23}, {15,23} and {16,23}. Further, close observation of this design shows that we may permute the points so that we have the following blocks: < a b c 6 23 > < d e f 15 23 > < g h i 16 23 >. In the first block replace 23 by 22, in the second replace 23 by 1 and in the third replace 23 by 5. Then it is easy to check that the above construction yields the blocks of a (24,5,9) minimal covering design.

The construction of a (v,5,9) minimal covering design for v = 44,64,84, is given in the following table.

V	Point Set	Base Blocks
44	$Z_{36} \cup H_8$	On $Z_{36} \cup H_5$ construct a B[41,5,5] with a hole of size5, say, H_5. Further, take the following blocks: <0 9 18 27 h_6>+i, i $\in$ Z_9. <0 7 18 25> $\cup$ {h_7,h_8}, half orbit. <0 2 10 16 h_1> <0 3 5 17 h_2> <0 1 3 11 h_3> <0 4 16 21 h_4><0 6 13 27 h_5> <0 1 12 16 h_6> <0 6 13 23 h_6> <0 1 2 5 h_7> < 0 3 9 23 h_7 > <0 4 9 28 h_8> <0 7 15 25 h_8>
64	$Z_{56} \cup H_8$	On $Z_{56} \cup H_5$ construct a B[61,5,7] with a hole of size 5, say, H_5. Furthermore, take the following blocks: <0 14 28 42 h_6>+i, i $\in$ Z_{14}. <0 4 8 26 h_6>

$\langle 0\ 1\ 3\ 11\ h_6\rangle$ $\langle 0\ 5\ 12\ 40\ h_7\rangle$ $\langle 0\ 6\ 15\ 32\ h_7\rangle$
$\langle 0\ 6\ 9\ 42\ h_8\rangle$ $\langle 0\ 9\ 25\ 36\ h_8\rangle$ $\langle 0\ 5\ 12\ 29\rangle \cup \{h_1, h_2\}$
$\langle 0\ 10\ 23\ 41\rangle \cup \{h_3, h_4\}$
$\langle 0\ 1\ 3\ 22\rangle \cup \{h_5, h_5, h_7, h_8\}$.

84 $Z_{76} \cup H_8$ On $Z_{76} \cup H_5$ construct a B[81,5,7] with a hole of size 5, say H_5. Furthermore, take the following blocks:
$\langle 0\ 19\ 38\ 57\ h_6\rangle + i,\ i \in Z_{19}$ $\langle 0\ 1\ 3\ 7\ 17\rangle$
$\langle 0\ 5\ 24\ 42\ 54\rangle$ $\langle 0\ 8\ 31\ 51\rangle \cup \{h_1, h_2\}$
$\langle 0\ 9\ 35\ 48\rangle \cup \{h_3, h_4\}$ $\langle 0\ 11\ 32\ 47\ h_6\rangle$
$\langle 0\ 1\ 3\ 11\ h_6\rangle$ $\langle 0\ 4\ 20\ 38\ h_7\rangle$ $\langle 0\ 6\ 32\ 49\ h_7\rangle$
$\langle 0\ 7\ 29\ 52\ h_8\rangle$ $\langle 0\ 12\ 25\ 40\ h_8\rangle$
$\langle 0\ 5\ 14\ 35\rangle \cup \{h_5, h_5, h_7, h_8\}$.

Lemma 6.2 Let $v \equiv 4 \pmod{20}$ be a positive integer greater than 4. Then $\alpha(v,5,9) = \phi(v,5,9)$.

Proof For v = 24,44,64,84 the result follows from Lemma 6.1. For v = 104 apply Theorem 2.14 with m = 5, h = 4 and $\lambda = 9$. For $v \geq 124$, $v \neq 144$, simple calculation shows that v can be written in the form v = 20m+4u+h+s where m, u, h and s are chosen so that:

1) there exists a (5,1) - RMGDD of type 5^m;
2) there exists a (5,9) - GDD of type $4^m s^1$;
3) 4u + h + s =24,44,64,84;
4) $0 \leq u \leq m-1$, $s \equiv 0 \pmod 4$ and h = 0 or 4.

Now apply Theorem 5.2 with $\lambda = 9$ and the result follows.
For v = 144 Apply Theorem 2.7 with n=7 h=4 and λ =9 and see Lemma 2.3 for a (6,9)-GD of type 5^7.

6.2 $v \equiv 8 \pmod{20}$

Lemma 6.3 $\alpha(v,5,9) = \phi(v,5,9)$ for v = 8,48,68,88.

Proof For v = 68 the construction is as follows:
take a T[6,9,3] [26] and delete one point from last group. Inflate the resultant design by a factor of 4, that is, replace the blocks of size 5 and 6 by the blocks of a (5,1) - GDD of type 4^5 and a (5,1) - GDD of type 4^6 respectively. Finally on the groups construct a (v,5,9) minimal covering design where v = 8,12. See Lemma 6.5 for a (12,5,9) minimal covering design.
For v = 8,48,88, see the following table.

V	Point Set	Base Blocks
8	Z_8	<0 2 4 6>+i, i $\in$ Z_2 <0 1 3 4 5> <0 1 2 3 5> twice
48	$Z_{40} \cup H_8$	Take three copies of a (47,5,2) minimal covering design with a hole of size 7, say, H_7 [14]. Furthermore, take the following blocks: <0 8 16 24 32>+i, i $\in$ Z_8, twice. <0 9 20 29 h_7> half orbit , <0 2 6 15 28> <0 6 10 22 h_8> <0 1 3 20 h_8> <0 1 2 7>$\cup\{h_i\}^4_{i=1}$ <0 3 14 21>$\cup\{h_i\}^8_{i=5}$ <0 1 5 10>$\cup\{h_1,h_2\}$ <0 1 7 28>$\cup\{h_3,h_4\}$ <0 3 11 26>$\cup\{h_5,h_6\}$
88	$Z_{80} \cup H_8$	Take three copies of an (87,5,2) minimal covering design with a hole of size 7, say H_7 [14]. Take also an (80,5,1) minimal covering design [35]. Furthermore, take the following blocks: <0 16 32 48 64>+i, i $\in$ Z_{16}, twice.

$<0\ 13\ 40\ 53\ h_7>$ half orbit. $<0\ 1\ 3\ 39\ 59>$

$<0\ 5\ 15\ 33\ 61>$ $<0\ 6\ 14\ 51\ 63>$ $<0\ 4\ 26\ 38\ h_8>$

$<0\ 4\ 25\ 31\ h_8>$ $<0\ 11\ 25\ 30>\cup\{h_i\}^4_{i=1}$

$<0\ 1\ 3\ 10>\cup\{h_i\}^8_{i=5}$ $<0\ 7\ 18\ 51>\cup\{h_1,h_2\}$

$<0\ 8\ 23\ 43>\cup\{h_3,h_4\}$ $<0\ 9\ 26\ 39>\cup\{h_5,h_6\}$.

Lemma 6.4 Let $v \equiv 8 \pmod{20}$ be a positive integer. Then $\alpha(v,5,9) = \phi(v,5,9)$ with the possible exception of v = 28.

Proof For v = 8,48,68,88 the result follows from the previous Lemma. For v $\geq$ 108, v $\neq$128,168,208,268 simple calculation shows that v can be written in the form v = 20m + 4u+h+s where m, u, h, and s are chosen as in Lemma 6.2 with the difference that 4u+h+s = 8,48,68,88, and h = 0 or 4. Notice that a (20,5,9) minimal covering design exists by Corollary 2.1.

Now apply Theorem 5.2 with $\lambda = 9$ and the result follows. For v = 128 apply Theorem 2.6 with n = 7, $\lambda = 9$, h = 0 and u = 2. For v = 168 apply theorem 2.11 with m = 8, $\lambda = 9$ and u = 1. For v = 208 take a T[5,9,40] [26]. Add eight points to the groups and on the first four groups construct a (48,5,9) minimal covering design with a hole of size 8 and on the last group construct a (48,5,9) minimal covering design.

For v = 268 apply Theorem 2.8 with m=13, h=0 t=4 and u=2. This construction gives a (268,5,9) minimal covering design with a hole of size 8. But $\alpha(8,5,9) = \phi(8,5,9)$. Hence, $\alpha(268,5,9) = \phi(268,5,9)$.

6.3 $v \equiv 12 \pmod{20}$

Lemma 6.5 $\alpha(v,5,9) = \phi(v,5,9)$ for v = 12,32,52,72,92.

Proof For v = 12,32,52,92 see next table. For v = 72 take a T[6,9,3] [26] and inflate it by a factor of 4, that is, replace all blocks which are of size 6 by the blocks of a (5,1) - GDD of type 4^6 . Finally on each group construct a (12,5,9) minimal covering design.

V	Point Set	Base Blocks
12	$Z_{10} \cup H_2$	<0 1 4 h_1 h_2> <0 1 5 7 h_1> <0 1 4 5 h_2> <0 1 2 3 4> <0 1 3 5 7><0 2 5 7>$\cup\{h_1,h_2\}$
32	Z_{32}	<0 1 2 4 11> 3 times <0 3 8 15 21> 3 times <0 4 10 19 24> 3 times <0 3 9 16 20> <0 1 2 3 16> <0 2 6 10 18> <0 5 10 16 25> <0 3 10 19 24>
52	$Z_{40} \cup H_{12}$	On $Z_{40} \cup H_{11}$ construct a B[51,5,4] with a hole of size 11, say, H_{11}. Such design can be constructed by taking a T[5,4,10] [26], add a new point to the groups and on the first four groups construct a B[11,5,4] and take this point with the last group to be the hole. Furthermore, take the following blocks: <0 8 16 24 32>+i, i $\in$ Z_8. <0 1 3 18>$\cup\{h_i\}^4_{i=1}$ <0 5 11 26>$\cup\{h_i\}^8_{i=5}$ <0 7 17 26>$\cup\{h_i\}^{12}_{i=9}$ <0 1 3 7 h_1> <0 5 15 27 h_2> <0 8 17 28 h_3> <0 1 3 7 h_4> <0 5 15 27 h_5> <0 8 17 28 h_6> <0 1 3 10 h_7> <0 4 18 24 h_8> <0 5 13 24 h_9> <0 1 3 14 h_{10}> <0 4 10 26 h_{11}> <0 4 17 25 h_{12}> <0 5 12 21 h_{12}>
92	$Z_{80} \cup H_{12}$	On $Z_{80} \cup H_{11}$ construct a B[91,5,6] with a hole of size 11, say, H_{11}. This design can be constructed

by taking a T[6,1,8] [26]. Delete 5 points from last group and inflate the resultant design by a factor of 2. Replace its blocks, which are of size 5 and 6, by the blocks of a (5, 6) - GDD of type 2^5 and 2^6 [26] respectively. Finally, add 5 points to the groups of size 16 and construct a B[21,5,6] with a hole of size 5 where the hole is the new 5 points. Now take these 5 points with the group of size 6 to be the hole. Further, take the following blocks:

<0 16 32 48 64>+i, $i \in Z_{16}$ 3 times.

<0 13 40 53 h_{12}> half orbit. <0 2 5 9 15>

<0 1 3 7 37> <0 8 20 34 58>

<0 8 28 47 57> <0 9 24 45 62> <0 11 25 54>$\cup\{h_i\}_{i=1}^{4}$

<0 17 35 58>$\cup\{h_i\}_{i=5}^{8}$ <0 5 26 39>$\cup\{h_i\}_{i=9}^{12}$

<0 8 20 38 h_{12}>

<0 1 3 28>$\cup\{h_1,h_2\}$ <0 12 31 59>$\cup\{h_3,h_4\}$

<0 4 9 65>$\cup\{h_5,h_6\}$ <0 6 23 37>$\cup\{h_7,h_8\}$

<0 7 29 40>$\cup\{h_9, h_{10}\}$<0 1 11 36>$\cup\{h_{11},h_{12}\}$

Lemma 6.6 Let $v \equiv 12 \pmod{20}$ be a positive integer. Then $\alpha(v,5,9) = \phi(v,5,9)$.

Proof For v = 12,32,52,72,92 the result follows from the previous Lemma. For $v \geq 112$, $v \neq 132$, simple calculation shows that v can be written in the form v = 20m+4u + h+s where m, u, h and s are chosen as in Lemma 6.2 with the difference that
4u+h+s = 12,32,52,72,92, and h = 0 or 4. Now apply Theorem 5.2 with $\lambda = 9$ to get the result.

For v = 132 apply Theorem 2.6 with n = 7, h = 4, u = 2, and $\lambda = 9$ and see Lemma 2.3 for a (6,9)-GD of type 5^7.

6.4 $v \equiv 2 \pmod{20}$

Lemma 6.7 (a) Let $v \equiv 2 \pmod{20}$, $v \geq 22$ be a positive integer. Then $\alpha(v,5,9) = \phi(v,5,9)$.

(b) There exists a (22,5,9) minimal covering design with a hole of size 2.

Proof For all positive integers $v \equiv 2 \pmod{20}$, $v \geq 22$, the construction is as follows:

1) Take a (v,5,8) covering design with $\phi(v,5,8)+1$ blocks [7]. Assume that the pair (v-1,v) appears at least nine times.

2) Take a (v,5,1) minimal covering design with a hole of size two, [29]. It is readily checked that the above two steps yield the blocks of a (v,5,9) minimal covering design.

For a (22,5,9) minimal covering design with a hole of size 2 take two copies of a (22,5,4) minimal covering design with a hole of size 2 [17]; and take a (22,5,1) minimal covering design with a hole of size two.

6.5 $v \equiv 14 \pmod{20}$

Lemma 6.8 Let $v \equiv 14 \pmod{20}$ be a positive integer. Then $\alpha(v,5,9) = \phi(v,5,9)$.

Proof For v = 14,34,54,74,94 the blocks of a (v,5,9) minimal covering design are the blocks of a (v,5,3) minimal covering design, each block taken three times [6]. For $v \geq 114$, $v \neq 134$, simple calculation shows that v can be written in the form v = 20m+4u+h+s, where m, u, h, and s are chosen as in Lemma 6.2 with the difference that 4u + h + s = 14,34,54,74,94, and h = 2.

Now apply Theorem 5.2 with $\lambda = 9$ to get the result.

For v = 134 apply Theorem 2.6 with n = 7, h = 2, u = 3, and $\lambda = 9$ and see Lemma 2.3 for a (6,9)-GDD of type 5^7.

6.6 v ≡ 18 (mod 20)

Lemma 6.9 Let v ≡ 18 (mod 20) be a positive integer. Then $\alpha(v,5,9) = \phi(v,5,9)$.

Proof The blocks of a (v,5,9) minimal covering design are the blocks of a (v,5,8) minimal covering design [7] together with the blocks of a (v,5,1) minimal covering design [29].

Theorem 6.1 Let v ≥ 5 be a positive integer. Then $\alpha(v,5,9) = \phi(v,5,9)$ with the possible exception of v = 28,56.

Proof The result follows from corollary 2.1 and Lemmas 6.1 - 6.9.

7. Covering With Index 10

Lemma 7.1 Let v ≡ 4 (mod 20) be a positive integer greater than 4. Then $\alpha(v,5,10) = \phi(v,5,10)$.

Proof The construction of a (v,5,10) minimal covering design, for all positive integers v ≡ 4 (mod 20), v ≥ 24, is as follows:

1) Take a (v,5,4) minimal covering design [7] [15]. In this design, each pair appears in exactly four blocks except one triple, say, {v-3,v-2,v-1}, the pairs of

Now apply Theorem 5.2 with $\lambda = 9$ to get the result.

For v = 134 apply Theorem 2.6 with n = 7, h = 2, u = 3, and $\lambda = 9$ and see Lemma 2.3 for a (6,9)-GDD of type 5^7.

6.6 $v \equiv 18 \pmod{20}$

Lemma 6.9 Let $v \equiv 18 \pmod{20}$ be a positive integer. Then $\alpha(v,5,9) = \phi(v,5,9)$.

Proof The blocks of a (v,5,9) minimal covering design are the blocks of a (v,5,8) minimal covering design [7] together with the blocks of a (v,5,1) minimal covering design [29].

Theorem 6.1 Let $v \geq 5$ be a positive integer. Then $\alpha(v,5,9) = \phi(v,5,9)$ with the possible exception of v = 28,56.

Proof The result follows from corollary 2.1 and Lemmas 6.1 - 6.9.

7. Covering With Index 10

Lemma 7.1 Let $v \equiv 4 \pmod{20}$ be a positive integer greater than 4. Then $\alpha(v,5,10) = \phi(v,5,10)$.

Proof The construction of a (v,5,10) minimal covering design, for all positive integers $v \equiv 4 \pmod{20}$, $v \geq 24$, is as follows:

1) Take a (v,5,4) minimal covering design [7] [15]. In this design, each pair appears in exactly four blocks except one triple, say, {v-3,v-2,v-1}, the pairs of

which appear in 6 blocks. Furthermore, assume we have the following two blocks <1 2 3 7 9> <a b c 8 10> where {1, 2, 3, 7, ..., 10} are arbitrary numbers and {1,2,3}, {a,b,c} are not necessarily disjoint, a, b, c $\neq$ v-1. In the first block change 9 to v and in the second change 10 to v.

2) Take a (v,5,2) minimal covering design [40] and assume that each of the pairs {7,9} and {8,10} appears at least three times in the blocks of this design.

3) Take a (v-1,5,2) optimal packing design [14]. In this design each pair appears in precisely two blocks except a triple, say, {v-3,v-2,v-1}, the pairs of which appear in zero blocks.

4) Take two copies of a B[v+1,5,1] and assume we have the two blocks <1 2 3 v v+1>, <a b c v v+1>. In the first block, change v+1 to 9 and in the second block change v+1 to 10. In all other blocks change v+1 to v. Now it is readily checked that the above four steps yield the blocks of a (v,5,10) minimal covering design for all $v \equiv 4 \pmod{20}$, $v \geq 24$.

<u>Lemma 7.2</u> Let $v \equiv 8 \pmod{20}$ be a positive integer. Then $\alpha(v,5,10) = \phi(v,5,10)$.

<u>Proof</u> Let $v \equiv 8 \pmod{20}$ be a positive integer. Then a (v,5,10) minimal covering design can be constructed as follows:

1) Take a (v,5,8) minimal covering design [7]. In this design each pair appears in eight blocks, except a triple, say, {v-2,v-1,v}, the pairs of which appear in 10 blocks.

2) Take a (v,5,2) minimal covering design with a hole of size 2. We may assume the hole to be {v-1,v}.

To complete the proof of this lemma we need to show that there exists a (v,5,2) minimal covering design with a hole of size 2 for all $v \equiv 8 \pmod{20}$, $v \geq 8$. For v = 8,28,48,68,88 see next table. For $v \geq 108$, $v \neq 128$, write v = 20m+4u+h+s where m, u, h and s are chosen the same as in Lemma 6.2 with the difference that 4u + h + s = 8,28,48,68,88 and h = 0. Now apply Theorem 5.2 with $\lambda = 2$ to get the result.

For v = 128 apply Theorem 2.6 with n = 7, u = 2, h = 0, and $\lambda = 2$ and see Lemma 2.3 for a (6,2)-GDD of type 5^7.

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v	Point Set	Base Blocks
8	$Z_6 \cup H_2$	<0 1 3 4> $\cup$ $\{h_1,h_2\}$
28	$Z_2 x Z_{13} \cup H_2$	<(0,0)(0,1)(0,4)(0,7)(1,0)><(0,0)(1,0)(1,1)(1,2)(1,4)>
		<(0,0)(0,2)(0,4)(1,7)(1,12)>
		<(0,0)(0,5)(1,2)(1,8)(1,11)>
		<(0,0)(0,1)(1,5)(1,10) h_1><(0,0)(0,3)(1,1)(1,7) h_2>
48	$Z_{46} \cup H_2$	<0 1 3 8 17> <0 4 10 22 35> <0 1 3 8 23>
		<0 4 13 20 32> <0 6 17 27>$\cup\{h_1,h_2\}$
68	$Z_{66} \cup H_2$	<0 1 3 7 21> <0 5 15 40 49> <0 8 24 36 47>
		<0 1 3 7 28> <0 5 13 23 35> <0 9 20 33 49>
		<0 5 19 34>$\cup\{h_1,h_2\}$
88	$Z_{86} \cup H_2$	<0 1 3 7 15> <0 5 21 38 63> <0 9 27 40 66>
		<0 10 32 51 62> <0 5 11 36 48> <0 1 3 16 24>
		<0 4 18 44 51> <0 5 27 37 57>
		<0 9 28 45>$\cup\{h_1,h_2\}$

Lemma 7.3 Let $v \equiv 12 \pmod{20}$ be a positive integer. Then $\alpha(v,5,10) = \phi(v,5,10)$.

Proof For v = 12,52,72 see the next table.
For all other values of v the construction is as follows:

1) Take a B[v-1,5,4]

2) Take a (v+1,5,2) minimal covering design [37]. This design exists for all v+1 ≡ 13 (mod 20), v+1 ≠ 13, with the possible exception of v = 53,73. Furthermore, in this design there is one pair, say, {v-1,v} that appears in six blocks. Assume in this design we have the two blocks: <1 2 3 v v+1> <a b c v v+1> where {1,2,3} and {a,b,c} are not necessarily disjoint a, b, c ≠ v-1. In the first block change v+1 to 10 and in the second block change v+1 to 11. In all other blocks change v+1 to v.

3) Take a (v+1,5,2) optimal packing design which exists for all v+1 ≡ 13 (mod 20), v+1 ≠ 13, [9]. Furthermore, in this design, there is a triple, say, {v-1,v,v+1}, the pairs of which appear in zero blocks. So change v+1 to v. Also assume we have the two blocks: <1 2 3 8 10> <a b c 9 11>, where 8 and 9 are arbitrary numbers. In the first block, change 10 to v and in the second change 11 to v.

4) Take a (v,5,2) minimal covering design [40] and assume each of the pairs {8,10} and {9,11} appear at least three times in the blocks of this design. It is readily checked that the above four steps yield the blocks of a (v,5,10) minimal covering design for all v ≡ 12 (mod 20), v ≠ 12,52,72.

v	Point Set	Base Blocks
12	$Z_2 x Z_5 \cup H_2$	<(0,0)(0,1)(0,2)(0,3)(0,4)> twice (orbit length 1)
		<(1,0)(1,1)(1,2)(1,3)(1,4)> (orbit length 1)
		<(0,0)(0,1)(0,2) h_1 h_2><(1,0)(1,1)(1,3) h_1 h_2)>
		<(0,0)(0,2)(1,4) h_1 h_2><(0,0)(1,2)(1,3) h_1 h_2)>
		<(0,0)(0,1)(1,0)(1,2) h_1><(0,0)(0,2)(1,0)(1,3) h_1>
		<(0,0)(0,2)(1,0)(1,4) h_2><(0,0)(0,2)(1,3)(1,4) h_2>
		<(0,0)(0,1)(1,0)(1,1)(1,2)><(0,0)(0,1)(0,2)(1,0)(1,1)>
		<(0,0)(0,2)(1,0)(1,1)(1,3)>

52 $Z_{40} \cup H_{12}$

On $Z_{40} \cup H_{11}$ construct a B[51,5,4] with a hole of size 11, say, H_{11}. Such design can be constructed by taking a T[5,4,10]. Add a point to the groups and on the first four groups construct a B[11,5,4] and take this point with the last group to be the hole. Furthermore, take the following blocks:

<0 1 5 22>$\cup\{h_1,h_2\}$ <0 3 14 23>$\cup\{h_3,h_4\}$
<0 6 13 25>$\cup\{h_5,h_6\}$ <0 1 5 22>$\cup\{h_7,h_8\}$
<0 3 14 23>$\cup\{h_9,h_{10}\}$ <0 6 13 25>$\cup\{h_{11},h_{12}\}$
<0 1 3 7 h_1> <0 2 14 24 h_2> <0 2 15 23 h_3>
<0 5 14 25 h_4> <0 1 3 13 h_5> <0 4 10 32 h_6>
<0 5 15 29 h_7> <0 7 16 29 h_8> <0 1 2 4 h_9>
<0 2 5 33 h_{10}> <0 4 10 34 h_{11}> <0 5 13 22 h_{12}>
<0 7 15 20 h_{12}>

72 $Z_{60} \cup H_{12}$

On $Z_{60} \cup H_{11}$ construct a B[71,5,8] with a hole of size 11, say, H_{11}. Such design can be constructed by taking a T[6,8,12]. Delete 3 points from last group and replace the blocks of the resultant design by the blocks of B[5,5,8] and B[6,5,8]. Add two points to the groups and on the first five groups take two copies of (14,5,4) minimal covering design with a hole of size 2, and take these two points with the last group to be the hole. Further, take the following blocks:

<0 12 24 36 48>+i, i $\in Z_{12}$ <0 1 6 19 35>
<0 3 17 26>$\cup\{h_1,h_2\}$ <0 1 3 8>$\cup\{h_3,h_4\}$
<0 4 15 33>$\cup\{h_5,h_6\}$ <0 6 23 39.$\cup\{h_7,h_8\}$
<0 7 20 45>$\cup\{h_9,h_{10}\}$ <0 9 19 30.$\cup\{h_{11},h_{12}\}$
<0 2 10 24 h_{12}> <0 4 20 32 h_{12}>

Lemma 7.4 (a) $\alpha(v,5,10) = \phi(v,5,10)$ for all v ∫ 2 or 14 (mod 20), $v \geq 14$.
(b) There exists a (22,5,10) minimal covering design with a hole of size 2, and a (26,5,10) minimal covering design with a hole of size 6.

Proof (a) For a (v,5,10) minimal covering design, v ∫ 2 or 14 (mod 20), $v \geq 14$, take a (v,5,4) minimal covering design [7], [15] and a (v,5,6) minimal covering design [5].
(b) A (22,5,10) minimal covering design with a hole of size 2 can be constructed as follows:
1) Take two copies of (22,5,4) minimal covering design with a hole of size 2, say, {21,22} [17];
2) Take a B[21,5,1];
3) Take a (23,5,1) minimal covering design with a hole of size 3, say, {21,22,23}, [38], then change 23 to 22.
For a (26,5,10) minimal covering design with a hole of size 6 proceed as follows:
1) Take a (26,5,8) minimal covering design with a hole of size 6. Such design can be constructed by taking a T[5,8,5]. Add a point to the groups and on the first four groups construct a B[6,5,8] and take the last group with the point to be the hole.
2) Take the blocks of a (26,5,2) minimal covering design with a hole of size 6 on $X = Z_2 \times Z_{10} \cup H_6$.
$\langle(a,0)\ (a,2)\ (a,4)\ (a,6)\ (a,8)\rangle + (-, i),\ i \in Z_2,\ a = 0, 1$
$\langle(0,0)\ (0,1)\ (1,0)\ (1,7)\ h_1\rangle$ mod (-, 10)
$\langle(0,0)\ (0,2)\ (1,5)\ (1,9)\ h_2\rangle$ mod (-, 10)
$\langle(0,0)\ (0,3)\ (1,1)\ (1,6)\ h_3\rangle$ mod (-, 10)
$\langle(0,0)\ (0,3)\ (1,4)\ (1,5)\ h_4\rangle$ mod (-, 10)
$\langle(0,0)\ (0,1)\ (0,5)\ (1,9)\rangle \in \{h_5,h_6\}$ mod (-, 10)
$\langle(0,0)\ (1,0)\ (1,2)\ (1,3)\rangle \in \{h_5,h_6\}$ mod (-, 10)

Lemma 7.5 $\alpha(v,5,10) = \phi(v,5,10)$ for v = 18,38,58,78,98.

Proof. The proof of this lemma is the same as Lemma 7.2: The blocks of a (v,5,10) minimal covering design are the blocks of a (v,5,8) minimal covering [7] together with the blocks of a (v,5,2) minimal covering design with a hole of size 2. For a (v,5,2) minimal covering design with a hole of size 2, v = 18,38,58,78, see next table. For v = 98, take a T[6,1,8] and inflate this design by a factor of 2. Replace the blocks of this design by the blocks of a (5,2) - GDD of type 2^6 [26]. Finally, adjoin two points to the groups and on each group construct an (18,5,2) minimal covering design with a hole of size 2.

v	Point Set	Base Blocks
18	$Z_2 \times Z_8 \cup H_2$	<(0,0) (0,1) (1,4) (1,7) h_1> <(0,0) (0,3) (1,0) (1,4) h_2> <(0,0) (0,2) (0,3) (0,6) (1,0)> <(0,0) (1,1) (1,2) (1,3) (1,7)>
38	$Z_{36} \cup H_2$	<0 1 4 11 19> <0 1 3 9 21> <0 2 6 13 26> <0 5 14 19>$\cup\{h_1,h_2\}$
58	$Z_{56} \cup H_2$	<0 1 3 9 31> <0 4 15 31 36> <0 7 17 30 44> <0 1 3 7 35> <0 5 11 41 49> <0 4 13 23>$\cup\{h_1,h_2\}$
78	$Z_{76} \cup H_2$	<0 1 3 16 39> <0 4 12 47 66> <0 6 27 34 52> <0 9 29 40 53> <0 1 3 7 49> <0 5 14 59 64> <0 8 18 33 44> <0 16 35 55>$\cup\{h_1,h_2\}$

Lemma 7.6 Let $v \equiv 18 \pmod{20}$ be a positive integer. Then $\alpha(v,5,10) = \phi(v,5,10)$.

Proof For v = 18,38,58,78,98 the result follows from Lemma 7.5. For $v \geq 118$, $v \neq 138$ write v = 20m+4u+h+s where m, u, h, and s are chosen the same as in Lemma 6.2 with the difference that 4u+h+s = 18,38,58,78,98 and h = 2 or 6. Now apply Theorem 5.2 with $\lambda = 10$ to get the result.
For v = 138 apply Theorem 2.6 with n = 7, h = 2, u = 4, and $\lambda = 10$ and see Lemma 2.3 for a (6,10)-GDD of type 5^7.

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To summarize this section and corrollary 2.1, we have shown:

Theorem 7.1 Let $v \geq 5$ be a positive integer. Then $\alpha(v,5,10) = \phi(v,5,10)$.

We like to remind the reader that from now on we only treat the cases $v \equiv 2,14$ or $18 \pmod{20}$, see Theorem 1.3 and Corrolary 2.1.

8. Covering with Index 11

Lemma 8.1 (a) $\alpha(v,5,11) = \phi(v,5,11)$ for v = 22, 42, 62, 82.
(b) There exists a (22,5,11) and a (26,5,11) minimal covering design with a hole of size 2 and 6 respectively.

Proof For v = 22 let $X = Z_{20} \cup \{a,b\}$. Then take the following base blocks under the action of the group Z_{20}.

< 0 1 9 a b >	< 0 1 2 3 9 >	< 0 1 5 10 14 >	< 0 2 6 12 15 >
< 0 2 7 10 13 >	< 0 1 2 3 7 >	< 0 1 6 9 11>	< 0 2 7 10 13 >
< 0 1 2 4 a >	< 0 2 7 14 a >	< 0 4 9 15 b >	< 0 3 6 10 b >

< 0 4 8 12 16 > + i, $i \in Z_4$, 4 times

For v = 42,62,82 the construction is as follows:

1) Take two copies of a (v,5,4) minimal covering design [15]. In this design there is a triple, say, {v-2,v-1,v} the pairs of which appear in 6 blocks.

2) Take a B[v-1,5,1].

3) Take a (v,5,1) optimal packing design [2]. The missing pairs form a 1-factor, Lemma 3.2. Assume that {v-1,v} is an edge of the 1-factor.

4) Take a (v+1,5,1) minimal covering design with a hole of size 3, say, {v-1,v,v+1}. The repeated pairs of this design form a two 1-factor [36] or [38]. In this design change v+1 to v. Now apply Theorem 3.1 to get a (v,5,11) minimal covering design for v = 42,62,82.

(b) For a (22,5,11) and a (26,5,11) minimal covering design with a hole of size 2, 6 respectively, take a (22,5,4), (26,5,4), (22,5,7), and a (26,5,7) minimal covering designs with a hole of size 2, 6 respectively [17] [6]. Notice that a (26,5,4) minimal covering design with a hole of size 6 can be constructed by taking a T[5,4,5]. Add a new point to the groups and on the first 4 groups, construct a B[6,5,4] and take this point with the last group to be the hole.

<u>Lemma 8.2</u> Let $v \equiv 2 \pmod{20}$, $v \geq 22$, be a positive integer. Then $\alpha(v,5,11) = \phi(v,5,11)$.

<u>Proof</u>. For v = 22,42,62,82 the result follows from the previous lemma.
For $v \geq 122$, $v \neq 142,182$ write v = 20m+4u+h+s where m, u, h, and s are chosen as in Lemma 6.2 with the difference that 4u+h+s = 22,42,62,82 and h = 2 or 6. Now apply Theorem 5.2 with $\lambda = 11$ to get the result.
For v = 102 apply Theorem 2.14 with m = 5, h = 2, and $\lambda = 11$.
For v = 142 apply Theorem 2.7 with n = 7, h = 2, and $\lambda = 11$ and see Lemma 2.3 for a (6, 11)-GDD of type 5^7.
For v = 182 apply Theorem 2.12 with m = s = 8, h = 6, and $\lambda = 11$.

Lemma 8.3 Let $v \equiv 14 \pmod{20}$ be a positive integer. Then $\alpha(v,5,11) = \phi(v,5,11)$.

Proof For v = 14,34,54,74,94 the construction is as follows:

1) Take a (v,5,4) minimal covering design [15]. This design has a triple, say, {v-2,v-1,v} the pairs of which appear in six blocks. Furthermore, assume in this design we have the blocks < a b c v-1 v-2 >, < d e f v v-2 > where {a,b,c}, {d,e,f} are arbitrary numbers, not necessarily disjoint. In the first block change v-2 to v and in the second change v-2 to 5.

2) Take a (v,5,3) minimal covering design [6] and assume that the pairs {v-1,v} and {5,9} appear four times.

3) Take a (v,5,4) optimal packing design and assume that {v-1,v} appears in zero blocks. Furthermore, assume we have the blocks < a b c 5 v >
< d e f 9 5 >.

In the first block change v to v-2 and in the second block change 5 to v-2.

For $v \geq 114$, $v \neq 134$, write v = 20m+4u+h+s where m, u, h, and s are chosen as in the previous lemma with the difference that 4u+h+s = 14,34,54,74,94. Now apply Theorem 5.2 with $\lambda = 11$ to get the result.

For v = 134 apply Theorem 2.6 with n = 7, h = 6, u = 2, and $\lambda = 11$ and see Lemma 2.3 for a (6, 11)-GDD of type 5^7.

Lemma 8.4. Let $v \equiv 18 \pmod{20}$ be a positive integer. Then $\alpha(v,5,11) = \phi(v,5,11)$.

Proof For v = 18 let $X = Z_{17} \cup \{\infty\}$. On Z_{17} construct a B[17,5,5]. Further, take the following blocks under the action of the group Z_{17}.

<0 1 2 3 4> <0 1 5 8 11> <0 1 5 10 12> <0 3 8 11 ∞> <0 2 6 10 ∞>
<0 2 7 11 ∞>

For all other values under 100 the blocks of a (v,5,11) minimal covering design are those of (v,5,3) and (v,5,8) minimal covering design [6], [7].

For $v \geq 118$, $v \neq 138$, write $v = 20m+4u+h+s$ where m, u, h, and s are chosen as in Lemma 8.2 with the difference that $4u+h+s = 18,38,58,78,98$. Now apply Theorem 2.10 with $\lambda = 11$ to get the result.

For $v = 138$ apply Theorem 2.6 with $n = 7$, $h = 6$, $u = 3$, and $\lambda = 11$.

9. Covering With Index 13

<u>Lemma 9.1</u> Let $v \equiv 2$, 14 or 18 (mod 20) $v \geq 14$ be a positive integer. Then $\alpha(v,5,13) = \phi(v,5,13)$.

<u>Proof</u> For $v \equiv 2$ (mod 20) the blocks of a (v,5,13) minimal covering design are the blocks of a (v,5,1) minimal covering design with a hole of size 2, say, {v-1,v} [29], together with the blocks of a (v,5,12) minimal covering design [7]. Assume in this design that the pair {v-1,v} appears at least 13 times.

For $v \equiv 14$ (mod 20) the blocks of a (v,5,13) minimal covering design are the blocks of a (v,5,9) and (v,5,4) minimal covering design [15].

For $v \equiv 18$ (mod 20) the blocks of a (v,5,13) minimal covering design are the blocks of a (v,5,12) and (v,5,1) minimal covering designs [7], [29].

10. Covering With Index 14

<u>Lemma 10.1</u>. Let $v \equiv 2$ (mod 20) $v \geq 22$ be a positive integer. Then $\alpha(v,5,14) = \phi(v,5,14)$.

(b) There exists a (26,5,14) minimal covering design with a hole of size 6.

<u>Proof</u> We first construct a (v,5,2) minimal covering design with a hole of size 2 by taking the blocks of a B[v-1,5,1] together with the blocks of a (v+1,5,1) minimal covering design with a hole of size 3, say, {v-1,v,v+1} [36]. Further, in this design we replace v+1 by v.

We now construct a (v,5,14) minimal covering design as follows:

1) Take a (v,5,4) optimal packing design [17]. This design has a pair, say, {v-1,v} that appears in zero blocks while each other pair appears in 4 blocks.

2) Take two copies of a (v,5,4) minimal covering design [7, 10]. In this design there is a triple the pairs of which appear in 6 blocks. Assume in both copies the triple is {v-2,v-1,v}.

3) Take a (v,5,2) minimal covering design with a hole of size 2, say, {v-2,v-1}.

(b) For a (26,5,14) minimal covering design with a hole of size 6 take a (26,5,10) minimal covering design with a hole of size 6 and a (26,5,4) minimal covering design with a hole of size 6, Lemma 8.1.

Lemma 10.2 Let $v \equiv 14 \pmod{20}$ be a positive integer. Then $\alpha(v,5,14) = \phi(v,5,14)$.

Proof For v = 14 the construction is as follows:

1) Take a (14,5,4) minimal covering design [13]. This design has a triple, say {12,13,14} the pairs of which appear in six blocks.

2) Take two copies of a (14,5,4) optimal packing design and assume in both copies that the pair {13,14} appears in zero blocks.

3) Take the (14,5,2) minimal covering design in [40]. Close observation of this design shows that there is a pair, say, {13,14} that appears in eight blocks [38].

For v = 34,54 the construction is as follows:

1) Take a (v,5,4) minimal covering design and assume that the pairs of the triple {1,2,3} appear in six blocks.

2) Take two copies of a (v,5,4) optimal packing design [17]. Assume that in the first copy the pair {1,2} appears in zero blocks and in second copy the pair is {1,3}.

3) Take a (v,5,2) minimal covering design with a hole of size 6, say,

{1,2, . . .,6}. Replace the hole by the blocks < 1 2 3 4 5 > < 1 2 3 4 6 > < 1 2 3 5 6 > < 1 2 4 5 6 >.

The above three steps give a design such that every pair appears in at least 14 blocks except the pair {1,3} which appear in exactly 13 blocks. To fix this, assume in the design in (1) we have the block < 1 8 9 10 6> and in the design in (2) we have the block < 2 8 9 10 3 >. In the first block change 6 to 3 and in the second block change 3 to 6.

To complete the construction of v = 34, 54 we need to show that there exists a (v,5,2) minimal covering design with a hole of size 6.

For v = 34 see [40].

For v = 54 let $X = Z_{48} \cup \{\infty_i\}_{i=1}^{6}$ then take the following blocks under the action of the group Z_{48}.

< 0 3 7 15 29 >< 0 1 11 17 31 >< 0 1 3 13 43 >< 0 2 21 25> $\cup \{\infty_1, \infty_2\}$

< 0 5 14 21 > $\cup \{\infty_3, \infty_4\}$ < 0 9 20 33 > $\cup \{\infty_5, \infty_6\}$

For v = 74, 94 the construction consists of the following three steps:

1) Take a (v,5,4) minimal covering design and assume that the pairs of {1,2,3} appears in six blocks, [15].

2) Take two copies of a (v,5,4) optimal packing design [17] and assume in both copies that the pair {1,2} appears in zero blocks.

3) Take the (v,5,2) minimal covering design in [40]. Close observation of these two designs shows that in each design there is one pair, say, {1,2} that appears in eight blocks.

For v $\geq$ 114, the proof is the same as Lemma 8.3 with h = 6.

Lemma 10.3 Let v $\equiv$ 18 (mod 20) be a positive integer. Then $\alpha(v,5,14) = \phi(v,5,14)$.

Proof For v = 18,38,58,78,98 the required construction are given in the following table.

For v $\geq$ 118 the proof is the same as Lemma 8.4 with h = 6.

v	Point Set	Base Blocks
18	Z_{18}	Take two copies of an (18,5,6) optimal packing design [10]. In these two copies each pair appears exactly 12 times except the pairs {i,i+9} i=0,8, which appear 8 times. Further, take the following blocks: < 0 1 2 7 10 > < 0 2 5 9 14 >
38	Z_{38}	Take two copies of a (38,5,6) optimal packing design [10]. In these tcopies each pair appears exactly 12 times except the pairs {i,i+17} i =0, . . .,38 which appear 10 times, so adjoin the following blocks: < 0 1 2 4 17 > < 0 3 8 19 28 > < 0 4 9 21 28 > < 0 6 12 20 27 >
58	Z_{58}	Take seven copies of the (58,5,1) minimal covering design in [29]. This means that the pairs {i,i+1} i = 0,. . .,57 and {j,j+29} j = 0, . . .,28 already appear 14 times. Furthermore, take the following blocks: < 0 2 5 18 28 > 3 times < 0 4 12 34 43 > 3 times < 0 6 17 31 38 > 3 times < 0 2 9 14 47 > twice < 0 3 16 22 37 > twice < 0 4 14 22 39 > twice < 0 2 4 7 54 > < 0 3 12 18 41 > < 0 5 15 35 47 > < 0 7 16 24 34 >
78	Z_{78}	Take seven copies of the (78,5,1) minimal covering design in [29]. This means that the pairs {i,i+2} i = 0, . . .,77 and {j,j+39} j = 0,. . .,38 appear 14 times. Furthermore, take the following blocks: < 0 1 4 9 42 > 3 times < 0 6 20 27 50 > 3 times < 0 10 22 47 62 > 3 times < 0 11 24 43 60 > 3 times < 0 1 4 12 58 > twice < 0 5 22 48 55 > twice

< 0 6 16 31 65 > twice, < 0 1 4 9 26> < 0 7 18 30 57 >
< 0 9 19 33 51 > < 0 13 27 47 62 > < 0 1 4 9 19 >
< 0 6 20 33 50 > < 0 6 22 29 60 > < 0 9 21 35 46 >

98 $Z_{80} \cup H_{18}$ On $Z_{80} \cup H_{17}$ construct a (97,5,10) minimal covering design with a hole of size 17. Such design can be constructed by taking a T[6,1,8]. Delete 2 points from last group and inflate the design by a factor of 2 and index 10. Add five points to the groups and on the first five groups construct a B[21,5,10] such that these 5 points are one block which we delete and take these five points with last group to be the hole of size 17. Furthermore on $Z_{80} \cup H_{13}$ construct a (93,5,1) minimal covering design with a hole of size 13, [25] and take the following blocks:

< 0 16 32 48 64 > + i, $i \in Z_{16}$, 3 times
< 0 20 40 60 h_{13} > + i, $i \in Z_{20}$ < 0 2 20 28 42 >
< 0 1 3 6 h_{14} > < 0 4 8 14 h_{15} > < 0 7 19 31 h_{16} >
< 0 9 27 52 h_{17} >< 0 11 34 56 h_{18} >< 0 13 30 60 h_{18} >
< 0 15 31 59 h_{18} > < 0 1 4 9 > $\cup \{h_1, h_2\}$
< 0 6 13 25 > $\cup \{h_3, h_4\}$ < 0 7 34 45 >$\cup \{h_5, h_6\}$
< 0 10 33 57 > $\cup \{h_7, h_8\}$ < 0 11 37 50 > $\cup \{h_9, h_{10}\}$
< 0 14 31 53 > $\cup \{h_{11}, h_{12}\}$
< 0 15 36 51 > $\cup \{h_{13}, h_{18}\}$
<0 1 10 39> $\cup \{h_i\}_{i=1}^{4}$ < 0 2 19 37 >$\cup \{h_i\}_{i=5}^{8}$
< 0 5 26 51 > $\cup \{h_i\}_{i=9}^{12}$

11. Covering with Index 15

Lemma 11.1 Let $v \equiv 2 \pmod{20}$, $v \geq 22$, be a positive integer. Then $\alpha(v,5,15) = \phi(v,5,15)$.

Proof We first construct a (22,5,15) and a (26,5,15) minimal covering design with a hole of size 2 and 6 by taking two copies of a (22,5,4) and a (26,5,4) minimal covering design with a hole of size 2, 6, Lemma 8.1, and one copy of a (22,5,7) and a (26,5,7) minimal covering design with a hole of size 2 and 6 respectively [6].

For a (22,5,15) minimal covering design let $X = Z_{20} \cup \{a, b\}$. On X take two copies of a (22,5,4) optimal packing design such that {a,b} appears in zero blocks. Furthermore, take the following blocks under the action of the group Z_{20}.

< 0 4 8 12 16 > + i, $i \in Z_4$	< 0 1 10 a b >	< 0 1 3 7 13 >
< 0 1 4 9 15 >	< 0 1 2 3 7 >	< 0 1 4 10 12 >
< 0 2 7 11 14 >	< 0 3 5 7 a >	< 0 3 8 13 b >

For v = 42,62,82 the construction is as follows:

1) Take two copies of a (v,5,4) minimal covering design and assume that in both copies the pairs of the triple {v-2,v-1,v} appear in six blocks.

2) Take a (v,5,4) optimal packing design and assume that the pair {v-2,v-1} appears in zero blocks [17].

3) Take a B[v-1,5,1].

4) Take a (v,5,1) optimal packing design [2]. The complement graph of this design is a 1-factor. Assume that {v-1,v} is an edge of the complement graph.

5) Take a (v+1,5,1) minimal covering design with a hole of size 3, say, {v-1, v, v+1}, [38]. In this design we replace v+1 by v. Furthermore, close observation of these designs shows that the complement graph contains a subgraph that is 1-factor on v-2 vertices.

Now apply Theorem 3.1 to get the constructions for v = 42,62,82.

For all other values of v the proof is the same as Lemma 8.2.

Lemma 11.2 Let $v \equiv 14$ or 18 (mod 20) be a positive integer. Then $\alpha(v,5,15) = \phi(v,5,15)$.

Proof For $v \equiv 18$ (mod 20) take the blocks of a (v,5,7) and a (v,5,8) minimal covering design [6], [7].

For v = 14,34,54,74,94 the construction is as follows:

1) Take two copies of a (v,5,4) minimal covering design and assume that in both copies the pairs of {a,b,c} appear in six blocks [15].

2) Take a (v,5,4) optimal packing design and assume that the pair {a,b} appears in zero blocks [17].

3) Take a (v,5,3) minimal covering design [6].

For $v \geq 114$ the proof is the same as Lemma 8.3.

12. Covering With Index 17

Lemma 12.1 (a) There exists a (22,5,17) and a (26,5,17) minnmal covering design with a hole of size 2 and 6 respectively. (b) Let $v \equiv 2 \pmod{20}$, $v \geq 22$, be a positive integer. Then $\alpha(v,5,17) = \phi(v,5,17)$.

Proof (a) We first construct a (22,5,17) minimal covering design with a hole of size two by taking six copies of a B[21,5,1], three copies of a (23,5,2) minimal covering design with a hole of size three then change 23 to 22, [14] together with a (22,5,5) minimal covering design with a hole of size 2 [4].

For a (26,5,17) minimal covering design with a hole of size 6 let $X = Z_{20} \cup H_6$. On X take three copies of a (26,5,4) minimal covering design with a hole of size 6, say, H_6, Lemma 8.1. Further, take the following blocks under the action of the group Z_{20}.

$<0\ 4\ 8\ 12\ 16> + i,\ i \in Z_4$, twice $\quad$ $<0\ 3\ 10\ 13\ h_6>$ half orbit

$<0\ 1\ 3\ 6> \cup \{h_i\}_{i=1}^{4}$ $\quad$ $<0\ 1\ 2\ 3\ h_1>$ $\quad$ $<0\ 2\ 7\ 9> \cup \{h_5, h_6, h_6, h_6\}$

$<0\ 1\ 8\ 12\ h_2>$ $\quad$ $<0\ 3\ 8\ 14\ h_3>$ $\quad$ $<0\ 3\ 9\ 13\ h_4>$ $\quad$ $<0\ 4\ 9\ 14\ h_5>$.

(b) For a (v,5,17) minimal covering design, v = 22,42,62,82 see next table. For all other values of v the proof is the same as Lemma 8.2.

v	Point Set	Base Blocks
22	Z_{22}	< 0 1 2 5 10 > 4 times, < 0 2 6 9 16 > 4 times < 0 1 3 4 11 > 3 times < 0 2 6 11 17 > 3 times < 0 1 3 10 15 > < 0 3 8 12 16 > < 0 1 3 13 17 > < 0 1 8 10 14 >.
42	Z_{42}	Take 13 copies of a (42,5,1) optimal packing design [2]. The complement graph of this design is a 1-factor, so add the following blocks: < 0 1 5 22 26 >< 0 2 14 21 29 >< 0 3 10 23 31 > < 0 3 9 21 32 > < 0 1 3 13 24 > < 0 3 10 23 31 > < 0 3 8 24 28 > < 0 5 11 26 33 >< 0 2 6 18 27 > < 0 1 2 7 10 >
62	Z_{62}	Take 13 copies of a (62,5,1) optimal packing design [2]. The complement graph of this design is a 1-factor, so add the following blocks: < 0 1 3 21 31 > < 0 4 9 40 55 > < 0 6 19 37 46 > < 0 6 14 37 49 > < 0 1 3 10 34 >< 0 5 13 36 46 > < 0 4 17 35 43 > < 0 5 16 30 45 >< 0 4 15 32 42 > < 0 1 3 37 42 > < 0 6 17 30 48 >< 0 1 3 7 15 > < 0 7 19 31 40 >
82	Z_{82}	Take 13 copies of a (82,5,1) optimal packing design [2]. The complement graph of this design is a 1-factor, so add the following blocks: < 0 1 9 30 41 >< 0 1 3 7 35 >< 0 9 22 47 68 > < 0 5 17 41 57 >< 0 8 22 37 63 >< 0 8 18 37 49> < 0 10 23 43 61 >< 0 1 3 10 44 >< 0 5 16 33 48 > < 0 4 9 26 71 >< 0 6 25 41 55 > < 0 1 3 7 27 >

< 0 11 23 51 69 >< 0 2 5 9 37 >< 0 10 23 41 53 >

< 0 6 21 48 62 >< 0 8 24 46 65 >< 0 10 23 41 53 >

Lemma 12.2 Let $v \equiv 14 \pmod{20}$ be a positive integer. Then $\alpha(v,5,17) = \phi(v,5,17)$.

Proof For v = 14,34,54,74,94 the construction is as follows:

1) Take a (v,5,4) minimal covering design and assume that the pairs of {a,b,c} appear in six blocks [15].

2) Take a (v,5,4) optimal packing design [17] and assume that {b,c} appears in zero blocks.

3) Take three copies of a (v,5,3) minimal covering design and assume in each copy the pair {b,c} appears four times [6].

It is readily checked that the above three steps yield a (v,5,17) minimal covering design for v = 14,34,54,74,94.

For $v \geq 114$ the proof is the same as Lemma 8.3.

Lemma 12.3 Let $v \equiv 18 \pmod{20}$ be a positive integer. Then $\alpha(v,5,17) = \phi(v,5,17)$.

Proof For all $v \equiv 18 \pmod{20}$ the construction is as follows:

1) Take a (v-1,5,4) minimal covering design, [7, 15] and assume that the pairs of {a,b,c} appears in six blocks where {a,b,c} are arbitrary numbers.

2) Take two copies of (v+1,5,4) optimal packing design [17] and in both copies we assume that the pair {v,v+1} appears in zero blocks, so change v+1 to v.

3) Take a (v-1,5,4) optimal packing design and assume that the pair {a,b} appears in zero blocks [17].

4) Take a (v,5,1) minimal covering design [29]. This design has 3v/2 repeated pairs, so if there is a pair, say, {a,b} that appears in three blocks then we are

done. Otherwise, we may assume that {a,b} and {5,b} appear in two blocks. Furthermore, assume in design (2) we have the two blocks < 1 2 3 a c > and < 1 2 3 5 b > where {1,2,3,5} are arbitrary numbers. In the first block change c to b and in the second block change b to c.
It is readily checked that the above construction yields a (v,5,17) minimal covering design for all v ≡ 18 (mod 20), v ≥ 18.

13. Covering With Index 18

<u>Lemma 13.1</u> Let v ≡ 2 (mod 20), v ≥ 22 be a positive integer. Then $\alpha(v,5,18) = \phi(v,5,18)$.

<u>Proof</u> We first construct a (22,5,18) and a (26,5,18) minimal covering design with a hole of size 2 and 6 respectively.
For a (22,5,18), (26,5,18) minimal covering design with a hole of size 2 or 6 respectively take two copies of a (22,5,4), (26,5,4) minimal covering design with a hole of size 2, 6 [17] and Lemma 8.1 together with a (22,5,10), (26,5,10) minimal covering design with a hole of size 2, 6 respectively, Lemma 7.4.
For a (22,5,18) minimal covering design let $X = Z_{22}$, then take the following blocks under the action of the group Z_{22}:

< 0 1 2 3 8 > 3 times	< 0 2 5 11 15 > 3 times	
< 0 3 7 11 17 > 3 times	< 0 1 2 4 13 > twice	
< 0 1 6 13 16 > twice	< 0 2 5 10 18 > twice	< 0 3 7 12 15 >
< 0 1 2 4 10 >	< 0 1 7 11 15 >	< 0 2 7 11 16 >

For v = 42,62,82 take a (v,5,17) minimal covering design together with a (v,5,1) optimal packing design, [2] and notice that the complement graph of the (v,5,1) optimal packing design is a 1-factor while the excess graph of the (v,5,17) minimal covering design has a subgraph that is 1-factor.
For v ≥ 102 the proof is the same as Lemma 8.2.

Lemma 13.2 Let $v \equiv 14$ or $18 \pmod{20}$ be a positive integer. Then $\alpha(v,5,18) = \phi(v,5,18)$.

Proof For $v \equiv 18 \pmod{20}$ the blocks of a (v,5,18) minimal covering design are those of a (v,5,4) and (v,5,14) minimal covering design [15], [16], [7].
For $v \equiv 14 \pmod{20}$ the construction consists of the following three steps:

1) Take two copies of a (v,5,4) minimal covering design, and assume that in both copies the pairs of the triple {a,b,c} appear in six blocks.
2) Take a (v,5,4) optimal packing design [17] and assume that the pair {a,b} appears in zero blocks.
3) Take a (v,5,6) minimal covering design [5].

14. Covering With Index 19

Lemma 14.1 (a) There exists a (22,5,19) and a (26,5,19) minimal covering design with a hole of size 2 and 6 respectively.
(b) Let $v \equiv 2 \pmod{20}$, $v \geq 22$, be a positive integer Then $\alpha(v,5,19) = \phi(v,5,19)$.

Proof (a) For a (22,5,19) minmal covering design with a hole of size 2 take 3 copies of a (22,5,4) minimal covering design with a hole of size 2 [17] together with a (22,5,7) minimal covering design with a hole of size 2 [6].
For a (26,5,19) minimal covering design with a hole of size 6 take 3 copies of a (26,5,4) minimal covering design with a hole of size 6, Lemma 8.1, together with a (26,5,7) minimal covering design with a hole of size 6 [6]
(b) For v = 22 let $X = Z_{20} \cup H_2$ then the blocks are the following under the action of the group Z_{20}.

< 0 1 3 7 12 > 5 times	< 0 1 3 9 13 > 5 times	
< 0 1 3 6 16 > 3 times	< 0 1 3 15 h_1 > twice	
< 0 2 6 15 h_1 > twice	< 0 1 2 5 h_2 >	< 0 1 5 10 h_2 >
< 0 1 8 13 h_2 >	< 0 2 6 13 h_2 >	< 0 4 10 h_1 h_2 >

For v = 42,62,82 take the blocks of a (v,5,1) optimal packing design [2] together with the blocks of a (v,5,18) minimal covering design, Lemma 13.1, and notice that the complement graph of the packing design is a 1-factor while the excess graph of the covering design is a two 1-factor.
For all other values of v the proof is the same as Lemma 8.3

Lemma 14.2 Let $v \equiv 14$ or 18 (mod 20) be a positive integer. Then $\alpha(v,5,19) = \phi(v,5,19)$.

Proof For $v \equiv 18$ (mod 20) the blocks of a (v,5,19) minimal covering design are those of a (v,5,7) and a (v,5,12) minimal covering design [6] [7].
For v = 14,34,54,74,94 the construction is as follows:

1) Take two copies of a (v,5,4) minimal covering design [7, 15], and assume in both copies the pairs of the triple {a,b,c} appear in six blocks.
2) Take two copies of a (v,5,4) optimal packing design, [17]. Assume that in the first copy the pair {a,b} appears in zero blocks, and in the second copy the pair is {a,c}.
3) Take a (v,5,3) minimal covering design [6].

For $v \geq 114$, the proof is the same as lemma 8.3.

Conclusion: In this paper we have shown that $\alpha(v,5,\lambda) = \phi(v,5,\lambda)+e$, where e is as before, for the values of v stated in Theorem 1.4. Furthermore, if the possible exceptions of Theorem 1.3 and Theorem 1.4 are removed then $\alpha(v,5,\lambda) = \phi(v,5,\lambda)+e$ for all $\lambda > 1$ and $v \geq 5$.

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