Edge Domination Critical Graphs

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Abstract

A set of edges D in a graph G is a dominating set of edges if every edge not in D is adjacent to at least one edge in D. The minimum cardinality of an edge dominating set of G is the edge domination number of G, denoted $D_E(G)$. A graph G is edge domination critical, or *EDC*, if for any vertex v in G we have $D_E(G-v) = D_E(G) - 1$. Every graph G must have an induced subgraph F such that F is *EDC* and $D_E(G) = D_E(F)$. In this paper we prove that no tree with more than 2 vertices is *EDC*, develop a forbidden subgraph characterization for the edge domination number of a tree, and we develop a construction that conserves the *EDC* property.

1 Preliminaries

We consider only finite, undirected graphs $G(V, E)$ where V is the vertex set and E is the edge set. All graphs are loopless and have no multiple edges. A **bridge** of a graph G is an edge e whose removal increases the number of connected components of G. A subgraph F of a graph G is a graph such that every vertex and edge of F is contained in G. We say F is an **induced subgraph** G if F is a subgraph of G and two vertices are adjacent in F if and only if they are adjacent in G. For other terminology used in this paper please see [1].

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A subset of edges D of E of a graph $G(V,E)$ is called an **edge dominating set** of G if each edge in $E-D$ is adjacent to at least one edge in D. The **edge domination number** of G, denoted $D_E(G)$, is the cardinality of a minimum edge dominating set of G. The edge domination number of a graph was first discussed in [2] and in [3]. Yannakakis and Gavril show that the problem of determining the edge domination number of bipartite graphs with degree at most three is NP-complete. However, as is shown by Mitchell and Hedetniemi [4] a minimum edge dominating set can be found for a tree in linear time. Forcade [5] discusses the edge domination of the n-cube and this work is generalized by Cutler [6]. Georges et. al. [7] find formulas for the edge domination number of many classes of graphs as well as give a forbidden subgraph characterization for graphs with edge domination number 1. In this paper we continue the study of forbidden subgraphs with respect to edge domination. We note that if v is vertex of a graph G then $D_E(G)-1 \leq D_E(G-v) \leq D_E(G)$. This leads us to our first definition.

Definition 1 *Let G be a graph with edge domination number $D_E(G)=k$. G is called a* **k-edge domination critical graph** *($k-EDC$) if $D_E(G-v)=k-1$ for every $v \in G$. We say a graph G is* **edge domination critical** *, or EDC, if it is* **k-edge domination critical** *for some k.*

It is easy to show that if F is an induced subgraph of G then $D_E(F) \leq D_E(G)$. This implies that every graph G has an EDC graph F as an induced subgraph such that F has the same domination number as G. If we determine all of the $k-EDC$ graphs then we have essentially characterized graphs whose edge domination number is less than k. Georges et. al. [7] find all of the $2-EDC$ graphs. Given the NP-completeness of the general edge domination problem it seems worthwhile to investigate EDC graphs. We note before we go on, that a graph G is EDC if and only if each component of G is EDC.

As a start, consider the complete graph K_n on n vertices. The edge domination number of K_n is $\frac{n}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd. So, K_n is EDC if and only if n is even. Jayram [8] computes the edge domination numbers for the path P_n on n vertices and for the cycle C_n on n vertices. For completeness we give the numbers here.

$$D_E(P_n) = \begin{cases} n \equiv 0(\text{mod}3) & \frac{n}{3} \\ n \equiv 1(\text{mod}3) & \frac{n+2}{3} \\ n \equiv 2(\text{mod}3) & \frac{n+1}{3} \end{cases}$$

$$D_E(C_n) = \begin{cases} n \equiv 0(\text{mod}3) & \frac{n}{3} \\ n \equiv 1(\text{mod}3) & \frac{n+2}{3} \\ n \equiv 2(\text{mod}3) & \frac{n+1}{3} \end{cases}$$

Given that when a vertex is removed from C_n the result is P_{n-1} the following proposition is straightforward.

Proposition 1 *C_n is EDC if and only if n is either 1 or 2* (mod3).

2 Examples of EDC Graphs

We begin with a proposition that examines the structure of *EDC* graphs.

Proposition 2 *Let G be a connected EDC graph with at least three vertices. If an edge e of G is a bridge of G then any non-trivial component of $G - e$ can not be a tree.*

Proof: Let G be $k - EDC$ and suppose for the sake of contradiction that e is a bridge and that one non-trivial component of $G - e$ is a tree. Note that this tree together with e is still a tree which we denote by T. Let P be a longest path in T and let x_1, x_2, and x_3 be the first three vertices on P. We may assume that the edge e is not incident with x_1. This means that in G the vertex x_1 has degree 1. Now, $G - x_3$ has edge domination number $k - 1$. Let $D = \{e_1, ..., e_{k-1}\}$ be a minimum edge dominating set for $G - x_3$. For D to be a dominating set one of the edges, say e_1, in D must either be x_1x_2 or x_2y for some vertex y. If e_1 is x_1x_2 then $D - e_1 \cup \{x_2x_3\}$ is an edge dominating set for G which is a contradiction. If e_1 is x_2y then y must be in T and y must have degree 1. Otherwise P would not be a longest path in T. Thus, $D - x_2y \cup \{x_2x_3\}$ is an edge dominating set of G which is a contradiction.
□

Since every tree with at least three vertices has a bridge whose removal leaves at least one non-trivial component we have the following corollary.

Corollary 1 *Any tree with at least three vertices is not EDC.*

So, the complete graph on two vertices, K_2, is the only tree which is EDC. We will denote by nK_2 the graph which consists of n independent edges. It is easy to see that nK_2 is *EDC*. Using this result, we have the next corollary.

Corollary 2 *If F is a forest then F is EDC if and only if F is isomorphic to nK_2 for some n.*

Given a tree T, T must have an induced subgraph F (which is a forest) such that F is *EDC* and $D_E(T) = D_E(F)$. However, F must be isomorphic to nK_2 by the above corollary. This, along with the fact that $D_E(nK_2) = n$ gives us the following proposition.

Proposition 3 *A tree T has edge domination number n if and only if T has nK_2 as an induced subgraph but does not have $(n+1)K_2$ as an induced subgraph.*

The previous three results first appeared in [9].

Although trees are not *EDC* graphs it certainly is not the case that all *EDC* graphs are two connected. We will now present several families of examples that show this. In what follows a **pendant edge** of a graph is an edge that contains a vertex of degree one. An **n-crown**, denoted CR_n, is a cycle on n vertices with a pendant edge attached to each vertex on the cycle. In [7] the following result is proved.

Proposition 4 *$D_E(CR_n)$ is $\frac{n+1}{2}$ if n is odd and $\frac{n}{2}$ is n is even.*

The next definition generalizes the idea of an n-crown.

Definition 2 *The* **partial crown on n+k vertices,** *denoted $PC_{n,k}$ where $k \leq n$, is a cycle with n vertices and k pendant edges incident to adjacent vertices on the cycle. See Figure 1 for examples of partial crowns.*

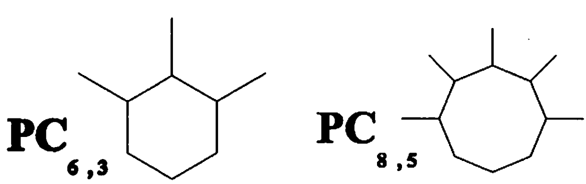


Figure 1: Two *EDC* partial crowns

If k is either 0, 1, or 2 then it is easy to see that $D_E(PC_{n,k}) = D_E(C_n)$

Proposition 5 *If k is odd and $k \geq 3$ then the edge domination number for $PC_{n,k}$ is as follows:*

$$D_E(PC_{n,k}) = \begin{cases} n-k \equiv 0(\text{mod}3) & \frac{2n+k+3}{6} \\ n-k \equiv 1(\text{mod}3) & \frac{2n+k+1}{6} \\ n-k \equiv 2(\text{mod}3) & \frac{2n+k-1}{6} \end{cases}$$

Proof: At least $\frac{k+1}{2}$ edges are needed to edge dominate the k pendant edges since each dominating edge can at most dominate two pendant edges. Use the edge incident to the first two pendant edges as the first dominating edge

and then use every other edge around the cycle until all k pendant edges are edge dominated by the minimal $\frac{k+1}{2}$ dominating edges. This leaves a path with $n-2k-1$ edges left to be dominated. Using the domination numbers for paths given previously, the desired result immediately follows.
□

Proposition 6 *If k is even and $k \geq 4$ then the edge domination number for $PC_{n,k}$ is as follows:*

$$D_E(PC_{n,k}) = \begin{cases} (n-k) \equiv 0(\text{mod}3) & \frac{2n+k}{6} \\ (n-k) \equiv 1(\text{mod}3) & \frac{2n+k-2}{6} \\ (n-k) \equiv 2(\text{mod}3) & \frac{2n+k+2}{6} \end{cases}$$

Proof: At least $\frac{k}{2}$ edges are needed to edge dominate the k pendant edges since each dominating edge can at most dominate two pendant edges. Use the edge incident to the first two pendant edges as a dominating edge. By using every other edge in the cycle as dominating edges, all pendant edges are edge dominated by the minimal $\frac{k}{2}$ dominating edges. These k edges dominate $2k+1$ of the cycle edges leaving a path with $n-2k-1$ edges left to be dominated. Using the domination numbers for paths given previously, the desired result immediately follows.
□

Proposition 7 *Let $k \geq 3$. $PC_{n,k}$ is EDC if and only if k is odd and $(n-k) \equiv 0(\text{mod}3)$.*

Proof: Suppose that $PC_{n,k}$ is EDC and k is even. By removing a degree one vertex from $PC_{n,k}$ it is possible to produce $PC_{n,k-1}$. Using Propositions 5 and 6 it is easy to see these two graphs have the same edge domination number, which is a contradiction. Thus, if k is even, $PC_{n,k}$ is not EDC. Therefore, for $PC_{n,k}$ to be EDC, k must be odd.

For k odd, assume $k = 3$. If $(n-k) \equiv 1(\text{mod}3)$ then remove a vertex from $PC_{n,3}$ to produce $PC_{n,2}$. Now, $D_E(PC_{n,2}) = \frac{n+2}{3}$ which by Proposition 6 is the edge domination of $PC_{n,3}$. This is a contradiction. A similar contradiction can be reached if we assume $(n-k) \equiv 2(\text{mod}3)$. So, in the case $k = 3$ we must have $(n-k) \equiv (\text{mod}0)$. Assume $k > 3$. If $n-k$ is either 1 or 2 (mod 3) then we can use Propositions 5 and 6 to show $D_E(PC_{n,k}) = D_E(PC_{n,k-1})$ which is a contradiction.

Suppose that k is odd and $(n-k) \equiv 0(\text{mod}3)$. Let e_p be any pendant edge. If there are an even number of pendant edges to either side of e_p then we can dominate those pendant edges with $\frac{k-1}{2}$ edges from the cycle and none of the edges will be incident with e_p but will dominate at least k

edges on the cycle. Let D_1 be the set containing these $\frac{k-1}{2}$ edges. We can dominate the remaining $n-k$ edges on the cycle with $\frac{n-k}{3}$ edges so that none of the edges are incident with e_p. Let D_2 be the set containing these $\frac{n-k}{3}$ edges. Let $D = D_1 \cup D_2 \cup \{e_p\}$. Every edge of $PC_{n,k}$ other than e_p is incident to an edge of $D - e_p$. This implies that if either vertex of e_p is removed from $PC_{n,k}$ the resulting graph has edge domination number $\frac{2n+k-3}{6}$. A similar result can be shown if there is an odd number edges to either side of e_p. Now, let v be a vertex in $PC_{n,k}$ not incident to a pendant edge. If v is removed from $PC_{n,k}$ the resulting graph can be dominated as follows: use $\frac{k+1}{2}$ from the remaining cycle edges to dominate the k pendant edges and at least $k+1$ remaining edges from the cycle. The remaining $n-k-3$ cycle edges can be dominated by $\frac{n-k-3}{3}$ edges. Thus, for every vertex removed from $PC_{n,k}$ the edge domination number of the resulting graph decreases by one.
□

Using Proposition 7 the next corollary immediately follows.

Corollary 3 *CR_n is EDC if and only if n is odd.*

We next consider what happens when we take a CR_n and add chords to the cycle by making pairs of vertices that are not adjacent on the cycle adjacent.

Definition 3 *A* **crown with chords on 2n vertices,** *denoted CRC_n is a graph with n pendant edges that has CR_n as a spanning subgraph.*

For example, Figure 2 shows all possible CRC_5 graphs.

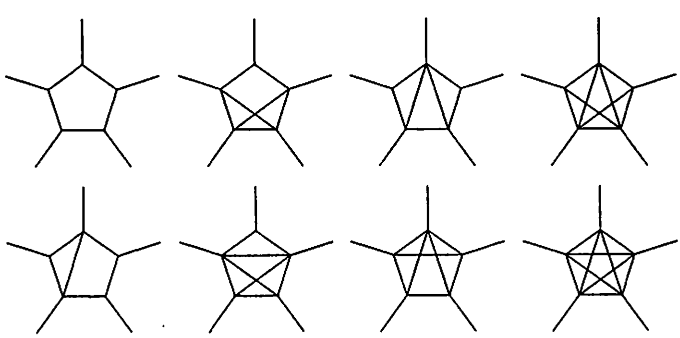


Figure 2: The eight CRC_5 graphs

Proposition 8 *A CRC_n graph is EDC if and only if n is odd.*

Proof: Let G be a CRC_n graph with n odd. Consider the set of edges, D, that contain $\frac{n-1}{2}$ disjoint edges from the cycle and the one pendant edge that is not incident to edges chosen from the cycle. Since any chord in the graph is incident to only 2 pendant edges, no fewer than $\frac{n+1}{2}$ edges can dominate G. Thus, $D_E(G) = \frac{n+1}{2}$. If any vertex is removed from G then the resulting graph can be dominated by $\frac{n-1}{2}$ disjoint edges from the cycle. So if n is odd, G is EDC.

If n is even then an argument similar to before shows that $D_E(G) = \frac{n}{2}$. Now, if a degree one vertex is removed from G there will be $n-1$ pendant edges left and at least $\frac{n}{2}$ are needed to dominate the pendant edges. Thus if n is even, G is not EDC.
□

3 A Construction, *EDC* structure, and Trees

In the previous section, we examined several classes of EDC graphs and proved that trees are not EDC. In this section, we will take a different approach to examine EDC graphs. Starting with a general construction that uses EDC graphs to generate new EDC graphs, we will show how to use trees to construct new EDC graphs.

The following construction will take two EDC graphs and generate another EDC graph from them.

Construction 1 *Let G be an EDC graph with $D_E(G) = n$ and pendant edge e_p incident to vertex v_1, and vertex v_p of degree 1. Let G' be an EDC graph with vertex v_2 and $D_E(G') = m$. Let H be the graph constructed from $G - v_p$ and G' with vertices v_1 and v_2 identified. H is an EDC graph with $D_E(H) = n + m - 1$.*

Proof: Label the identified vertex in H, v_y. Since $H - v_y$ is the union of $G - v_y$ and $G' - v_y$, $D_E(H - v_y) = n + m - 2$. Therefore, for G to be an EDC graph, $D_E(H) = n + m - 1$ and $D_E(H - v) = n + m - 2$ for all vertices in H.

The pendant edge e_p in G has two vertices v_1 and v_p, where vertex v_p has degree 1. Since graph G is EDC, $D_E(G - v_p) = n - 1$. In addition, for G to be EDC, every minimum edge dominating set that dominates $G - v_p$ with $n - 1$ edges cannot have an edge incident to vertex v_1 otherwise G would be dominated by $n - 1$ edges. Identifying vertices v_1 and v_2 allows no additional reduction in edge domination number since no dominating edge of $G - v_p$ is incident to v_1. Therefore $D_E(H) = n + m - 1$. Removing any vertex v other than v_y in H yields either:

I. $G - v_p$ and $G' - v$ or

II. $G - v - v_p$ and G'

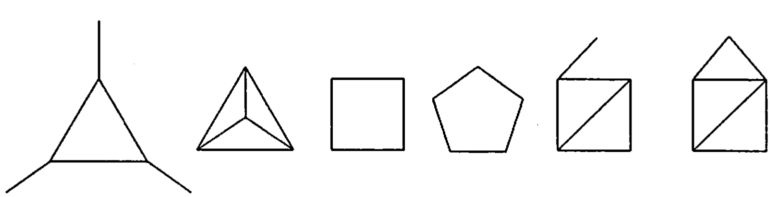
with vertices v_1 and v_2 identified. Since $D_E(H - v) \geq n + m - 2$, to prove that H is EDC we must find a minimum dominating edge set that dominates $H - v$ with $n + m - 2$ edges for these two cases.

Case I: Since G and G' are EDC $D_E(G' - v) = m - 1$ and $D_E(G - v_p) = n - 1$ and therefore H can be dominated by $n + m - 2$ edges.

Case II: Since G' is EDC, $D_E(G' - v_2) = m - 1$. This means that all of the edges of G' can be dominated by $m - 1$ edges except for edges incident to vertex v_2. All of G' would be dominated if there exists a minimum dominating edge set with an edge in G incident to vertex v_1. Since G is EDC, $G - v$ can be dominated by $n - 1$ edges. For edge e_p to be dominated, one of the $n - 1$ dominating edges must be incident to vertex v_1. Therefore subgraph G' is dominated by the $m - 1$ edges that dominate $G' - v_2$ plus the edge in G incident to v_1. G is dominated by $n - 1$ edges. So $H - v$ is dominated by $n + m - 2$ edges.

Combining Cases I and II proves that for all vertices in H, $D_E(H - v) = n + m - 2$. Therefore H is an EDC graph.
□

Construction 1 is an extremely powerful and general way to create new EDC graphs. As an example of the speed and versatility of this approach, we include 24 EDC graphs with domination number 3 that can be generated using this construction. To create $3 - EDC$ graphs out of other EDC graphs, we need to join two $2 - EDC$ graphs. The six connected $2 - EDC$ graphs found by [7] are shown in Figure 3.



CR_3 K_4 C_4 C_5 A_1 A_2

Figure 3: The 6 connected $2 - EDC$ graphs

To make new graphs using Construction 1 one of the original EDC graphs must contain a pendant edge. This only leaves CR_3 and A_1 to be used as the pendant graphs which can then be joined to other $2-EDC$ graphs. These graphs are shown in Figure 4.

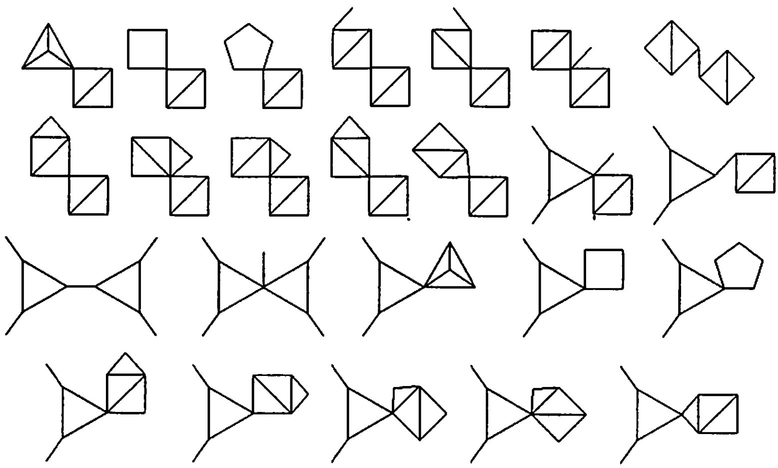


Figure 4: $3-EDC$ graphs

This example immediately shows that there are an immense amount of graphs that the edge domination number can be immediately determined using Construction 1. Using just these graphs as seed graphs, the number of graphs that can be generated this way grows explosively as the domination number increases.

Though no tree with more than three vertices is EDC, we can use trees and Construction 1 to construct new EDC graphs. For a start, take a path P_n with vertices $v_1, ..., v_n$. Each vertex in the path corresponds to a CR_3 graph, and each edge in the path corresponds to an identified pendant edge between two CR_3 graphs.

To show that this is indeed EDC, we will start by induction with a path of length zero, a single vertex. This is associated to CR_3 which is EDC. Assume now that a P_n has an associated EDC as described above, P_{n+1} can be generated from P_n using Construction 1 as follows. Let a new CR_3 be graph G in Construction 1. By removing one of the pendant edges from CR_3 and identifying the degree 2 vertex with a degree 1 vertex on the last

CR_3 associated with P_n, a new EDC graph for P_{n+1} is formed with the properties described above.

Examples of the associated EDC graphs for P_2, P_3 and P_6 are shown in Figure 5.

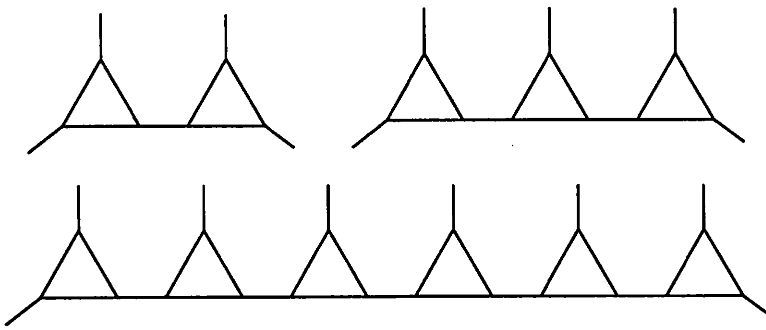


Figure 5: EDC graphs associated with P_2, P_3 and P_6

The graphs described above are just one possible associated EDC graph for each P_n. A nice stuctural property of these graphs is that although the edge domination number can be arbitrarily large, each of these graphs have maximum circumference 3.

Picking the CR_3 graph as the seed graph was somewhat arbitrary since any EDC crown graph can be used to be associated with each of the vertices in the path. So for example, EDC graphs associated with P_3 can also be any of the graphs shown in Figure 6.

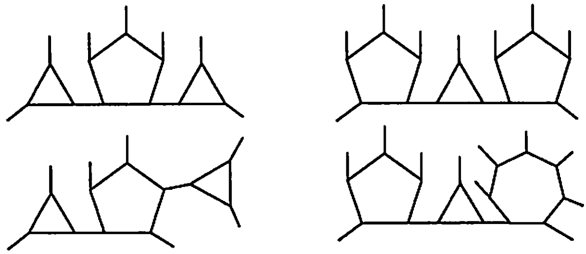


Figure 6: additional EDC graphs associated with P_3

The two graphs on the left of Figure 6 also show that the choice of edge to which the EDC seed graphs are attached is arbitrary as well.

The same stuctures examined with respect to path graphs can be generalized to apply to any arbitrary tree. To do this we start with a given tree T. To generate the associated EDC graph, each vertex v in T corresponds to a crown CR_n such that n is odd and $n \geq$ degree v. This ensures that there are enough pendant edges available to join the associated crowns of adjacent vertices in T.

Take any vertex in T as a starting point. For each adjacent vertex in T, use Construction 1 (as we did for paths) to join the crowns associated with the two vertices in T together. This intermediate graph is EDC. Continue joining crowns associated with adjacent vertices in T until all of the crowns are part of one large connected graph. Since Construction 1 was used multiple times creating EDC graphs at each step, this final graph is EDC.

Due to this process of construction, there are numerous forms of EDC graphs associated with each tree, and all of these graphs share similar properties.

In conclusion, we will examine one consequence of Construction 1 and end with a prediction about the maximum size of $k - EDC$ graphs. Note that each crown graph G with $D_E(G) = k$ has $4k-2$ vertices. This property is preserved under Construction 1.

Proposition 9 *Let G be a $k - EDC$ graph associated with some tree T. G has $4k - 2$ vertices.*

Proof: Using Construction 1, if we start with two graphs with l and m vertices respectively, the new graph will contain $l + m - 2$ vertices since one vertex is removed and one vertex is identified. A crown graph with edge domination number d has $4d - 2$ vertices. Joining two crown graphs with domination numbers n and p, yields a graph with $4n + 4p - 6$ vertices. This graph has domination number $d' = n + p - 1$, so the number of vetices is $4d' - 2$. Since the EDC graph associated with the tree is generated using crowns and Construction 1, at every stage of the construction, the two graphs that are joined conserve this property.
□

We believe this to be an upper bound on all EDC graphs, and all of the associated tree graphs presented in this paper achieve this upper bound. So in conclusion we end with the following Conjecture.

Conjecture 1 *Any $k - EDC$ graph has at most $4k - 2$ vertices.*

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