On the Bipacking Numbers $g_2^{(4)}(v)$

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ABSTRACT. The quantity $g_2^{(k)}(v)$ is the minimum number of blocks in a family of blocks from a v-set that covers all $\binom{v}{2}$ pairs exactly twice, given the restriction that the longest block in the covering family has length k (there may be many blocks of length k). We give certain results for the case $k = 4$.

1 Introduction

Suppose we have a set of v elements and we wish to cover all pairs from this set exactly twice by a family F of blocks subject to the restriction that the longest block (not necessarily unique) in F has length k. There will be many such covering families but we wish to determine the minimum cardinality of such a family F in the case $k = 4$. We denote this minimum cardinality by $g_2^{(4)}(v)$.

The results for $k = 2$ and $k = 3$ are straightforward (cf. [2]). Also, if $k = 3n + 1$, there is a BIBD $(3n + 1, \frac{n(3n+1)}{2}, 2n, 4, 2)$ and so there exists a bicovering with all blocks of size 4. Thus $g_2^{(4)}(3n + 1) = \frac{n(3n+1)}{2}$.

We next look at the case $v = 3n$.

2 An Example with $v = 24$

The behaviour for $v = 3n$ is most readily understood by looking at an example. Suppose we take $n = 8$; then $v = 24$, and we know there is a BIBD $(25, 100, 16, 4, 2)$. Obviously, if we delete one point we have a bicovering for $v = 24$ and this bicovering contains 84 quadruples and 16 triples. So $g_2^{(4)}(24) \leq 100$. If we were to use fewer than 84 quadruples, we would need more than 100 blocks. Also, the maximum number of quadruples in a bipacking is given by

$$D_2(2,4,24) \leq \frac{24}{4}\left\lfloor \frac{23(2)}{3} \right\rfloor = 90.$$

So we need merely consider cases when the number Q of quadruples lies in the range $84 \leq Q \leq 90$. We construct the following table.

Q	PNC	PUV	AV	AB	RB
90	12	24	0	0	102
89	18	20	8	4	103
88	24	16	16	6	102
87	30	12	24	8	101
86	36	8	32	12	102
85	42	4	40	14	101
84	48	0	48	16	100

In this table, PNC represents the number of pairs that are not covered by quadruples. Thus

$$PNC = 24(23) - 6Q.$$

The column PUV represents the minimum number of elements of unit valence among the PNC pairs. Since the valencies in the defect graph formed by the missing pairs are necessarily $1, 4, 7, \ldots$, we can immediately calculate that

$$PUV(90) = 24,$$

and, in general, PUV will decrease by 4 units as Q decreases by one unit until $PUV(84) = 0$.

For any value of Q, the minimum number of blocks will be achieved if we use $\frac{PUV}{2}$ pairs and distribute the other pairs in as many triples as possible. So we form a column of available pairs AV (having used PUV pairs in $\frac{PUV}{2}$ blocks). Clearly $AV = PNC - \frac{PUV}{2}$. The AV pairs then could produce $\lfloor \frac{AV}{3} \rfloor$ triples and $AV - 3\lfloor \frac{AV}{3} \rfloor$ pairs if we use as many triples as possible. This number is recorded in the column AB (additional blocks). The final column (minimum number of required blocks) is given by

$$RB = Q + \frac{PUV}{2} + AB.$$

This table establishes that $g_2^{(4)}(24) = 100$.

In the next section, we show that this procedure is quite general, that is,

$$g_2^{(4)}(3n) = \frac{n(3n+1)}{2}.$$

3 General Discussion of the Case $v = 3n$

We proceed as in the last section and use the BIBD $(3n+1, \frac{n(3n+1)}{2}, 2n, 4, 2)$. Then deletion of a single point leaves

$$\frac{n(3n+1)}{2} - 2n = \frac{3n(n-1)}{2} = 3\binom{n}{2}$$

quadruples and $2n$ triples; so

$$g_2^{(4)}(3n) \leq \frac{n(3n+1)}{2}.$$

Now suppose the number of quadruples is given by $Q = 3\binom{n}{2} + i$. These quadruples cover $18\binom{n}{2} + 6i$ pairs. Then

$$\begin{aligned} PNC &= 3n(3n-1) - 18\binom{n}{2} - 6i \\ &= 6n - 6i = 6(n-i). \end{aligned}$$

Also

$$PUV = 4i.$$

Hence

$$\begin{aligned} AV &= 6(n-i) - 2i \\ &= 6n - 8i \\ &= 3(2n-3i) + i. \end{aligned}$$

We now write $i = 3a + \lambda$ ($\lambda = 0, 1,$ or 2). Then

$$AB = 2n - 3i + a + \lambda$$

So the minimum number of blocks is at least

$$\begin{aligned} Q + \frac{PUV}{2} + AB &= 3\binom{n}{2} + i + 2i + 2n - 3i + a + \lambda \\ &= 3\binom{n}{2} + 2n + a + \lambda. \end{aligned}$$

Since $a + \lambda > 0$, we have established that

$$\begin{aligned} g_2^{(4)}(3n) &= 3\binom{n}{2} + 2n \\ &= \frac{n(3n+1)}{2}. \end{aligned}$$

4 The Case $v = 3n + 2$, n even

If $n = 2m$, then we have $6m + 2$ elements. Since

$$D_2(2, 4, 6m + s) = \left\lfloor \frac{6m+2}{4} \left\lfloor \frac{12m+2}{3} \right\rfloor \right\rfloor = m(6m + 2),$$

the optimal bicovering would have $Q = m(6m+2)$ quadruples. This would leave $6m + 2$ pairs uncovered, and so the best we could hope for would be $P = 2$ pairs and $T = 2m$ triples. So such an optimal bicovering would have $6m^2 + 4m + 2$ blocks.

To illustrate the structure of such a bicovering, let us take an example with $m = 3$ ($P = 2$, $T = 6$, $Q = 60$). Let an element occur λ_i times in blocks of length i ($i = 2, 3, 4$). Then $\lambda_2 + 2\lambda_3 + 3\lambda_4 = 38$. The possible values for $(\lambda_2, \lambda_3, \lambda_4)$ are $(2, 0, 12), (0, 1, 12), (1, 2, 11), \ldots$. Suppose there are a_{12} blocks with $\lambda_4 = 12$, a_{11} blocks with $\lambda_4 = 11$, etc. Then

$$a_{12} + a_{11} + a_{10} + \ldots = 20,$$
$$12a_{12} + 11a_{11} + 10a_{10} + \ldots = 60(4) = 240.$$

Multiply the first equation by 12 and subtract. The resulting equation shows that $a_i = 0$ for $i < 12$. Hence $a_{12} = 20$, and we immediately find that there are 2 elements with $(\lambda_2, \lambda_3, \lambda_4) = (2, 0, 12)$ and 18 elements with $(\lambda_2, \lambda_3, \lambda_4) = (0, 1, 12)$. This procedure is quite general. The general values for $(\lambda_2, \lambda_3, \lambda_4)$ are $(2, 0, 4m)$, $(0, 1, 4m)$, $(1, 2, 4m - 1)$, etc. If there are a_{4m-i} elements with $\lambda_4 = 4m - i$, we have

$$\sum_{i \geq 0} a_{4m-i} = 6m + 2,$$
$$\sum_{i \geq 0} (4m - i) a_{4m-i} = m(6m + 2)(4).$$

Multiply the first equation by $4m$, and we at once deduce that $a_{4m-i} = 0$ for $i > 0$. Hence the only possiblilities for $(\lambda_2, \lambda_3, \lambda_4)$ are $(2, 0, 4m)$ and $(0, 1, 4m)$. It follows that there are 2 elements of the first type and $6m$ elements of the second type.

The case $m = 1$ ($v = 8$) is exceptional; it was shown in [2] that the bound $6m^2 + 4m + 2$ can not be met, and the solutions for a bicovering in 13 blocks were given. However, for $m > 1$, Griggs and Grannell [1] have obtained solutions in $6m^2 + 4m + 2$ blocks.

5 The Case $v = 3n + 2$, n odd

In this case, let $n = 2m + 1$; then $v = 6m + 5$. Then

$$\begin{aligned} D_2(2, 4, 6m+5) &= \left\lfloor \frac{6m+5}{4} \left\lfloor \frac{12m+8}{3} \right\rfloor \right\rfloor \\ &= \left\lfloor \frac{6m+5}{4}(4m+2) \right\rfloor \\ &= \left\lfloor \frac{24m^2+32m}{4} + 10 \right\rfloor \\ &= 6m^2 + 8m + 2. \end{aligned}$$

This leaves a total of $(6m+5)(6m+4) - 6(6m^2+8m+2) = 6m+8$ pairs uncovered, and so the optimal bicovering would have $P = 2$, $T = 2m+2$. The total number of blocks would then be $6m^2 + 10m + 6$.

Let us again start with an example, say $m = 3$. Then we have 23 elements and we want 2 pairs, 8 triples, and 80 quadruples. In this case, using the previous notation, $\lambda_2 + 2\lambda_3 + 3\lambda_4 = 44$, and $(\lambda_2, \lambda_3, \lambda_4)$ can be $(2, 0, 14)$, $(0, 1, 14)$, $(1, 2, 13)$, $(2, 3, 12)$, $(0, 4, 12)$, etc. We use a_i as before and have

$$\begin{aligned} a_{14} + a_{13} + a_{12} + \ldots &= 23, \\ 14a_{14} + 13a_{13} + 12a_{12} + \ldots &= 80(4) = 320. \end{aligned}$$

It follows that

$$a_{13} + 2a_{12} + 3a_{11} + \cdots = 14(23) - 320 = 2.$$

Hence $a_{13} = 2$ or $a_{12} = 1$. This leads to three possibilities.

Case 1. Two elements with pattern $(1, 2, 13)$, 21 elements with patterns $(2, 0, 14)$ or $(0, 1, 14)$.

Case 2. One element with pattern $(2, 3, 12)$, 22 elements with patterns $(2, 0, 14)$ or $(0, 1, 14)$.

Case 3. One element with pattern $(0, 4, 12)$, 22 elements with patterns $(2, 0, 14)$ or $(0, 1, 14)$.

These cases then become:

(1) 2 elements of pattern (1,2,13),
1 element of pattern (2,0,14),
20 elements of pattern (0,1,14).

(2) 1 element of pattern (2,3,12),
1 element of pattern (2,0,14),
21 elements of pattern (0,1,14).

(3) 1 element of pattern (0,4,12),
2 elements of pattern (2,0,14),
20 elements of pattern (0,1,14).

If we now look at the general case, we have $P = 2$, $T = 2m + 2$, $Q = 6m^2 + 10m + 6$. Then $\lambda_2 + 2\lambda_3 + 3\lambda_4 = 12m + 8$. The possibilities for $(\lambda_2, \lambda_3, \lambda_4)$ are $(2, 0, 4m + 2), (0, 1, 4m + 2), (1, 2, 4m + 1), \ldots$. We obtain the equations

$$a_{4m+2} + a_{4m+1} + a_{4m} + \ldots = 6m + 5,$$
$$(4m + 2)a_{4m+2} + (4m + 1)a_{4m+1} + \ldots = 4(6m^2 + 8m + 2).$$

It follows that $a_{4m+1} + 2a_{4m} + \cdots = 2$, and so $a_{4m+1} = 2$ or $a_{4m} = 1$.

As in the example, we end up with 3 cases, namely

(1) 2 elements of pattern $(1, 2, 4m + 1)$,
1 element of pattern $(2, 0, 4m + 2)$,
$6m + 2$ elements of pattern $(0, 1, 4m + 2)$.

(2) 1 element of pattern $(2, 3, 4m)$,
1 element of pattern $(2, 0, 4m + 2)$,
$6m + 3$ elements of pattern $(0, 1, 4m + 2)$.

(3) 1 element of pattern $(0, 4, 4m)$,
2 elements of pattern $(2, 0, 4m + 2)$,
$6m + 2$ elements of pattern $(0, 1, 4m + 2)$.

It is, of course, obvious that, for $m = 0$ (when $P = T = Q = 2$), the second case can not occur since one can not have $\lambda_3 = 3$. Also the third case can not occur since $\lambda_2 = 4$ is not possible. It immediately follows that, for $m = 0$ ($v = 5$), there is the unique solution given by

xy	yza	$xyab$
xz	yzb	$xzab$

There do exist optimal solutions for the case $m = 1$ ($v = 11$, $P = 2$, $T = 4$, $Q = 16$). Here is one solution for Case 2.

xy	$x12$	$x789$	$y725$	7135
xy	$x34$	$x714$	$y736$	7246
	$x56$	$x856$	$y814$	8136
	789	$x923$	$y823$	8245
			$y915$	9126
			$y946$	9345

And here a solution for Case 3.

xy	$a16$	$x138$	$ya37$	1234
xy	$a25$	$x247$	$ya48$	1278
	$a38$	$x458$	$y235$	3456
	$a47$	$x367$	$y146$	5678
		$xa15$	$y157$	
		$xa26$	$y268$	

These examples lead to the conjecture that there is always a solution in $6m^2 + 10m + 6$ blocks. A complete discussion of the case $v = 11$ will be given in a later paper.

References

[1] M.J. Grannell and T.S. Griggs, personal communication.

[2] R.G. Stanton, Non-isomorphic Minimal Bicovers of K_8, *Ars Combinatoria* **62** (2002), 137–144.