Note
On Forest Isomorphic Decomposition of Cayley Digraphs

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Abstract

We prove that if S is a quasiminimal generating set of a group Γ and F is an oriented forest with $|S| > 2$ arcs, then the Cayley graph $Cay(\Gamma, S)$ can be decomposed into $|\Gamma|$ arc-disjoint subdigraphs, each of which is isomorphic to F.

1 Introduction

A *decomposition* of a directed graph G is a set $P = \{H_1, \ldots, H_k\}$ of pairwise arc disjoint subdigraphs of G whose sets of arcs partition the set of arcs of G. When all digraphs in P are isomorphic to a digraph H, then G is said to be *H-decomposable* and P an *H-decomposition* of G.

The subject of decompositions of graphs and digraphs has been widely studied in the literature. When H is a tree and G is a complete graph, the study of such decompositions is related to some well-known conjectures, particularly the *Graceful* tree conjecture. This conjecture asks for a particular labeling f of the vertices of a tree T of m edges with the set of integers modulo $(2m + 1)$ such that the differences $|f(x) - f(y)|$ for each edge xy of the tree are pairwise different. Such a labeling leads to a T-decomposition of K_{2n+1} (see,

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for instance, the survey of Gallian [1] and the references therein). The problem can be stated in terms of H-decompositions of Cayley graphs.

Recall that, given a group Γ (written multiplicatively) and a set $S \subset G \setminus \{1\}$ of generators of Γ, the Cayley digraph $Cay(\Gamma, S)$ has the elements of Γ as vertices and (x, xg) is an arc of $Cay(\Gamma, S)$ for each $x \in \Gamma$ and each $g \in S$. It is understood that the digraph comes with a coloring of the arcs by the elements in S so that the arc (x, xg) has color g. The undirected Cayley color graph of Γ with respect to S is obtained from $Cay(G, S \cup S^{-1})$ by replacing each pair of arcs $(x, xg), (xg, x)$ by an undirected edge $\{x, xg\}$.

The Ringel-Kotzig conjecture can be stated as saying that each tree T with m edges admits a coloring of its edges with elements of the cyclic group C_{2m+1} such that the Cayley digraph $Cay(C_{2m+1}, C_{2m+1} \setminus \{1\})$ has a *colored* T-decomposition. Even if the known results for decompositions of Cayley graphs are far from what is required by the conjecture, Ruiz [5] proved that $Cay(C_{2m}, C_{2m} \setminus \{1\})$ has a colored F–decomposition for each *linear* forest F, that is, a forest whose connected components are paths. Fink [2] and Ramras [4] prove that the n-cube, a Cayley graph, can be decomposed into an arbitrary tree of n edges. Fink [3] generalizes the result to arbitrary Cayley digraphs when the generating set S is minimal, that is, $S \setminus \{g\}$ generates a proper subgroup of Γ for each $g \in S$, and T is an oriented tree.

In this note we show that the condition of minimality can be relaxed to quasiminimal generating sets. A generating set S of a group Γ is *quasiminimal* if there is an ordering $S = \{g_1, \ldots, g_s\}$ of the elements in $\Gamma \setminus \{1\}$ such that the group generated by the first k elements in S is a proper subgroup of the one generated by the first $(k+1)$ for each $k < s$. On the other hand, we show that the result can be extended to oriented forests with $|S| > 2$ edges (and no isolated vertices). We prove the following result.

Theorem 1 *Let Γ be a group and S a quasiminimal generating set of Γ and let F be an oriented forest with $|S|$ edges and no isolated vertex. Then the graph $Cay(\Gamma, S)$ is F-decomposable, unless $|S| = 2$ and $|V(F)| = |\Gamma| = 4$.* □

2 Decomposing Cayley Digraphs

Lemma 2 *Let Γ be a group and S a quasiminimal generating set of Γ. For each forest F with $|S|$ edges and no isolated vertices, there exists a one-to-one labeling $f : V(F) \to \Gamma$, such that*

$$\{[f(u)]^{-1} f(v) : (u, v) \in E(F)\} = S,$$

unless $|S| = 2$ and $|V(F)| = |\Gamma| = 4$.

Proof: We use induction on the cardinality of $S = \{g_1, \cdots, g_s\}$. Note that, by the quasiminimality of S, the subgroup Γ_k generated by the first $k < s$ generators is a proper subgroup of Γ_{k+1}, the one generated by the first $k+1$. Therefore, we have $|\Gamma| \geq 2^s$.

The result clearly holds when $s = 1$. When $s = 2$ we have $3 \leq |V(F)| \leq 4$. Figure 2 (a), (b) and (c) show the labelings of the three oriented forests with 3 vertices. Suppose that $|V(F)| = 4$. If $|\Gamma| = 4$ then, we have a situation equivalent to either $\Gamma = \mathbb{Z}_4$ and $S = \{2, 1\}$ or $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S = \{(0,1), (1,0)\}$. In either case, it is immediate to see that the labeling f does not exist. If $|\Gamma| > 4$ then either $|\Gamma_1| \geq 3$ or $|\Gamma : \Gamma_1| \geq 3$. Let $x = g_1$ in the first case and $x \notin \Gamma_1 \cup g_2\Gamma_1$ in the second one. Then, the set $\{1, g_2, x, xg_1\}$ has cardinality 4. A labeling for the oriented forest with four vertices is illustrated in Figure 2 (d).

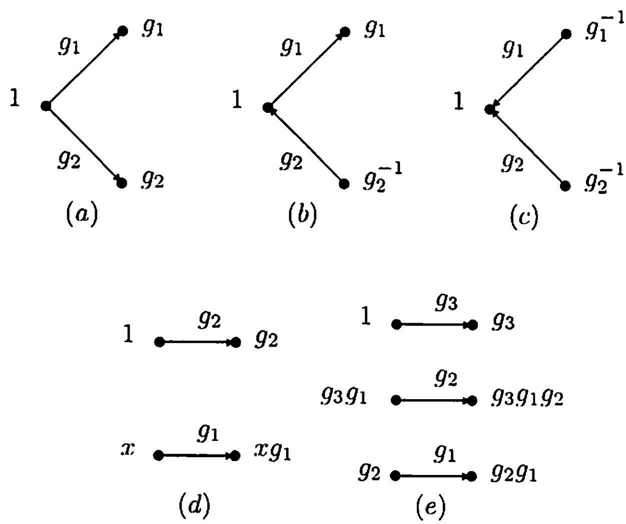

Figure 1: Labelings of oriented forests with $s = 2, 3$.

Let F be a forest of $s \geq 3$ edges. Let Γ_{s-1} the subgroup of Γ generated by the elements of $S \setminus \{g_s\}$. We consider two cases,

Case 1. F has an isolated arc $(a, b) \in E(F)$. Let $F^* = F - \{a, b\}$, which has $s-1 \geq 2$ edges and at most $2(s-1)$ vertices. Suppose first that $|V(F^*)| < |\Gamma_{s-1}|$. By the induction hypothesis, there is a one-to-one labeling $f^* : V(F^*) \to \Gamma_{s-1}$, such that

$$\{[f^*(u)]^{-1} f^*(v) : (u, v) \in E(F^*)\} = S \setminus \{g_s\}.$$

Since $|Im(f^*)| < |\Gamma_{s-1}|$, there is an element $z \in \Gamma^*$ such that $z \notin Im(f^*)$. We define the following labeling, $f : V(F) \to \Gamma$, on the vertices of F

$$f(u) = \begin{cases} f^*(u), & u \in V(F) \setminus \{a, b\} \\ z, & u = a \\ zg_s, & u = b. \end{cases}$$

It is straightforward to check that f is well defined and satisfies the conditions of the Lemma.

Since $2^{2-1} \leq |\Gamma_{s-1}| = |V(F^*)| = |\Gamma_{s-1}| \leq 2s - 2$, it follows that $s = 3$ and $|V(F^*)| = 4$. Consequently, $|V(F)| = 6$ and F consists of 3 isolated arcs. The labeling shown in Figure 2 (d) satisfies then the requirements of the Lemma.

Case 2. F has no isolated arcs. Let $u_0, v_0 \in V(F)$ be adjacent vertices such that v_0 is a pendant vertex of F. Consider the forest $F^* = F - v_0$, which has $s - 1 \geq 2$ arcs and at most $\frac{3s}{2} - 1$ vertices. Since $|V(F^*)| \leq \frac{3s}{2} - 1 < 2^{s-1} \leq |\Gamma_{s-1}|$ for all $s \geq 3$, it follows from the induction hypothesis that there is a one-to-one labeling $f^* : V(F^*) \to \Gamma_{s-1}$, such that

$$\{[f^*(u)]^{-1} f^*(v) : (u, v) \in E(F^*)\} = S \setminus \{g_s\}.$$

Define the labeling, $f : V(F) \to \Gamma$, on the vertices of F as follows,

$$f(u) = \begin{cases} f^*(u), & u \in V(F) \setminus v_0 \\ f^*(u_0) g_s, & u = v_0 \quad \text{and} \quad (u_0, v_0) \in E(F) \\ f^*(u_0) g_s^{-1}, & u = v_0 \quad \text{and} \quad (v_0, u_0) \in E(F). \end{cases}$$

Then f is injective and $\{(f(u))^{-1} f(v), (u, v) \in E(F)\} = S$. □

Proof of Theorem 1 By the above Lemma, there exists a one-to-one labeling $f : V(F) \to \Gamma$, such that

$$\{[f(u)]^{-1} f(v) : (u, v) \in E(F)\} = S.$$

Consider the coloring of the arcs of F with the elements in S given by the labeling f, that is, the arc (x, y) has the color $(f(x))^{-1} f(y)$. Then f can be regarded as a color-preserving map from the arc-colored digraph F into the Cayley color digraph $G = Cay(\Gamma, S)$. Clearly, $F \simeq f(F)$. Denote by F_1 the subdigraph $f(F)$. For each $h \in \Gamma$, the map $\varphi_h(x) = hx$ is a digraph automorphism of $Cay(\Gamma, S)$ which preserves the colors. Suppose that $(x, xg), (y, yg')$ are two arcs of F_1 and $(\varphi_h(x), \varphi_h(xg)) = (\varphi_k(y), \varphi_k(yg'))$ for some $h, k \in \Gamma$. Then we have $g = g'$ and, since there is an only arc in F_1 colored g, we also have $x = y$ and thus $h = k$. Hence, when $h \neq k$, the subdigraphs $\varphi_h(F_1)$ and $\varphi_k(F_1)$ are arc-disjoint. Moreover, for each arc (x, xg) of G there is an arc (y, yg) in F_1 and $(x, xg) \in E(\varphi_{xy^{-1}}(F_1))$. Hence, the set

$$\{\varphi_h(F_1) : h \in \Gamma\}$$

is an F-decomposition of G. □

As a final remark, we note that Theorem 1 extends to decompositions of Cayley color graphs into (unoriented) forests when S is antisymmetric ($S \cap S^{-1} = \emptyset$). In this case we can apply Theorem 1 to any orientation of F.

References

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