A Complete Set of Type 0 Hypercubes not Equivalent to a Latin Set

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Abstract

A set of $n+1$ orthogonal squares of order n is known to be equivalent to a complete set of $n-1$ mutually orthogonal latin squares of order n together with canonical row and column squares. In this note we show that this equivalence does not extend to orthogonal hypercubes of dimensions $d > 2$ by providing examples of affine designs that can be represented by complete sets of type 0 orthogonal hypercubes but not by complete sets of orthogonal latin hypercubes together with canonical hypercubes that generalize the row and column squares in the case where $d = 2$. These examples also clarify the relationship between affine designs and orthogonal hypercubes that generalize the classical equivalence between affine planes and complete sets of MOLS.

We conclude with the statement of a number of conjectures regarding some open questions.

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1 Introduction

In developing a hierarchy of complete sets of orthogonal structures linearly ordered by inclusion that began with affine planes and ended with (t, m, s)-nets, Laywine and Mullen [4] left unanswered the open question:

Is the existence of every set of $(n^d - 1)/(n - 1)$ type 0 hypercubes of dimension d and order n equivalent to the existence of d canonical and $(n^d - 1)/(n - 1) - d$ latin hypercubes of dimension d and order n?

Essentially the same question had been posed earlier by these authors in the last sentence of Section 9.2 in [6, page 161]. In addition to the formulation above, based on the relationship between types of hypercubes, the question has other origins which are mentioned below.

Kishen [2] was among the first to study latin hypercubes. Among other things he showed that a complete set of orthogonal latin hypercubes of dimension d and prime power order n, constructed using linear polynomials with d variables over F_n, the field of n elements, had $(n^d - 1)/(n - 1) - d$ members. In a paper that examined a variety of generalizations of Bose's equivalence between complete sets of MOLS and affine planes, Laywine and Mullen [5, Corollary 4.2] extended Kishen's results to state an equivalence between complete sets of orthogonal latin hypercubes and affine designs. The designs presented in this paper demonstrate that that result should have been restricted to type 0 hypercubes as given by Theorem 1 below.

Later Laywine, Mullen and Whittle [7] showed that the size of a complete set of hypercubes given by Kishen's construction was not dependent on that method of construction. In addition they classified the strength of the latin property in orthogonal hypercubes by describing a hypercube of dimension d and order n to be of *type* j with $0 \leq j \leq d - 1$ if each of the n symbols occurs n^{d-j-1} times in any subarray determined by fixing j of the coordinates.

According to this definition all hypercubes of type j are also hypercubes of type $k < j$ and Kishen's latin hypercubes correspond to those hypercubes of type $j \geq 1$. Consequently latin squares are latin hypercubes of type 1 and dimension 2, and a complete set of order n consists of $(n^2 - 1)/(n - 1) - 2 = n - 1$ latin members and 2 canonical squares.

The concept of hypercube type led to the following theorem which appeared as Theorem 9.11 in [6].

Theorem 1 *A set of $(n^d - 1)/(n - 1)$, d-dimensional orthogonal hypercubes of order n and type 0 is equivalent to an $AD(n^{d-1}, n)$, that is an affine* *resolvable design of order n with blocksize n^{d-1}.*

By this equivalence each class of the design is equivalent to a one hypercube so that the terms *class* and *hypercube* are used interchangably. For Corollary 4.2 of [5] to be consistent with Theorem 1 above, would require that $(n^d-1)/(n-1)$, d-dimensional orthogonal hypercubes of order n and type 0 be equivalent to $(n^d-1)/(n-1)-d$ d-dimensional orthogonal hypercubes of order n and type 1 together with d canonical hypercubes.

Niederreiter [10, Lemma 5.5] showed exactly this result for $d = 2$ by demonstrating that any set of $n+1$ orthogonal squares of order n could be transformed into $n-1$ MOLS together with a canonical row and column square.

The design that follows in Section 3 shows that Niederreiter's result can not be extended to all $d > 2$, and that Corollary 4.2 of [5] should have referred to complete sets of type 0 hypercubes rather than latin hypercubes. The existence of this design implies the following.

Theorem 2 *The class of complete sets of $(n^d-1)/(n-1)$ type 0 hypercubes of dimension d and order n contains a member that is not equivalent to a set of d canonical and $(n^d-1)/(n-1)-d$ latin hypercubes of dimension d and order n.*

Theorem 6 of [4] states that if d members of a set of $(n^d-1)/(n-1)$, d-dimensional orthogonal hypercubes of order n and type 0 are canonical, in the sense of the row and column squares when $d = 2$, then the remaining $(n^d-1)/(n-1)-d$ hypercubes are type 1 or latin. In [4] it was shown that coordinates could be assigned to the points of an affine design of order n and blocksize n^{d-1} so as to give d canonical classes (i.e. hypercubes) if the complete set contains a subset of d d-orthogonal hypercubes. This result is based on the concept of k-orthogonality introduced in a paper by Mullen and Whittle, [9]. They defined a set of $s \geq k$ hypercubes of dimension d to be k-orthogonal if superimposition of any subset of k of the hypercubes gave each k-tuple exactly once.

From this it follows that:

Corollary 3 *Not every complete set of d-dimensional orthogonal hypercubes contains a d-orthogonal subset.*

Combining Theorem 1 and Theorem 2 above gives:

Corollary 4 *For $d > 2$ there exists an $AD(n^{d-1}, n)$ that can not be represented by d canonical and $(n^d - 1)/(n - 1) - d$ latin hypercubes of dimension d and order n.*

Yet another formulation involves the notion of independent transversals introduced by Mavron [8]. In this context one is interested in determining whether there exists an $AD(n^{d-1}, n)$ not possessing d parallel classes such that every point is uniquely given as the intersection of the blocks constituting an independent transversal. The design displayed in Section 3 demonstrates the following.

Corollary 5 *For $d > 2$ there exists an $AD(n^{d-1}, n)$ with no set of d classes all of whose transversals are independent.*

2 Special Tuples and Characteristic Number

Following the approach of Bhat and Shrikhande [1] we begin with a symmetric (v, k, λ) BIBD design with $v = 2^d - 1$, $k = 2^{d-1} - 1$, and $\lambda = 2^{d-2} - 1$ and extend it to obtain an $AD(2^{d-1}, 2)$, ie an affine resolvable BIBD with blocksize 2^{d-1} and order 2. The affine design is obtained by adding one new point to the symmetric design, adding this new point to each existing block, and then taking the complement of each of these blocks with respect to the 2^d points to give $2^d - 1$ new blocks.

In the symmetric case any set of $2^{d-2} - 1$ points common to 3 blocks is called a special $(2^{d-2} - 1)$-tuple and the number of such sets is known as the *characteristic number* of the design. Similarly in the affine case the characteristic number is given by the number of sets of 2^{d-2} points common to 3 nonparallel blocks. A simple argument in [1] shows that if the characteristic of a $(2^d - 1, 2^{d-1} - 1, 2^{d-2} - 1)$ symmetric design is α then the characteristic of the derived $AD(2^{d-1}, 2)$ is 4α. We will consider primarily the affine case since these are the designs that relate directly to orthogonal hypercubes. While our examples are affine designs of order two it should be realized that concept of characteristic number is relevant to all affine designs. In the case of an arbitrary $AD(n^{d-1}, n)$ a special tuple will be a set of n^{d-2} points that are in the intersection of $n + 1$ blocks.

Designs with different characteristic numbers are nonisomorphic. (See again [1].) In fact in the affine case for any prime power order the characteristic number is a rough measure of the degree to which the structure of any $AD(n^{d-1}, n)$ resembles that of the affine geometry $AG(d, n)$ since the intersection of any 2 nonparallel hyperplanes in the geometry gives a special n^{d-2}-tuple. Accordingly affine geometries taken as designs always have a maximum characteristic number for all affine designs with the same set of parameters. The intersection of a prime block in an $AD(n^{d-1}, n)$ with any other nonparallel block defines a special n^{d-2}-tuple so that affine designs containing prime blocks will possess structure that locally resembles a geometry and such designs will have relatively a high characteristic number.

Moreover these special n^{d-2}-tuples are crucial in developing relationships between affine designs and other combinatorial structures. Specifically Laywine [3] showed that the construction of a complete set of MOFS from an affine design depended on the existence of certain families of these special tuples. Also the mutual intersections of the set of d d-orthogonal classes that Laywine and Mullen [4] used to convert an $AD(n^{d-1}, n)$, or alternatively a complete set of $(n^d - 1)/(n - 1)$ type 0 hypercubes, to a complete set of $(n^d - 1)/(n - 1) - d$ latin hypercubes resembled the mutual intersections of independent prime classes. While the useful property arising from these mutual intersections did not directly involve special tuples it seemed reasonable to anticipate a correlation between the presence of many special tuples and that of d d-orthogonal classes. Alternatively it seemed logical to expect that an $AD(n^{d-1}, n)$ that could not be represented by a complete set of latin hypercubes would possess a small characteristic number. Indeed this proved to be the case as the two examples that were found both had characteristic zero.

3 The Designs

Using Bhat and Shrikhande [1] we constructed the 22 non-isomorphic $(31, 15, 7)$ symmetric designs, which they designated as F_1 to F_{22} in [1].

Using each of these designs we then constructed the corresponding affine designs with parameters (32,62,31,16,15) ie the corresponding $AD(16, 2)$ designs. We constructed these using the simple construction method of Bhat and Shrikhande mentioned in Section 2 of this paper. To reduce the possibility of error we also produced the characteristic number for each F_i and confirmed the results of Bhat and Shrikhande.

Each of the affine designs was exhaustively searched for a set of 5 classes which would permit each of the 32 points to be uniquely represented by the intersection of a representative from each of the classes. Equivalently we checked whether in each affine design there existed 5 classes containing an independent transversal. In all cases but one there were an abundance of such sets of classes with independent transversals. The only design which contained no such set of 5 classes was the design produced using a difference set consisting of quadratic residues mod 31 which, as noted previously, has characteristic 0. That design is displayed at the end of this section. In this table the 32 points are numbered $1, ..., 31, \infty$; each row gives the points constituting one block; and one block from each of the 31 classes is displayed so that the missing blocks are simply the complements of those that are listed.

The first block is made up of the 15 quadratic residues mod 31 together with the 32nd point which we label ∞. The $i+1$st block, $i = 1, \ldots, 30$, is constructed by adding 1 mod 31 to the points other than ∞ in block i.

Having noted that the quadratic residue design was the only one to produce no set of 5 classes with an independent transversal, the next design considered was based on the (127,63,31) symmetric design constructed using quadratic residues mod 127. As in the earlier case with quadratic residues mod 31 this design was found to have characteristic zero. The corresponding affine design had parameters (128,254,127,64,63), ie it was an $AD(64, 2)$. Again by exhaustively considering all sets of 7 classes we determined that no such set contained an independent transversal. Alternatively the 127 type 0 hypercubes of dimension 7 and order 2 that correspond to the 127 classes of this $AD(64, 2)$ can not be represented as a complete set of 120 latin hypercubes of dimension 7 and order 2 together with 7 canonical hypercubes.

4 Some Conjectures

The two affine designs with the required properties were both derived from symmetric designs of characteristic zero. In turn these symmetric designs were derived using a difference set based on quadratic residues $\bmod 2^d - 1$ in the case where $d = 5$ or 7. In the case where $d = 3$ quadratic residues mod 7 give a (7,3,1) symmetric design where every pair of blocks intersect at a single point, and every point lies in 3 blocks and is a special 1-tuple. In the corresponding $AD(4, 2)$ every pair of points is a special 2-tuple so that this design has maximum characteristic, and, in fact, is the geometry $AG(3, 2)$. Clearly this phenomenom reflects the small blocksize. In fact both the (7,3,1) symmetric design, and the $AD(4, 2)$ derived from it, are unique designs with their respective parameters. The cases where $d = 5$ or 7 suggest the following.

1	2	4	5	7	8	9	10	14	16	18	19	20	25	28	∞
2	3	5	6	8	9	10	11	15	17	19	20	21	26	29	∞
3	4	6	7	9	10	11	12	16	18	20	21	22	27	30	∞
4	5	7	8	10	11	12	13	17	19	21	22	23	28	31	∞
1	5	6	8	9	11	12	13	14	18	20	22	23	24	29	∞
2	6	7	9	10	12	13	14	15	19	21	23	24	25	30	∞
3	7	8	10	11	13	14	15	16	20	22	24	25	26	31	∞
1	4	8	9	11	12	14	15	16	17	21	23	25	26	27	∞
2	5	9	10	12	13	15	16	17	18	22	24	26	27	28	∞
3	6	10	11	13	14	16	17	18	19	23	25	27	28	29	∞
4	7	11	12	14	15	17	18	19	20	24	26	28	29	30	∞
5	8	12	13	15	16	18	19	20	21	25	27	29	30	31	∞
1	6	9	13	14	16	17	19	20	21	22	26	28	30	31	∞
1	2	7	10	14	15	17	18	20	21	22	23	27	29	31	∞
1	2	3	8	11	15	16	18	19	21	22	23	24	28	30	∞
2	3	4	9	12	16	17	19	20	22	23	24	25	29	31	∞
1	3	4	5	10	13	17	18	20	21	23	24	25	26	30	∞
2	4	5	6	11	14	18	19	21	22	24	25	26	27	31	∞
1	3	5	6	7	12	15	19	20	22	23	25	26	27	28	∞
2	4	6	7	8	13	16	20	21	23	24	26	27	28	29	∞
3	5	7	8	9	14	17	21	22	24	25	27	28	29	30	∞
4	6	8	9	10	15	18	22	23	25	26	28	29	30	31	∞
1	5	7	9	10	11	16	19	23	24	26	27	29	30	31	∞
1	2	6	8	10	11	12	17	20	24	25	27	28	30	31	∞
1	2	3	7	9	11	12	13	18	21	25	26	28	29	31	∞
1	2	3	4	8	10	12	13	14	19	22	26	27	29	30	∞
2	3	4	5	9	11	13	14	15	20	23	27	28	30	31	∞
1	3	4	5	6	10	12	14	15	16	21	24	28	29	31	∞
1	2	4	5	6	7	11	13	15	16	17	22	25	29	30	∞
2	3	5	6	7	8	12	14	16	17	18	23	26	30	31	∞
1	3	4	6	7	8	9	13	15	17	18	19	24	27	31	∞
1	2	4	5	7	8	9	10	14	16	18	19	20	25	28	∞

Table 1: The $AD(16,2)$ derived using quadratic residues.

Conjecture 1 *For all Mersennes primes* $2^d - 1$ *with* $d > 3$, *quadratic residues* $\mathrm{mod}(2^d - 1)$ *give a symmetric* $(2^d - 1, 2^{d-1} - 1, 2^{d-2} - 1)$ *design with characteristic zero.*

Conjecture 2 *No* $AD(2^{d-1}, 2)$ *design with characteristic zero can be represented by a complete set of* $(2^d - 1)/(2 - 1) - d$ *latin hypercubes of dimension* d *and order* 2 *together with* d *canonical hypercubes.*

Conjecture 3 *No* $AD(n^{d-1}, n)$ *design with characteristic zero can be represented by a complete set of* $(n^d - 1)/(n - 1) - d$ *latin hypercubes of dimension* d *and order* n *together with* d *canonical hypercubes.*

Conjecture 4 *Any* $AD(2^{d-1}, 2)$ *that can not be represented by a complete set of* $(2^d - 1)/(2 - 1) - d$ *latin hypercubes of dimension* d *and order* 2 *together with* d *canonical hypercubes has a characteristic number of zero.*

Conjecture 5 *Any* $AD(n^{d-1}, n)$ *that can not be represented by a complete set of* $(n^d - 1)/(n - 1) - d$ *latin hypercubes of dimension* d *and order* n *together with* d *canonical hypercubes has a characteristic number of zero.*

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