A Note on Balanced Arrays of Strength Eight

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Abstract

In this paper we derive a necessary existence condition involving the parameters of a balanced array (B-array) with two symbols and of strength $t = 8$. Consequently, we demonstrate that the existence condition derived here can provide us with useful information on the maximum number of constraints for B-arrays with a given number of columns.

1. Introduction and Preliminaries.

For ease of reference, we state here the definition of a balanced array (B-array) of strength eight and with two symbols (say, 0 and 1).

Definition 1.1. A balanced array (B-array) T with m rows (constraints), N columns (runs, treatment-combinations), two symbols (say, 0 and 1), and of strength $t = 8$ with index set $\underline{\mu}'=(\mu_0,\mu_1,\mu_2,\dots, \mu_8)$ is merely a matrix T of size $(m \times N)$ with elements 0 and 1 (also called levels) such that in every $(8 \times N, m \geq 8)$ submatrix T^* of T, every (8×1) vector $\underline{\alpha}$ of weight i ($0 \leq i \leq 8$; the weight of a vector with elements 0 and 1 is the number of 1's in it) appears the same number of times (say) μ_i. The vector $\underline{\mu}'=(\mu_0,\mu_1,\mu_2,\dots, \mu_8)$ is called the index set of the array, and the B-array is sometimes denoted by $BA(m, N, s = 2, t = 8;\ \underline{\mu}')$.

Clearly $N = \sum_{i=0}^{8}\binom{8}{i}\mu_i$ is known once we know the μ_i 's.

JCMCC **41** (2002), pp. 133-138

Remark: The above definition can be easily extended to B-arrays of strength t and with s symbols.

Definition 1.2. If we set $\mu_i = \mu$ for each i, then the B-array is called an orthogonal array (O-array), and here $N = \mu 2^8 = 256\mu$. Thus, O-arrays are special cases of the B-arrays.

O-arrays and B-arrays have been extensively used in the construction of symmetrical as well as asymmetrical factorial designs. These arrays for different values of the strength t give rise to fractional factorial designs of various resolutions. For example, B-arrays with $t = 8$, under certain conditions, give rise to factorial designs of resolution nine which allow us to estimate all the effects up to, and including, four factor interactions under the assumption that the higher order interactions are negligible. The rows of T correspond to factors, and columns to treatment-combinations. Furthermore, other combinatorial structures closely related to B-arrays are balanced incomplete block designs, doubly balanced designs, etc. To gain further insight into the importance of O-arrays and B-arrays, the interested reader is referred to the list of references at the end (by no means an exhaustive list) of this paper, and also further references listed therein.

From the above discussion it is quite obvious that the total number of treatment-combinations (runs) N has to be a multiple of 2^t, for a given t, for an O-array to exist. This restricts the options available to a researcher in design of experiments. This has led to replacing the combinatorial structure on O-arrays by a weaker combinatorial condition giving rise to B-arrays–providing the experimenter, in general, more than one B-array for a given N. Also, it is evident that for a given $\underline{\mu}'(\mu)$ and m, one may not be able to construct a B-array (O-array). The problem of constructing such arrays, for a given $\underline{\mu}'$, with the maximum possible number of constraints m is very important–both in statistical design of experiments and combinatorics. This problem for O-arrays has been studied, among others, by Bose and Bush [1], Rao [11, 12], Seiden and Zemach [15], Yamamoto et al [17], etc.; while the corresponding problem for B-arrays has been investigated, among others, by Chopra [5, 7], Chopra and Dios [8], Saha et al [14], Yamamoto et al [18], etc. In this paper we consider B-arrays with strength $t = 8$, and obtain a necessary existence condition in the form of an inequality involving the parameters $\mu_0, \mu_1, \ldots, \mu_8$ and m. As a consequence of this inequality we obtain, for a given vector $\underline{\mu}'$, the maximum value of m (the number of constraints) for which the B-array may possibly exist.

2. Main Results

First of all, in this section, we state some results which we use later to obtain the necessary existence condition for B-arrays with $t = 8$, and provide some illustrative examples indicating the importance of the existence condition.

Lemma 2.1. A B-array with $m = t = 8$ always exists.

Lemma 2.2. A B-array T with index set $\underline{\mu}'=(\mu_0, \mu_1, \mu_2, \ldots, \mu_8)$ is also of strength t' where t' satisfies $0 \leq t' \leq 8$, with its new index set $\underline{\mu}'(t') = (A_{j,t'}; j = 0, 1, 2, ..., t')$, each $A_{j,t'}$ being a linear function of the μ_i's ($i = 0, 1, 2, ..., 8$) given by $A_{j,t'} = \sum_{i=0}^{8-t'} \binom{8-t'}{i} \mu_{i+j}, \; j = 0, 1, 2, ..., t'$.

Remark: It is not difficult to see that $t' = 0$ corresponds to the total number of columns N of T,and $t' = 8$ corresponds to the index set $\underline{\mu}'$.

Lemma 2.3. Let $x_j (0 \leq j \leq m)$ be the number of columns of weight j in a B-array T with m constraints and with index set $\underline{\mu}'=(\mu_0, \mu_1, \mu_2, \ldots, \mu_8)$. Then the following results are true:

$$
\begin{aligned}
R_0 &= \sum_{j=0}^{m} x_j = N \qquad (2.1)\\
R_1 &= \sum j x_j = m_1, A_{1,1} \text{ and}\\
R_k &= \sum_{j=0}^{m} j^k x_j = \sum_{l=1}^{k-1} (-1)^{l+k-1} a_{l,k} R_l + m_k A_{k,k} \quad 2 \leq k \leq 8
\end{aligned}
$$

Remark: Clearly there are 9 equalities in (2.1) where m_l stands for $m(m-1)(m-2)...(m-l+1)$, and the$a_{l,k}$ appear in the process of deriving (2.1) and are known constants. It is not difficult to observe that (2.1) represents moments of order k of the weights of the column vectors of T in terms of the parameters of the array T.

Next we note the following result from Lakshmanamurti [10].

Result: Let $z_i (i = 1, 2, ...n)$ be reals satisfying $\sum_{i=1}^{n} z_i = 0$ and $\sum_{i=1}^{n} z_i^2 = n$. Set $\alpha_m = \frac{1}{n} \sum z_i^m$. Then we have

$$\alpha_8 \geq \alpha_4^2 + \alpha_5^2 \qquad (2.2)$$

We use (2.1) and (2.2) in order to obtain the necessary existence condition for a B-array.

Theorem 2.1. Consider a B-array T with m constraints and with index set $(\mu_0, \ldots, \mu_8)$. Then the following inequality is true:

$$L_2 L_8 \geq L_2 L_4^2 + L_5^2 \qquad (2.3)$$

where L_i's ($i = 2, 4, 5$, and 8) are given by

$$L_2 = \sum_{k=1}^{2}(-1)^k a_k N^{k-1} R_k R_1^{2-k} (a_1 = a_2 = 1 \text{ here}),$$
$$L_4 = \sum_{k=1}^{4}(-1)^k b_k N^{k-1} R_k R_1^{4-k} \text{ (here } b_1 = +3, b_k = \tbinom{4}{k}, 2 \le k \le 4),$$
$$L_5 = \sum_{k=1}^{5}(-1)^{k-1} c_k N^{k-1} R_k R_1^{5-k} \text{ (here } c_1 = 4, c_k = \tbinom{5}{k}, 2 \le k \le 5),$$
$$L_8 = \sum_{k=1}^{8}(-1)^k d_k N^{k-1} R_k R_1^{8-k} \text{ (here } d_1 = 7, d_k = \tbinom{8}{k}, 2 \le k \le 8),$$

where R_k's are defined in Lemma (2.3).

Proof outline: Here x_j's are the frequencies of the weights j of the column vectors of T. We set $\frac{\sum jx_j}{N} = M$ (the mean weight), and $s^2 = \frac{1}{N}\sum(j-M)^2 x_j$. It is quite clear that $\sum\left(\frac{j-M}{s}\right)x_j = 0$ and $\sum\left(\frac{j-M}{s}\right)^2 x_j = N$. Next we set $\alpha_k = \frac{1}{N}\sum\left(\frac{j-M}{s}\right)^k x_j$. Substituting the values of α_4, α_5, and α_8 in (2.2), and after some simplification, we obtain

$N\ s^2\ \sum(j-M)^8 x_j \ge s^2\left[\sum(j-M)^4 x_j\right]^2 + \left[\sum(j-M)^5 x_j\right]^2$. Substituting the value of s^2 and expanding $\sum(j-M)^k x_j (k = 4, 5$, and 8), we obtain the desired result (2.3) after some simplification.

Remark: In order to check (2.3), a computer program is prepared. If (2.3) is contradicted for a given $\underline{\mu}'$ and $m(> t)$, then the B-array does not exist for these parameters. On the other hand if (2.3) is satisfied, then the B-array may (or may not) exist. In this sense (2.3) can aptly be described as a "non-existence condition" for B-arrays. Obviously (2.3) provides us with an upper bound on m for a given $\underline{\mu}'$, i.e., starting with $m = 9$, we look for the first value of m (say, $m = k + 1$) where (2.3) is contradicted, thus giving us $m = k$ as an upper bound.

Remark: For computational ease, we list here the values of $a_{l,k}$ (for various values of l and k) which appear in (2.1). The values of $a_{l,k}$ (listed in order beginning with $l = 1$ and ending with $l = k - 1$) are: $(k = 2; a_{1,2} = 1)$, ($k = 3$; values are $-2, 3$), $(k = 4; 6, 11, 6)$, $(k = 5; 24, 50, 35$, and $10)$, ($k = 6$; values are given by 120, 274, 225, 85, 15),($k = 7$; here we have the values as 720, 1764, 1624, 735, 175, 21), and ($k = 8$; values are 5040, 13068, 13132, 6769, 1960, 322, and 28).

Next we give some illustrative examples. A computer program was prepared to obtain results here.

Example 1. Take a B-array T with $\underline{\mu}'$ =(1, 3, 6, 4, 1, 7, 5, 1, 2). Applying condition (2.3) and starting with $m = 8$, we find a contradiction for the first time when $m = 12$ since, for this case, $RHS = 1.815041E+16$ and $LHS = -1.220586E+18$. Thus for the above array T, we find T does not exist for $m \geq 12$, and may exist for $8 < m \leq 11$.

Example 2. Take a B-array T with $\underline{\mu}'$ =(1, 3, 2, 2, 1, 5, 5, 2, 2). Here (2.3) is contradicted when $m = 11(RHS = 1.320199E + 16$, $LHS = 1.712767E + 14$, because $LHS \not\geq RHS$. Thus the maximum m for this array is 10.

Example 3. Consider T with $\underline{\mu}'$=(1, 4, 3, 3, 2, 8, 4, 1, 1). Using (2.3) with $m = 10$, we find $LHS = 4.26325E + 16$, $RHS = 1.017709E + 17$. Thus LHS is less than RHS, hence a contradiction. Therefore m for this array is ≤ 9.

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