Conjugacy Graphs

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Abstract

Let Γ be a finite group and let X be a subset of Γ such that $X^{-1} = X$ and $1 \notin X$. The conjugacy graph $\text{Con}(\Gamma; X)$ has vertex set Γ and two vertices $g, h \in \Gamma$ are adjacent in $\text{Con}(\Gamma; X)$ if and only if there exists $x \in X$ with $g = xhx^{-1}$. The components of a conjugacy graph partition the vertices into conjugacy classes (with respect to X) of the group. Sufficient conditions for a conjugacy graph to have either vertex-transitive or arc-transitive components are provided. It is also shown that every Cayley graph is the component of some conjugacy graph.

1 Introduction

The most studied graph associated with a given group is the Cayley graph. For a finite group Γ and generating set Δ for Γ (having $\Delta^{-1} = \Delta$ and $1 \notin \Delta$), the *Cayley graph* $\text{Cay}(\Gamma; \Delta)$ is the graph whose vertex set is Γ and where $g, h \in \Gamma$ are adjacent if and only if $h = g\delta$ for some $\delta \in \Delta$. If adjacency is defined by $h = \delta g$ for some $\delta \in \Delta$, then the graph is called the *left Cayley graph*. The left Cayley graph is isomorphic to the Cayley graph.

In this paper, a new graph, which is called the conjugacy graph, associated with a given group is defined. The graph models conjugation by specific elements in the group. Our investigation focuses on Cayley graph

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constructions as components of conjugacy graphs. We were primarily motivated by the problem of determining which Cayley graphs can be realized as conjugacy graphs. For basic graph theory and group theory used in this paper, we refer the reader to [1] and [2], respectively.

Let Γ be a finite group and let X be a subset of Γ such that $X^{-1} = X$. Then the *conjugacy digraph* $\overline{\text{Con}}(\Gamma; X)$ is the directed graph whose vertices are the elements of Γ and there is a directed edge from g to h if $h = xgx^{-1}$ for some element $x \in X$. If X generates Γ, then the components of $\overline{\text{Con}}(\Gamma; X)$ partition the vertices into the conjugacy classes of Γ, while in general, the components partition the vertices into conjugacy classes of $\langle X \rangle$. We denote the conjugacy class of an element g in Γ by $\text{cl}_\Gamma(g)$, or more simply, by $\text{cl}(g)$ when the group is understood.

Recall that the *center* of Γ consists of those elements that commute with every element of Γ. If $g \in \Gamma$ is in the center of Γ, then g is an isolated vertex of $\overline{\text{Con}}(\Gamma; X)$ with $|X|$ incident directed loops. In general, the number of directed loops at a vertex $g \in \Gamma$ is equal to the number of elements in X that commute with g. The *centralizer* of an element $g \in \Gamma$ is the set of all elements of Γ that commute with g and is denoted by $C_\Gamma(g)$ or, again, $C(g)$ if the group is understood. Hence when $X = \Gamma$, the number of loops at g is $|C(g)|$. For a subgroup Ω of Γ and $g \in \Gamma$, we define $C_\Omega(g) = \Omega \cap C_\Gamma(g)$ (here g need not be an element of Ω).

Since there is a natural bijection between the conjugacy class $\text{cl}(g)$ of g and the left cosets of $C(g)$, we may view the components of a conjugacy digraph as "coset digraphs". For a group Γ, let X be a generating set such that $X^{-1} = X$ and let Ω be a subgroup of Γ. (It is possible that X contains the identity element of Γ.) The (left) *Schreier coset digraph* $\overline{S}(\Gamma/\Omega; X)$ is the directed graph whose vertices are the left cosets of Ω and there is a directed edge from $g\Omega$ to $h\Omega$ if $h\Omega = xg\Omega$ for some element $x \in X$.

Proposition 1 *Let Γ be a finite group and let X be a subset of Γ such that $X^{-1} = X$. Let G be a component of the conjugacy digraph $\overline{\text{Con}}(\Gamma; X)$ and let g be a fixed vertex of G. Then G is isomorphic to $\overline{S}(\langle X \rangle / C_{\langle X \rangle}(g); X)$.*

Proof Since $g \in V(G)$, we have $V(G) = \{zgz^{-1} : z \in \langle X \rangle\}$. Let $C = C_{\langle X \rangle}(g)$ and define $\Psi : V(G) \to V(\overline{S}(\langle X \rangle / C; X))$ by $\Psi(zgz^{-1}) = zC$. Then Ψ is well-defined and one-to-one since $z_1 g z_1^{-1} = z_2 g z_2^{-1}$ if and only if $z_1 C = z_2 C$. To see that Ψ preserves directed edges, consider $z_1 g z_1^{-1}$ adjacent to $z_2 g z_2^{-1}$ in G. Then there exists $x \in X$ such that $z_2 g z_2^{-1} = x z_1 g z_1^{-1} x^{-1}$. Hence $(z_1^{-1} x^{-1} z_2) g = g (z_1^{-1} x^{-1} z_2)$ so that $z_1^{-1} x^{-1} z_2 \in C$ or $z_2 C = x z_1 C$. Therefore $z_1 C$ is adjacent to $z_2 C$ in $\overline{S}(\langle X \rangle / C; X)$. Since these steps are reversible, the result follows. □

The underlying graph of the conjugacy digraph is the *conjugacy graph* for Γ and X and is denoted by $\mathrm{Con}(\Gamma; X)$. Similarly, the Schreier coset graph is the underlying graph of the Schreier coset digraph and is denoted by $S(\Gamma/\Omega; X)$. By underlying graph, we mean multiple arcs are replaced by a single edge and loops are omitted. Of course, when $X = \Gamma$, the components of $\mathrm{Con}(\Gamma; X)$ partition the vertices into the conjugacy classes of Γ, and furthermore, if $j_1, j_2, \ldots, j_n$ are the orders of the conjugacy classes in Γ, then $\mathrm{Con}(\Gamma; X)$ is the union of n complete graphs having orders $j_1, j_2, \ldots, j_n$, where the isolated vertices are those elements in the center of Γ. So if Γ is abelian, then $\mathrm{Con}(\Gamma; X)$ consists of $|\Gamma|$ isolated vertices. If $\langle X \rangle = \Gamma$, i.e., X generates Γ, then the components are not necessarily complete but still partition the vertices into the conjugacy classes of Γ. More generally the components partition the vertices into the orbits in Γ under the conjugation action of $\langle X \rangle$.

Again, a primary focus of this work is on constructing Cayley graphs as components of conjugacy graphs. For example, a Cayley graph for the symmetric group S_n can be seen as a component of a conjugacy graph for S_{n+1}.

Proposition 2 *Let X be a generating set for the symmetric group S_n, where $n \geq 3$ such that $X^{-1} = X$. Then the (left) Cayley graph $Cay(S_n; X)$ is isomorphic to the component of $Con(S_{n+1}; X)$ whose vertex set consists of the cycles of length $n+1$.*

Proof Let $\sigma = (12\ldots n{+}1)$ and let G denote the component of $\mathrm{Con}(S_{n+1}; X)$ containing σ. So $V(G) = \{z\sigma z^{-1} : z \in S_n\}$ consists of the cycles of length $n+1$ in S_{n+1}. Define $\Phi : V(\mathrm{Cay}(S_n; X)) \to V(G)$ by $\Phi(z) = z\sigma z^{-1}$. Then Φ is one-to-one since $\Phi(y) = \Phi(z)$ if and only if $y^{-1}z \in C_{S_{n+1}}(\sigma) = \langle \sigma \rangle$ and every nontrivial power of σ moves $n+1$ while $y^{-1}z \in S_n$ and fixes $n+1$. To see that Φ preserves adjacencies, let y and z be adjacent in $\mathrm{Cay}(S_n; X)$. So there exists $x \in X$ with $z = xy$. Consider $\Phi(z) = z\sigma z^{-1} = (xy)\sigma(xy)^{-1} = x(y\sigma y^{-1})x^{-1} = x\Phi(y)x^{-1}$. Therefore $\Phi(y)$ is adjacent to $\Phi(z)$ in G. Since these steps are reversible, Φ is an isomorphism. □

Before continuing with other such realizations of Cayley graphs, we look at some basic properties of conjugacy graphs. The final section of this paper is concerned with constructions like that of Proposition 2, but where the order of the group used for the conjugacy graph is minimized.

2 Properties of Conjugacy Graphs

While it is well-known that every Cayley graph is vertex-transitive, such is not the case for the components of a conjugacy graph. The following result provides a sufficient condition for a conjugacy graph to have vertex-transitive components.

Proposition 3 *Let* Γ *be a finite group. If* X *consists of a union of conjugacy classes of* $\langle X\rangle$, *then every component of* $Con(\Gamma; X)$ *is vertex-transitive.*

Proof Let G be an arbitrary component of $\mathrm{Con}(\Gamma; X)$, and let $g, h \in V(G)$. Since g and h are conjugate elements with respect to $\langle X\rangle$, there exists an element z in $\langle X\rangle$ such that $h = zgz^{-1}$. We show that the inner automorphism $\phi_z : V(G) \to V(G)$ defined by $\phi_z(a) = zaz^{-1}$ is a graph automorphism and maps g to h. Let a and b be adjacent in G. So there exists $x \in X$ such that $b = xax^{-1}$. Since X is a union of conjugacy classes of $\langle X\rangle$, it follows that $zxz^{-1} \in X$, say $y = zxz^{-1}$, and so $zx = yz$. Consider $\phi_z(b) = zbz^{-1} = z(xax^{-1})z^{-1} = (zx)a(zx)^{-1} = (yz)a(yz)^{-1} = y(zaz^{-1})y^{-1} = y\phi_z(a)y^{-1}$. Thus $\phi_z(a)$ is adjacent to $\phi_z(b)$. In a similar manner ϕ preserves nonadjacencies. □

Next, we provide a sufficient condition for a component of $\mathrm{Con}(\Gamma; X)$ to be arc-transitive.

Proposition 4 *Let* G *be a component of* $Con(\Gamma; X)$, *where* X *is a union of conjugacy classes of* $\langle X\rangle$, *and let* g *be a fixed vertex of* G. *If for every pair* $x, y \in X$, *there exists* $z \in C_{\langle X\rangle}(g)$ *such that* $yz \in zxC_{\langle X\rangle}(g)$, *then* G *is arc-transitive.*

Proof Since G is vertex-transitive, it suffices to show that for any two neighbors xgx^{-1} and ygy^{-1} of g, there exists a graph automorphism that fixes g and maps xgx^{-1} to ygy^{-1}. By the hypothesis, there exists $z \in \langle X\rangle$ such that $yz \in zxC_{\langle X\rangle}(g)$. We consider $\phi_z : V(G) \to V(G)$ defined by $\phi_z(h) = zhz^{-1}$. Then ϕ_z is a graph automorphism. Since $z \in C_{\langle X\rangle}(g)$, we have $\phi_z(g) = g$ and since $x^{-1}z^{-1}yz \in C_{\langle X\rangle}(g)$, we have

$$(x^{-1}z^{-1}yz)g = g(x^{-1}z^{-1}yz),$$

or

$$yzgz^{-1}y^{-1} = zxgx^{-1}z^{-1},$$

or

$$ygy^{-1} = z(xgx^{-1})z^{-1} = \phi_z(xgx^{-1}).$$

Hence ϕ_z maps xgx^{-1} to ygy^{-1} as desired. □

For example, consider the symmetric group $\Gamma = S_4$ with X the pair of transpositions $(12), (34)$. Then $\langle X\rangle$ is the Klein four-group and so X is a union of conjugacy classes of $\langle X\rangle$ and $\mathrm{Con}(S_4; X) = 4K_1 \cup 2K_2 \cup 4C_4$ has vertex-transitive components. As another example, $\mathrm{Con}(S_3; \{(12), (13)\}) = K_1 \cup K_2 \cup P_3$. Here X generates S_3 and is not a union of conjugacy classes since $(23) \notin X$. The component P_3 is not regular. It is straightforward to determine the degree of a given vertex of $\mathrm{Con}(\Gamma; X)$.

Proposition 5 *Let Γ be a finite group and let $X \subset \Gamma$ with $X^{-1} = X$. For each $g \in \Gamma$, the degree of g in $Con(\Gamma; X)$ is given by*

$$deg\ g = |\{xC(g) : x \in X - C(g)\}|,$$

that is, the number of distinct cosets of the centralizer of g having a representative in $X - C(g)$.

Proof First note that if $x \in X \cap C(g)$, then x contributes nothing to the degree of g. Next consider $x, y \in X - C(g)$. Then $xgx^{-1} = ygy^{-1}$ if and only if $x^{-1}y \in C(g)$, that is, if and only if $xC(g) = yC(g)$. □

The next result is an immediate consequence of Proposition 5.

Corollary 6 *Let Γ be a finite group, $X \subset \Gamma$ with $X^{-1} = X$, and G be a component of $Con(\Gamma; X)$. If $g \in V(G)$, then*

$$V(G) = \{zgz^{-1} : z \in \langle X \rangle\} = cl_{\langle X \rangle}(g).$$

Furthermore, G is $|X|$-regular if and only if for every vertex $h \in V(G)$, (i) $X \cap C(h) = \emptyset$ and (ii) $xC(h) \neq yC(h)$ for every $x, y \in X$.

In the proof of Proposition 3, we saw that when X is a union of conjugacy classes of $\langle X \rangle$, every inner automorphism ϕ_z, where $z \in \langle X \rangle$, is a graph automorphism on each component of $\mathrm{Con}(\Gamma; X)$. We now establish necessary and sufficient conditions for a group automorphism of Γ to induce an automorphism of a component of $\mathrm{Con}(\Gamma; X)$. In the following proof we use the easily verified fact that in a group Γ, if $b = cac^{-1}$ and $d \in \Gamma$, then $cd \in C(b)$ if and only if $c^{-1}d^{-1} \in C(a)$.

Proposition 7 *Let α be a group automorphism of a finite group Γ and let $X \subset \Gamma$ with $X^{-1} = X$. Let G be a component of $Con(\Gamma; X)$ with $g \in V(G)$. Then α is an automorphism of G if and only if $X \cap \alpha(x)C(\alpha(g)) \neq \emptyset$ for every $x \in X - C(g)$.*

Proof Suppose that α is an automorphism of G and let $x \in X - C(g)$. Then g is adjacent to xgx^{-1}. Hence $\alpha(g)$ is adjacent to $\alpha(xgx^{-1})$ so that there exists $y \in X$ such that $y\alpha(g)y^{-1} = \alpha(xgx^{-1}) = \alpha(x)\alpha(g)\alpha(x^{-1})$. Thus $(\alpha(x)^{-1}y)\alpha(g) = \alpha(g)(\alpha(x)^{-1}y)$ and so $\alpha(x)^{-1}y \in C(\alpha(g))$ or $y \in \alpha(x)C(\alpha(g))$. So $X \cap \alpha(x)C(\alpha(g)) \neq \emptyset$.

For the converse, let α be a group automorphism of Γ and suppose that $X \cap \alpha(x)C(\alpha(g)) \neq \emptyset$ for every $x \in X - C(g)$. First we show that if $h \in V(G)$, then $X \cap \alpha(x)C(\alpha(h)) \neq \emptyset$ for every $x \in X - C(h)$. Since g was chosen as a fixed but arbitrary vertex of G and G is connected, it suffices to verify this claim for vertices h that are adjacent to g. Let $x \in X - C(h)$, and consider $h = x^{-1}gx$. Because h and g are distinct vertices of G, it follows that $x, x^{-1} \in X - C(g)$ and $x, x^{-1} \in X - C(h)$ Since α is an automorphism of Γ, we have $\alpha(g) = \alpha(x)\alpha(h)\alpha(x)^{-1}$. From the hypothesis, since $x^{-1} \in X - C(g)$, there exists $y \in X$ such that $y \in \alpha(x^{-1})C(\alpha(g))$ so that $\alpha(x)y \in C(\alpha(g))$. From the observation made prior to the proposition, we have $\alpha(x)^{-1}y^{-1} \in C(\alpha(h))$ so that $y^{-1} \in \alpha(x)C(\alpha(h))$. Further, $y^{-1} \in X$. Therefore, it now follows that $X \cap \alpha(x)C(\alpha(h)) \neq \emptyset$ for every pair h, x, where $h \in V(G)$ and $x \in X - C(h)$.

Finally we show that α is a graph automorphism of G. Let a and b be adjacent vertices of G. Thus there exists $x \in X - C(b)$ such that $a = xbx^{-1}$. Since $b \in \Gamma$ and $x \in X - C(b)$, it follows that there exists $y \in X$ such that $\alpha(x)^{-1}y \in C(\alpha(b))$. Hence $\alpha(x)^{-1}y\alpha(b) = \alpha(b)\alpha(x)^{-1}y$ or $y\alpha(b)y^{-1} = \alpha(x)\alpha(b)\alpha(x)^{-1} = \alpha(a)$. So $\alpha(a)$ is adjacent to $\alpha(b)$. □

3 Cayley Graphs and Conjugacy Graphs

In a Cayley graph, a particular generator induces a subgraph that is either a 1-factor (when the generator is an involution) or a disjoint union of cycles having length equal to the order of the generator. The situation is clearly different for a conjugacy graph. For elements $g \in \Gamma$ and $x \in X$, if k is the minimum integer such that $x^k \in C(g)$, then $g, xgx^{-1}, x^2gx^{-2}, \ldots, x^{k-1}gx^{-(k-1)}, g$ is a k-cycle in $\mathrm{Con}(\Gamma; X)$. For each positive integer k, let $C_k(x)$ consist of the elements of Γ that commute with x^k and with no smaller positive power of x. Thus if $o(x)$ is the order of x, then $C_{o(x)}(x)$ consists of the elements that commute with no nontrivial power of x. For each $x \in X$, the group Γ may be partitioned into the disjoint union $\Gamma = C_1(x) \cup C_2(x) \cup \cdots \cup C_{o(x)}(x)$. Let $n_k = |C_k(x)|$ $(1 \leq k \leq o(x))$. Then the subgraph induced by x in $\mathrm{Con}(\Gamma; X)$ is $n_1 K_1 \cup n_2/2\ K_2 \cup_{i=3}^{o(x)} n_i/i\ C_i$.

Let Ω be a finite group and Δ be a generating set for Ω with $\Delta^{-1} = \Delta$. Certainly the (left) Cayley graph $\mathrm{Cay}(\Omega; \Delta)$ is isomorphic to the (left) Schreier coset graph $S(\Omega/\{e\}; \Delta)$. We now consider whether there exists a group Γ and subset X of Γ with $X^{-1} = X$ such that $\mathrm{Cay}(\Omega; \Delta)$ is isomorphic to some component G of the conjugacy graph $\mathrm{Con}(\Gamma; X)$. If so, then by Proposition 1, it follows that G is isomorphic to $S(\langle X\rangle / C_{\langle X\rangle}(g); X)$, where $g \in V(G)$ and $C_{\langle X\rangle}(g) = \langle X\rangle \cap C_\Gamma(g)$. In this case $|\Omega| = |\langle X\rangle| / |C_{\langle X\rangle}(g)|$. Since

$$|\Gamma| = |C_{\langle X\rangle}(g)|\frac{|\Gamma|}{|\langle X\rangle|}\frac{|\langle X\rangle|}{|C_{\langle X\rangle}(g)|} = |C_{\langle X\rangle}(g)|\frac{|\Gamma|}{|\langle X\rangle|}|\Omega|$$

and $|C_{\langle X\rangle}(g)|(|\Gamma|/|\langle X\rangle|)$ is an integer, it follows that $|\Gamma| = k|\Omega|$ for some positive integer k. Actually, since the identity element of Γ is always an isolated vertex of $\mathrm{Con}(\Gamma; X)$, when $\Omega \neq \{e\}$, we have $|\Gamma| = k|\Omega|$ for some integer $k \geq 2$.

First we use Proposition 2 to show by that every Cayley graph is the component of some conjugacy graph and then we explore constructions that obtain minimum values for k.

Let Ω be a finite group and let $L : \Omega \to S_\Omega$ denote the left regular representation of Ω (that is, L is the homomorphism with $L(a) = L_a$ given by $L_a(g) = ag$ for every $g \in \Omega$). Let X be a generating set for Ω with $X^{-1} = X$. So $L(X)$ is a subset of $L(\Omega)$ in S_Ω and the Cayley graph $\mathrm{Cay}(\Omega; X)$ is now seen to be the Cayley graph $\mathrm{Cay}(L(\Omega); L(X))$, which is a connected component of the "Cayley" graph $\mathrm{Cay}(S_\Omega; L(X))$ (here $L(X)$ generates $L(\Omega)$, which is a subgroup of S_Ω and so $\mathrm{Cay}(S_\Omega; L(X))$ is disconnected, but what is important is that adjacency is defined as in left Cayley graphs). By Proposition 2, $\mathrm{Cay}(S_\Omega; L(X))$ is isomorphic to the components of the conjugacy graph for the symmetric group on $|\Omega| + 1$ objects whose vertices are the cycles of length $|\Omega| + 1$. By restricting to the vertices $L(\Omega)$ that correspond to the elements of Ω, we obtain an isomorphism between $\mathrm{Cay}(L(\Omega); L(X))$ or $\mathrm{Cay}(\Omega; X)$ and a conjugacy graph component and the group that is used for the conjugacy graph has order $(|\Omega| + 1)!$.

Proposition 8 *Every Cayley graph is isomorphic to a component of a conjugacy graph.*

In constructing a conjugacy graph $\mathrm{Con}(\Gamma; X)$ that has a component isomorphic to the Cayley graph $\mathrm{Cay}(\Omega; \Delta)$, we have seen that $|\Gamma| = k|\Omega|$ for some integer $k \geq 2$. For a group Ω and generating set Δ, we define the *conjugacy number* $\mathrm{con}(\Omega; \Delta)$ as the minimum integer k for which there exists a (necessarily) nonabelian group Γ of order $k|\Omega|$ and subset X of Γ such that $\mathrm{Cay}(\Omega; \Delta)$ is isomorphic to a component of $\mathrm{Con}(\Gamma; X)$. By the comments preceeding Proposition 8, $\mathrm{con}(\Omega; \Delta) \leq (|\Omega|+1)(|\Omega|-1)!$ and from Proposition 2, $\mathrm{con}(S_n; \Delta) \leq n + 1$. Certainly $\mathrm{con}(\Omega; \Delta) \leq \mathrm{con}(\Omega; \Omega - \{e\})$, where e is the identity of Ω and Δ is any generating set.

Next we show that if Ω has an automorphism ϕ that fixes only the identity, then $\mathrm{con}(\Omega; \Delta) = 2$. We use Z_n to denote the additive cyclic group of order n whose elements are $0, 1, 2, \ldots, n-1$. The conjugacy graph construction uses the *semi-direct product* $\Omega \rtimes Z_2$ *of* Ω *by* Z_2 defined as $\Omega \rtimes Z_2 = \{(g, i) : g \in \Omega, i \in Z_2\}$ with group multiplication given by

$$(g,i)(h,j) = \begin{cases} (gh, i+j) & if \ i=0 \\ (g\phi(h), i+j) & if \ i=1. \end{cases}$$

For simplification, let $\theta : Z_2 \to Aut\ \Omega$ be defined by $\theta(0) = \theta_0 = id_\Omega$, the identity on Ω, and $\theta(1) = \theta_1 = \phi$. Then $(g,i)(h,j) = (g\theta_i(h), i+j)$.

Theorem 9 *Let Ω be a finite group with generating set Δ. If Ω has an automorphism that fixes only the identity, then $con(\Omega;\Delta) = 2$.*

Proof Let e denote the identity of Ω. Define $\Gamma = \Omega \rtimes Z_2$ and let $X = \Delta \times \{0\}$. So $X^{-1} = X$ and $\langle X \rangle = \Omega \times \{0\}$. Let G be the component of $\mathrm{Con}(\Omega \rtimes Z_2; X)$ containing $(e,1)$. Notice that $C(e,1) = \{(e,0),(e,1)\}$ and $gC(e,1) = \{(g,0),(g,1)\}$ for every $g \in \Omega$ so that by Corollary 5, G is $|X|$-regular. We show that $\mathrm{Cay}(\Omega;\Delta) \cong G$. Define $\Psi : V(Cay(\Omega;\Delta)) \to G$ by $\Psi(g) = (g\phi(g^{-1}),1)$. Let g be adjacent to h in $\mathrm{Cay}(\Omega;\Delta)$ so that there exists $\delta \in \Delta$ such that $h = \delta g$. Then $(\delta,0) \in X$ and $(\delta,0)\Psi(g)(\delta,0)^{-1} = [(\delta,0)(g\phi(g^{-1}),1)](\delta^{-1},0) = (\delta g\phi(g^{-1}),1)(\delta^{-1},0) = (\delta g\phi(g^{-1})\phi(\delta^{-1}),1) = (\delta g\phi((\delta g)^{-1}),1) = (h\phi(h^{-1}),1) = \Psi(h)$. Thus $\Psi(g)$ is adjacent to $\Psi(h)$ and since these steps are reversible, $\mathrm{Cay}(\Omega;\Delta) \cong G$. □

Let D_n denote the dihedral group of order $2n$ with presentation $D_n = \langle x,y | y^2 = x^n = 1 = (xy)^2\rangle$. If Ω is a finite abelian group of odd order, then $\phi : \Omega \to \Omega$ defined by $\phi(g) = g^{-1}$ is an automorphism fixing only the identity and in the case that Ω is cyclic, $\Omega \rtimes Z_2 \cong D_n$.

Corollary 10 *Let Ω be a finite abelian group of odd order and let Δ be a generating set for Ω. Then $con(\Omega;\Delta) = 2$.*

Since there are no nonabelian groups of order 4 or 5, we know that $\mathrm{con}(Z_2;\{1\}) \geq 3$ and since $\mathrm{Con}(S_3;\{(12)\})$ has a component isomorphic to K_2, it follows that $\mathrm{con}(Z_2;\{1\}) = 3$. Using straightforward calculations, one can also determine that $\mathrm{con}(Z_4;\{1,2,3\}) = 3$ and thus $\mathrm{con}(Z_4;\Delta) = 3$ for every generating set Δ of Z_4. (There are three nonabelian groups of order 12, namely, the dihedral group D_6, the alternating group A_4, and the dicyclic group Q_6; and A_4 is the only one that works here. In particular, the component of $\mathrm{Con}(A_4;\{(12)(34),(13)(24),(14)(23)\})$ containing (123) is the desired component.) Next we show that the conjugacy number of a cyclic group of even order is at most 4.

Proposition 11 *Let $n \geq 6$ be an even integer. Then $con(Z_n;\Delta) \leq 4$ for every generating set Δ of Z_n.*

Proof Let Δ be a generating set for Z_n. We show that $\mathrm{Cay}(Z_n;\Delta)$ is isomorphic to a component of a conjugacy graph for D_{2n}. Define $X = \{x^k :$

$k \in \Delta\}$ and let G be the component of $\mathrm{Con}(D_{2n}; X)$ containing y. Then $V(G) = \{x^{2i}y : 0 \leq i \leq n-1\}$. Define $\Psi : V(\mathrm{Cay}(Z_n; \Delta)) \to V(G)$ by $\Psi(i) = x^{2i}y$. Suppose that i is adjacent to j in $\mathrm{Cay}(Z_n; \Delta)$. So there exists $k \in \Delta$ with $j \equiv (i+k) \mathrm{mod}\ n$. So $2j \equiv 2(i+k) \mathrm{mod}\ 2n$ so that $x^{2j}y = x^{2(i+k)}y$ in D_{2n}. Hence $x^{2j}y = x^k(x^{2i}y)x^{-k}$ and thus it follows that $x^{2i}y$ is adjacent to $x^{2j}y$ in G. Since these steps are reversible, $\mathrm{Cay}(Z_n; \Delta) \cong G$ and $\mathrm{con}(Z_n; \Delta) \leq 4$. □

In general, we have the bound $\mathrm{con}(\Omega; \Delta) \leq (|\Omega| + 1)(|\Omega| - 1)!$. Using the following result, we can obtain much better bounds for the remaining finite abelian groups of even order. We use $\times$ for the direct product of two groups as well as for the cartesian product of two graphs. The proof is straightforward and therefore omitted.

Proposition 12 *For each $i = 1, 2$, let Γ_i be a finite group with X_i a subset of Γ_i such that $X_i^{-1} = X_i$, and let e_i be the identity of Γ_i. Then*

$$Con(\Gamma_1 \times \Gamma_2; (X_1 \times \{e_2\}) \cup (\{e_1\} \times X_2)) \cong Con(\Gamma_1; X_1) \times Con(\Gamma_2; X_2).$$

An immediate consequence follows.

Corollary 13 *For each $i = 1, 2$, let Ω_i be a finite group with generating set Δ_i and let e_i be the identity of Ω_i. Then*

$$con(\Omega_1 \times \Omega_2; (\Delta_1 \times \{e_2\}) \cup (\{e_1\} \times \Delta_2)) \leq con(\Omega_1; \Delta_1) con(\Omega_2; \Delta_2).$$

Proof For each $i = 1, 2$, let Γ_i be a group and X_i a subset of Γ_i with $X_i^{-1} = X_i$ such that $\mathrm{Cay}(\Omega_i; \Delta_i)$ is isomorphic to a component of $\mathrm{Con}(\Gamma_i; X_i)$. So $|\Gamma_i| = \mathrm{con}(\Omega_i; \Delta_i)|\Omega_i|$. Now consider $\mathrm{Cay}(\Omega_1 \times \Omega_2; (\Delta_1 \times \{e_2\}) \cup (\{e_1\} \times \Delta_2))$. Since $\mathrm{Con}(\Gamma_1; X_1) \times \mathrm{Con}(\Gamma_2; X_2)$ has a component isomorphic to $\mathrm{Cay}(\Omega_1; \Delta_1) \times \mathrm{Cay}(\Omega_2; \Delta_2) \cong \mathrm{Cay}(\Omega_1 \times \Omega_2; (\Delta_1 \times \{e_2\}) \cup (\{e_1\} \times \Delta_2))$, by Proposition 12, the proof is complete. □

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References

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Third ed., Chapman & Hall, New York, 1996.

[2] J. J. Rotman, *An Introduction to the Theory of Groups*, Fourth ed., Springer-Verlag, New York, 1995.