On Edge-balanced Multigraphs*

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Abstract A new graph labeling problem on simple graphs called edge-balanced labeling is introduced by Kong and Lee [11]. They conjectured that all trees except $K_{1,n}$ where n odd and all connected regular graphs except K_2 are edge-balanced. In this paper we extend the concept of edge-balanced labeling to multigraphs and completely characterize the edge-balanced multigraphs. Thus we proved that the above two conjectures are true. A byproduct of this result is proof that the problem of decision a graph is edge-balanced does not belong to N P-hard.

1. The concept of edge-balanced graphs.

In a bi-racial country, it is desired that the numbers of government ministers from the two races should differ by at most one. Moreover, the numbers of pairs of ministries which interact directly with each other and both headed by ministers of one race should differ by at most one from that of the other race.

We can naturally model this situation by graph. Each vertex represents a ministry. Two vertices are joined by an edge if and only if the ministries they represent interact directly with each other. We seek a partition of the vertices into two sets satisfies certain conditions of balance.

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Let G = (V,E) be a simple graph. A mapping f from V(G) to {0,1}will induce an edge labeling f^*: E(G) → {0,1} as follows: $f^*(\{x,y\}) = |f(x) - f(y)|$. In 1987, Cahit [2] called a graph **cordial** if $|v_f(0) - v_f(1)| \le 1$, $|e_f(0) - e_f(1)| \le 1$.where $v_f(i)$ = the number of nodes with label i, and $e_f(i)$ = the number of edges with label i. Thus a graph on n vertices is said to be cordial if there exists a labelling f of the vertex set using *as equal as possible a number of zeros and ones*, so that the the numbers of induced edge labellings that are zero and one differ by at most one. The edge labeling induced by vertices i, j is defined by $|f(i)-f(j)|$.

In 1988, a dual concept of cordial graphs which is called "edge-cordial" was introduced by the third author and Ng [14]. Intuitively speaking, a graph G is said to be **edge-cordial** if its edges can be labelled either 0 or 1 so that half of the edges are labelled 0; half are labelled 1; and half of the vertices meet an even number of edges labelled 1, while the other half meet an odd number of edges labelled 1. Ng and Lee conjectured that all trees not of order n=2 (mod 4) are edge-cordial. The problem was solved by Collin and Hovey [6] in 1993.They showed that most graphs are edge-cordial.

In 1989, Lee, Liu and Tan [13] introduced another type of graph labeling. Given a graph G with vertex set V(G) and edge set E(G). Any binary labeling f : V(G) →{0,1} induces a partial binary labeling f+: E(G) →{0,1} as follows. If f(x)=f(y), then f+({x,y}) is assigned their common value. If not, then f+({x,y}) is undefined. Similarly to Cahit's definition of cordial graph, a graph G is called balanced if $|v_f(0) - v_f(1)| \le 1$, $|e_{f+}(0) - e_{f+}(1)| \le 1$. The concept of balanced labelings is dual analogy to cordial labelings.

A similar graph labeling, although not related to the concept of cordial labeling, was introduced by Kong and Lee [13]. For a given binary edge labeling g : E(G) →{0,1}, the edge labeling g induces a partial vertex labeling g*: V(G) →{0,1} such that g*(v)= 1 (0) iff the number of 1-edges (0-edges) is strictly greater than the number of 0-edges (1-edges) incident to v, otherwise g*(v) is undefined. The mapping g is an edge-balanced labeling of G if and only if $|v_{g^*}(0) - v_{g^*}(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$.Furthermore, g is strongly edge-balanced iff $|v_{g^*}(0) - v_{g^*}(1)| = 0$, $|e_g(0) - e_g(1)| = 0$.

Although these concepts superficially resemble each other, they are apparently not related. Figure 1 illustrates the difference between them. For example, $K_{1,5}$ is cordial, but neither balanced nor edge-balanced and C_6 is not cordial, but balanced and edge-balanced.

Cahit [2] showed that all trees are cordial. But deciding cordiality of a graph is NP-complete has been proved to be true very recently by N. Cairnie, and E. Keith [4]. Liu Shih-San proved all trees except star $K_{1,2n+1}$, are edge-balanced. In this paper we show that decide edge-balanced of graph is belong to **P**, i.e. we have a polynomial time algorithm to decide whether a graph is edge-balanced or not.

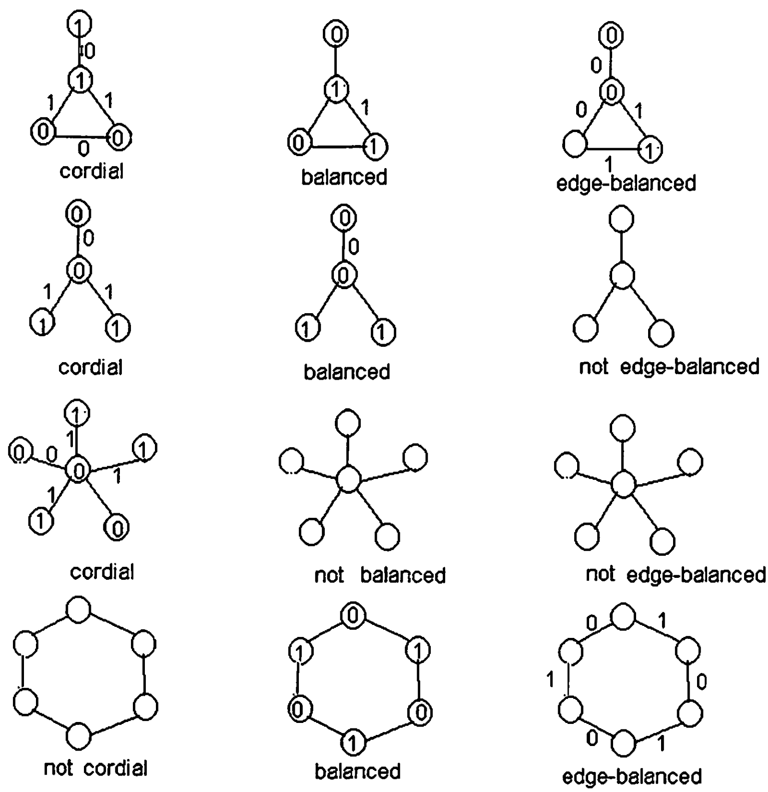

Figure 1.

The following two conjectures are proposed at the end of the article [11].

Conjecture 1: All trees except star $K_{1,2n+1}$ are edge-balanced.

Conjecture 2: All connected regular graphs except K_2 are edge-balanced.

The readers are refered to [1, 4, 9, 10, 12] for cordial graphs and their generalization [3, 15, 16].

In this paper we extend the concept of edge-balanced labeling to multigraphs and completely characterize the edge-balanced multigraphs. Thus we proved that the above two conjectures are true.

2. Basic properties.

In this paper, we define edge-balanced in a little different way from [11]. We will label the edges by {+1, -1} instead of {0,1}. In this paper the word graph will mean an undirected (multi)graph without loop. Given G = (V, E) and an edge labeling f from E (G) to {1, -1} and for each vertex v, define $\sigma(v)$ to be the sum of f (e) which e is incident to v.

Let $V_+ = \{v \in V: \sigma(v) > 0\}$, $V_- = \{v \in V: \sigma(v) < 0\}$, $V_0 = \{v \in V: \sigma(v) = 0\}$, $E_+ = f^{-1}(1)$, and $E_- = f^{-1}(-1)$. Assume $f_V = |V_+| - |V_-|$, $f_E = |E_+| - |E_-|$ and $D(G) = \{(m,n) \in Z \times Z$: f is an edge labeling of G with $f_E = m$ and $f_V = n\}$.

Definition1. A graph G is an edge-balanced graph if $|f_V| \leq 1$ and $|f_E| \leq 1$. It is called strongly-edge-balanced if $f_V = 0$ and Fe = 0.

Thus, G is edge-balanced if (man) $\in$D(G) where $|m| \leq 1$and $|n| \leq 1$ and strongly edge-balanced if (m,n) $\in$ D(G). For simplicity, we will denote the class of all edge-balanced graphs by **EB** and the class of all strongly edge-balanced graphs by **SEB**. Then **SEB** $\subset$ **EB**.

Example 1. We observe that for
(1) K_2. If edge is labeled by +1, then $f_V = 2$ and $f_E = 1$. If the edge is labeled by -1 , then $f_V = -2$ and $f_E = -1$. Thus, $K_2 \notin$ **EB**
(2) K_3 The graph is edge-balanced, for we have an edge-balanced labeling f with $f_V = 1$ and $f_E = 1$. However, it can not be strongly edge-balanced.
Thus, $K_3 \in$ **EB\SEB**.

Remark 1. In the literature, there exists another notion of balance in signed graph which was first introduced by Harary [7], based on a method of analyzing social interactions due to Heider [8]. A signed graph G is a graph in which every edge has been assigned either a + or -. G is said to be balanced (in the sense of Harary) if the product of the signs around each cycle in G is positive.

However, this concept is independent with the above concept which we just considered.

Proposition 2. Let P_n be a path with order n. Then
(a) $f_v = 2$ if n = 2.
(b) $f_v = 0$ or 3 if n = 3.
(c) $f_v = 1, 2$ or 4 if n = 4.
(d) The value f_v can be 0, 1 or 2 if $n \geq 5$.

Proof. It is not difficult to see that (a),(b) and (c) are all true.
(d) We consider two cases.

Case 1. $n = 2k \geq 5$.
$f_v = 0$: + - + - ... + -
$f_v = 1$: + - - + - ... + - +
$f_v = 2$: + + + - + - ... + -

Case 2. $n = 2k+1 \geq 5$.
$f_v = 0$: - + + + - + - ... + -
$f_v = 1$: + + - + - ... + - +
$f_v = 2$: + - + + - ... + - +.

Theorem 3. Let G be a connected multigraph. Then $G \in$ **SEB** if and only if $|E(G)|$ is even.

Proof. By definition, if G is in **SEB** then G has an even number of edges. Conversely, suppose G has an even number of edges. It is well-known that G has an even number of odd vertices, say $v_1, v_2, \ldots, v_{2k-1}, v_{2k}$ ($k \geq 0$). We add k new vertices $u_1, u_2, \ldots, u_k$, and each u_i joining to v_{2i-1}, and v_{2i}. The new graph H is obviously eulerian with even size. Let C be an eulerian tour of H. Label the edges of C by – and + alternatively. Deleting the vertices $u_1, u_2, \ldots, u_k$, it is easy to see G is in **SEB** by using this labeling. Q. E. D.

Lemma 4. If G is a connected simple graph with an odd number of edges except $K_{1,2m+1}$, then $G \in$ **EB.**

Proof. If $|V(G)| \leq 3$ then G is K_3 which is in **EB**. We may assume $|V(G)| \geq 4$.

Case 1. G contains an even vertex, say v_0. Using the method in Theorem 3, we have an eulerian tour C with origin and terminus v_0. At the beginning of the edge joined to v_0, we label the edges of C by + and – alternatively. It is a routine matter to check that G is in **EB** with $(1,1) \in D(G)$.

Case 2. G contains no even vertices. Then $|V(G)|$ is even. Since G is not isomorphic to $K_{1,2m+1}$, G contains a path with length 3, say $P = v_1v_2v_3v_4$. Let $G(1) = G\backslash\{v_2v_3, v_3v_4\}$. Using the method in Theorem 3, we may add $|V(G)|/2 - 1$ vertices into G(1) and form an eulerian graph H (the vertex v_4 may be an isolated vertex of H). Let $C = x_1x_2\ldots x_nx_1$ be an eulerian tour of H, where $x_1 = v_1$ and $x_2 = v_2$. Consider the new eulerian tour $C^* = v_1v_2v_3v_4ux_2\ldots x_nx_1$, where u is a new vertex. Label the edges of C^* at the beginning of v_1v_2 by +, +, -, +, - ,..., and so on. Now, deleting all the vertices which are not in G. Again, G is in **EB** with $(f_E,f_v) = (1,0)$ by using this labeling. Q.E.D.

Remark 5. If we replace the sign-labeling of C^* by +, -, +, +, -, +, -, ... in the case 2, then $(1,2) \in D(G)$.

3.Main Results

Combine the Theorem 3 and Lemma 4, we have the following.

Theorem 6. If G is simple and connected except $K_{1,2m+1}$ where $m \geq 0$, then G is in **EB**.

The above theorem proved the two conjectures in [11].

In the following we want to extend the above result to the edge-balanced multigraphs.

Lemma 7. If G is a connected multigraph with an odd number of edges, then G is in **EB** if and only if G is neither $K_{1,2m+1}$ nor $K_2(2n+1)$, where $K_2(2n+1)$ is a multigraph with 2 vertices joined by 2n+1 parallel edges.

Proof. It is easy to see that G is not in **EB** if it is either $K_{1,\,2m+1}$ or $K_2(2n+1)$. Conversely, assume G is neither $K_{1,\,2m+1}$ or $K_2(2n+1)$.

If $|V(G)| = 3$, then G has an even vertex. By the same argument in Theorem 3, there is a sign-labeling f such that $(f_E,f_v) = (1,1)$.

Assume $|V(G)| \geq 4$. If G contains a path with length 3, then G is in **EB** by the same argument in Lemma 4.

If G does not contain a path with 3, then the underline graph must be $K_{1,s}$ for some $s \geq 3$. Let $\{e_1, e_2, \ldots, e_s\}$ be the edge set of $K_{1,s}$. Consider the multiplicity of e_i. First, assume that there is an even vertex. Then there is an edge with even multiplicity, say e_1. Label the multiple edges of $e_1, e_2, \ldots, e_s$ in order by +, +, -, +, -,..., -. Then G is in **EB** with $(f_E,f_v) = (1,1)$ by using this sign-labeling. Secondly, assume that there is no even vertex. Then s must be an odd number and at least 3, and each multiplicity of e_i is odd. Since G is not $K_{1,2m+1}$ for some $m \geq 1$, there is an edge e_i with multiplicity at least 3, say e_1. Label the multiple edges of $e_1, e_2, \ldots, e_s$ in order by the following: label the multiple edges of e_1 and e_2 by -, +, -,...,-. Then G is in **EB** with $(f_E,f_v) = (1,0)$ by using this sign-labeling. Q.E.D.

Corollary 8. Let G be a connected graph. Then there is a sign-labeling f such that

(a) $(f_E,f_v) = (0,0)$ if $|E(G)|$ is even,
(b) $(f_E,f_v) = (1,1)$ if $|E(G)|$ is odd with an even vertex,
(c) $(f_E,f_v) = (1,0)$ if $|E(G)|$ is odd without even vertex and neither isomorphic to $K_{1,2m+1}$ nor $K_2(2n+1)$,
(d) $(f_E,f_v) = (1,2)$ if G isomorphic to $K_{1,2m+1}$ or $K_2(2n+1)$.

Lemma 9. If G is a connected multigraph with even size, then $D(G) \cap \{(2,1), (2,2), (2,3)\}$ is nonempty.

Proof. By Theorem 3, we have a sign-labeling of G such that $(f_E,f_v) = (0,0)$ and $|\sigma(v)| \le 1$ for any vertex v. Assume that the V_0 is nonempty. Then there is a vertex v belongs to V_0 and the edge e incident to v with f(e) = -1. Changing the sign-labeling of e by +, we have $(f_e,f_v) \in \{(2,1), (2,2), (2,3)\}$. If the set of V_0 is empty, then $V= V_+ \cup V_-$. If f(e)=+1, then change the sign-labeling for all edges except e. If f(e)=-1, then change the sign-labeling of e. Again, we have $(f_E,f_v) = (2,2)$ by using the new sign-labeling. Q. E. D.

Lemma 10.

If $(i,j) \in D(G)$, then $(-i,-j) \in D(G)$.

If $(i,j) \in D(G)$ and $(i',j') \in D(H)$, then $(i+i', j+j') \in D(G \cup H)$.

Proof. If $(i,j) \in D(H)$, then there is a sign-labeling h of G such that $(h_E, h_v) = (i,j)$. Let f(e) = -h(e) for all $e \in E(G)$. Then f is a sign-labeling of G with $(h_E,h_v) = (-i,-j) \in D(G)$.

If $(i,j) \in D(G)$ and $(i',j') \in D(H)$, then there are two sign-labelings g of G and h of H such that $(g_E,g_V)=(i,j)$ and $(h_E,h_V)=(i',j')$. Let f(e) = g(e) for all $e \in E(G)$ and f(e) = h(e) for all $e \in E(H)$. Then f is a sign-labeling of $G \cup H$ with $(f_E, f_v) = (i+i',j+j') \in D(G \cup H)$. Q.E.D.

Theorem 11. If G is a multigraph, then G is in **EB** if and only if G is neither $K_{1,2m+1}$ nor $\bigcup_{i=1}^{2t+1} K_2(2n_i+1)$, where $n_i \ge 0$ and $t \ge 0$.

Proof. It is not difficult to see that G is not in **EB** if G is either $K_{1,2m+1}$ or $\bigcup_{i=1}^{2t+1} K_2(2n_i+1)$

Conversely, suppose G is neither $K_{1,2m+1}$ nor $\bigcup_{i=1}^{2t+1} K_2(2n_i+1)$

Let $G_1 \cup ... \cup G_{r+t}$, where G_i is a connected component of G for all $1 \le i \le r+t$ and G_j is isomorphic to $K_{1,2m+1}$ or $K_2(2n_j+1)$ for all $r+1 \le j \le r+t$.

Assume that t is even. By Corollary 8 and Lemma 10, we have $(0,0) \in D(G_{r+1} \cup ... \cup G_{r+t})$. By Corollary 8 we have $D(G_i) \cap \{(0,0), (1,1), (1,0)\}$ is nonempty for all $1 \le i \le r$. Again, by Lemma 7, we have G is **EB**.

Assume that t is odd and there is a component G_s $(1 \le s \le r)$ with an even vertex and $|E(G_s)|$ is odd. Since $(1,1) \in D(G_s)$, we have $(0,1) \in D(G_s \cup G_{r+1} \cup ... \cup G_{r+t})$. By Lemma 10 , we have G is in **EB**.

Assume that t is odd and there is a component G_s ($1 \le s \le r$) with no even vertex and $|E(G_s)|$ is odd. By Remark 5, we have a sign-labeling f of G_s such that $(f_E, f_v) = (1,2) \in D(G_s)$. Then we have $(0,0) \in D(G_s \cup G_{r+1} \cup \ldots \cup G_{r+t})$. By Lemma 10, we have G is in **EB**.

Assume that t is odd and each component G_i with an even size for all $1 \le i \le r$. By Lemma 9, we have $D(G_i) \cap \{(2,1), (2,2), (2,3)\}$ is nonempty. Then we have $G_1 \cup G_{r+1} \cup \ldots \cup G_{r+t}$ is in EB. Again, by Lemma 10, we have G is in **EB.**

Assume that t is odd and r = 0. In this case, $t \ge 3$. Since G is not a disjoint union of an odd number of $K_2(2n_j+1)$, there is a component which is isomorphic to $K_{1,2m+1}$, where $m \ge 1$. Label the edges of $K_{1,2m+1}$ by +, +, +, +, -, +, -,... . Then $(3,4) \in D(K_{1,2m+1})$ and by Lemma 10, we have $(1,0) \in D(G)$. Q.E.D.

The above results completely characterize edge-balanced multigraphs.

Using Theorem 11 one can show that contrary to the case of cordial graphs the problem to decide a graph is edge-balanced is not belong to **NP-hard**.

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