On The Integer-magic Spectra of The Power of Paths*

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Abstract: For k>0, we call a graph G=(V,E) as Z_k-magic if there exists an edge labeling $l:E(G) \rightarrow Z_k^*$ such that the induced vertex set labeling $l^+: V(G) \rightarrow Z_k$ defined by

$$l^+(v)=\Sigma \{l(u,v) : (u,v) \text{ in } E(G)\}$$

is a constant map. We denote the set of all k such that G is k-magic by IM(G). We call this set as the **integer-magic spectrum** of G. This paper deals with determining the integer-magic spectra of power of paths Pn^k for k=2 and 3. We also show that $IM(P_{2k}^k) = N\backslash\{2\}$ for all odd integer k >1.Finally, a conjecture for $IM(P_n^k)$ for $k \geq 4$ is proposed.

1. **Introduction**. We begin with a few definitions and some notations. For any abelian group A, written additively we denote $A^* = A-\{0\}$. Given a graph G=(V,E), any mapping $l: E(G) \rightarrow A^*$ is called a labeling. Given a labeling on edge set of G we can induced a vertex set labeling $l^+: V(G) \rightarrow A$ as follows:

$$l^+(v)=\Sigma \{l(u,v) : (u,v) \text{ in } E(G)\}$$

A graph G is called A-magic if there is a labeling $l: E(G) \rightarrow A^*$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; i.e., $l^+(v) = c$ for some fixed c in A. We will called <G,l> a A-magic graph with index c. In general, a graph G may admits more than one labeling to become a A-magic graph.

When A = Z, the Z-magic graphs were considered in Stanley [20,21]; he pointed out that the theory of magic labelings can be put into the more general context of linear homogeneous Diophantine equations. When the group is Z_k, we shall refer to the Z_k – magic graph as *k- magic* (Figure 1). Graphs which are k-magic had been studied in [2,6,9,11,12,15].

At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs [14]. The original concept of A-magic graph is due to J. Sedlacek [17,18], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

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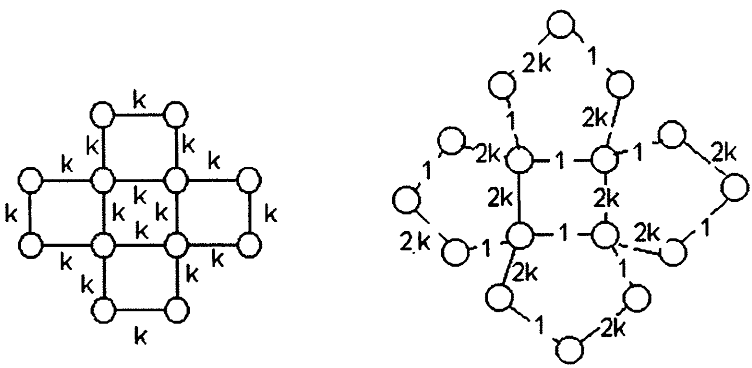


$C_4 @ C_4$ is 2k-magic $\qquad$ $C_4 @ C_5$ is 2k+1 -magic

Figure 1.

The definition was different from the one introduced by Rosa and Kotzig [10]. In literature, graphs had been considered as magic if it is **N**-magic where N ={1,2,3,...}, the set of natural numbers. It is well-known that a graph G is N-magic if and only if each edge of G is contained in a 1-factor (a perfect matching) or a (1,2)-factor ([8,16,24]). Some special classes of N-magic graphs which are called supermagic graphs had been considered in the literature.(see [4],[22],[23]). A generalzation of supermagicness was introduced by the first author in [12]. Interested readers can read other related works in [13,19].

For convenience, we will consider Z-magic as 1-magic. Given a graph G, we denote the set of all k> 0 such that G is k-magic by IM(G). We call this set as **integer-magic spectrum** of G. Likewise, we denote AM(G)={ A ε **Ab**: G is A-magic} the **group-magic spectrum** of G. In general, it is difficult to determine the sets AM(G) and IM(G) for a given graph G. We investigate these sets for general graphs in [15].

In this paper we want to investigate the integer-magic spectra sets for kth power of paths.

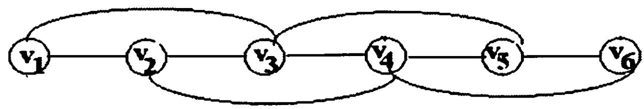
2. Kth-power of paths.

The kth power of the path P_n, denote by P_n^k, is the graph resulting from joining every pair of vertices of P_n whose distance from each other is k.. For the sake of convenience, the graph of P_n^k is drawn in the following manner (see Figure 2 for n=6 and k=2)

In P_n^k the edges of the path P_n are called the middle edges, the edges v_iv_j, where j=i+k for i=1,3,5,..., are called the top edges, and the edges of v_iv_j, where j=i+k, for k=2,4,6,..., are called the bottom edges.

A special case of P_n^k is P_n^{n-1} for n >2, which is a cycle of n vertices It is obvious that the integer magic specrum of P_n^{n-1} is N.

We have the following result.



P_6^2

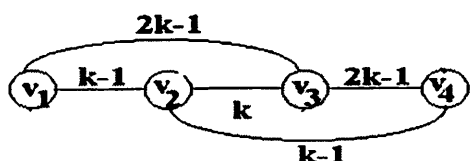
Figure 2

Theorem 1. The integer magic spectrum $IM(P_4^2) = 2N\backslash\{2\}$

Proof. It is not difficult to show that P_4^2 is not Z-magic. According to the theorem of [14], a graph G is 2-magic if and only if each vertex of G has the same parity. Thus we see that P_4^2 is not 2-magic.

Now for any even number $2k \geq 4$, we want to show that P_4^2 is 2k-magic·

This result can be achieved by showing P_4^2 has the following 2k-magic labeling (see Figure 3.)



P_4^2 **is 2k-magic**

Figure 3

To see that it is not 2k+1-magic for all $k \geq 1$. We prove by contradiction. Assume P_4^2 has a magic labeling $l(v_1v_3) = x$, $l(v_1v_2) = y$, $l(v_2v_3) = z$, $l(v_3v_4) = w$, $l(v_2v_4) = u$. Then we have $x+y = x+z+w$ and $w+u = y+z+u$. Thus $y = z+w$ and $w = y+z$. This implies that $y = z+w = z+y+z = 2z+y$. Thus $2z = 0$ which is not possible in Z_{2k+1}.

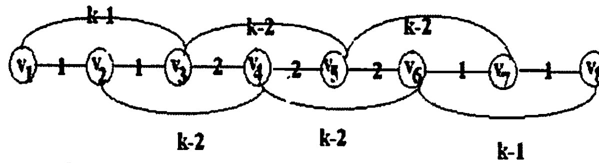
Theorem 2. The integer magic spectrum $IM(P_n^2) = N\backslash\{2\}$ for all $n > 5$.

Proof. Since the degree set of P_n^2 is {2,3,4} we see that P_n^2 is not 2-magic. To see that it is k-magic for all other $k > 0$, we proceed as follows:

Case 1. n is even. Assume $n = 2t$, $t > 2$.

For $k > 2$, we label the top edges from left to right by k-1 and t-2 (k-2)'s consecutively and the bottom edges from left to right by t-2 (k-2)'s consecutively and k-1 for the last one. The middle edges from left to right are labeled in the following way 1,1, and 2t-5 copies of 2, and then 1, 1.

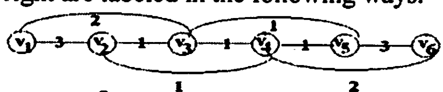
The index of the labeling is 0. (see Figure 4 for n=8)



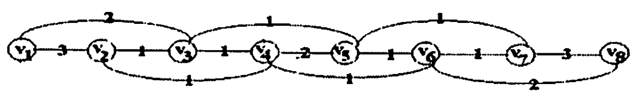
P_8^2 is k-magic for k>2

Figure 4.

To see that it is Z-magic, we label in the following way: the top edges from left to right are labeled by 2 and t-2 1's consecutively and the bottom edges from left to right by t-2 1's consecutively and 2 for the last. The middle edges from left to right are labeled in the following ways:



P_6^2 is Z-magic



P_8^2 is Z-magic

Figure 5

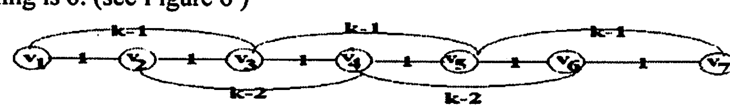
(1) if n=6, we label 3,1,1,1,3

(2) if n≥8, we label 3,1 , and t-3 (1,2)'s consecutively and the remaining three edges by 1,1,3. The index of this labeling is 5. (see Figure 5)

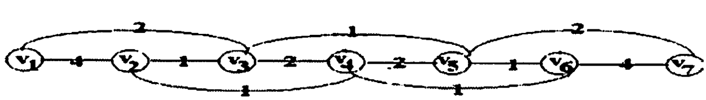
<u>**Case 2. n is odd**</u>. Assume n=2t+1.

For k>2, we label all the top edges by k-1 , all the bottom edges by k-2 and all the middle edges by 1 We see that it is k-magic with the index of the labeling is 0. (see Figure 6 for n=7)

To see that it is Z-magic, we label the edges in the following way: the top edges from left to right are labeled by 2 and t-2 1's consecutively and 2 for the last one. All the bottom edges are labeled by 1. The middle edges from left to right are labeled by 4,1, and 2t-4 2's and the remaining two by 1,4. The index of this labeling is 6. (see Figure 6)



P_7^2 is k-magic



P_7^2 is Z-magic

Figure 6

3. Integer-magic spectra for 3rd-power of paths.

In [11], Saba, Sun and the first author showed that

Theorem 4. The kth power P_n^k is not N-magic if

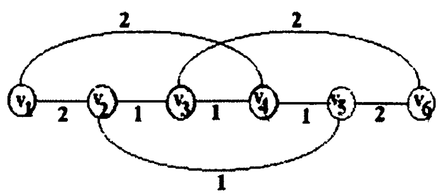
(1) n and k are both odd and $2<k<n$.

(2) n and k are both even and $k\geq n/2$.

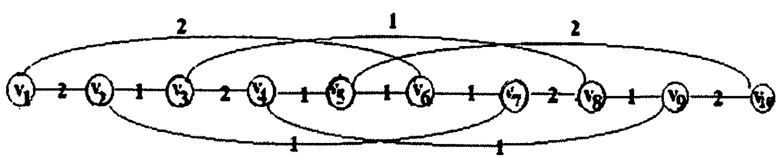
Here we can show that

Theorem 5. The integer magic spectrum $IM(P_{2d}^d)$ is $N\backslash\{2\}$ for all odd integer $d>1$.

Proof. Let $d=2t+1$. We label in the following way: the top edges from left to right are labeled by 2 and t-1 1's consecutively and 2 for the last one. All the bottom edges are labeled by 1. The middle edges from left to right are labeled in the following way: t (2 1)'s consecutively, followed by 1, and then t (1 2)'s consecutively. The index is 4. (see Figure 7)



P_6^3 is k-magic for k>2 and Z-magic



P_{10}^5 is k-magic for k>2 and Z-magic

Figure 7.

Using the above results, we can state that

Theorem 5. The integer magic spectrum $IM(P_n^3)$ is

(1) N if n=4

(2) $N\backslash\{1, 2\}$ for all odd $n\geq 5$.

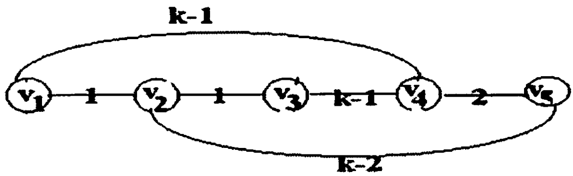
(3) $N\backslash\{2\}$ for all even $n\geq 6$.

Proof. (1) If n=4, then as P_4^3 is a 4-cycle. Thus $IM(P_n^3)=N$.

(2) Assume now n=5. We see that P_5^3 has the following k-magic labeling for all $k>2$.(see Figure 8).

By Theorem 4, it is not Z-magic.

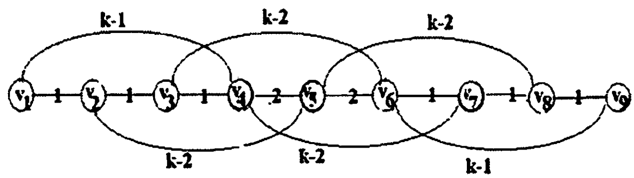
Thus $IM(P_5^3)=N\backslash\{1,2\}$



P_5^3 is k-magic

Figure 8.

For n=2t+1 > 5. For k>2, we label the top edges from left to right by k-1 and t-2 (k-2)'s consecutively and the bottom edges from left to right by t-2 (k-2)'s consecutively and k-1 for the last one. The middle edges from left to right are labeled in the following way 1,1, 1, and 2t-6 copies of 2, and then 1,1,1. The index of the labeling is 0. (see Figure 9 for n=9).



P_9^3 is k-magic for k>2

Figure 9.

(3)For n=2t . We see that if t=3, the 3rd power of P_6 has $IM(P_6^3) = N \setminus \{2\}$ by Theorem 5.

Now assume n=2t $\geq$ 8. To see that P_n^3 is 3-magic we label the top edges from left to right by 2 and t-3 1's consecutively and the last edge by 2. All the bottom edges by 1. The middle edges from left to right are labeled in the following way 1,1, 1, and 2t-7 copies of 2, and then 1,1,1. The index of the labeling is 0. (see Figure 10 a for n=10).

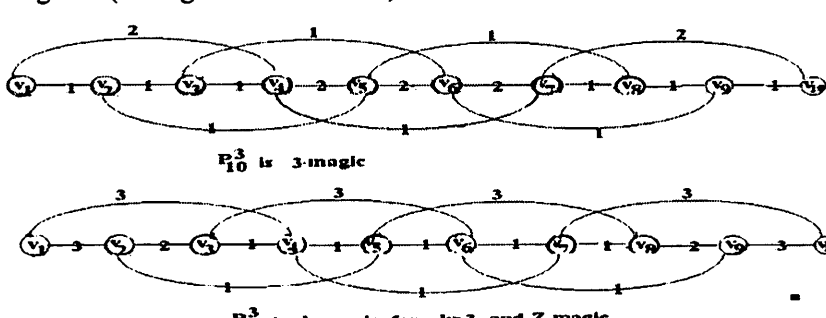


Figure 10

If we label all the top edges by 3 and all the bottom edges by 1. The middle edges from left to right are labeled in the following way 3,2 and 2t-5 copies of 1, and then 2,3 for the last two edges. We see that the index of the labeling is 6. Thus $\mathbf{P_n}^3$ is not only Z-magic but also k-magic for all k>3. (see Figure 10 b for n=10).

4. Conjecture

We conjecture that the integer magic spectrum $IM(P_n^k)$ for k >3 is one of the following three types:

(1) N if n=k+1

(2) N\{1, 2} for all odd n ≥k and k odd **or** n and k are both even and k≥n/2.

(3) N\ {2} for all even n ≥k and k odd **or** n and k are both even and k≤n/2.

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