A Computational Approach

for

the Ramsey Numbers $R(C_4, K_n)$

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Abstract

For graphs G and H, the Ramsey number $R(G, H)$ is the least integer n such that every 2-coloring of the edges of K_n contains a subgraph isomorphic to G in the first color or a subgraph isomorphic to H in the second color. Graph G is a (C_4, K_n)-graph if G doesn't contain a cycle C_4 and G has no independent set of order n. Jayawardene and Rousseau showed that $21 \leq R(C_4, K_7) \leq 22$. In this work we determine $R(C_4, K_7) = 22$ and $R(C_4, K_8) = 26$, and enumerate various families of (C_4, K_n)-graphs. In particular, we construct all (C_4, K_n)-graphs for $n < 7$, and all (C_4, K_7)-graphs on at least 19 vertices. Most of the results are based on computer algorithms.

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1. Introduction

We shall only consider graphs without multiple edges or loops, on a nonempty set of vertices. For graphs G and H, a (G, H)-graph is a graph F without a subgraph isomorphic to G, and such that the complement $\overline{F}$ has no subgraph isomorphic to H. A $(G, H; n)$-graph is a (G, H)-graph of order n. Similarly, a $(G, H; n, e)$-graph is a $(G, H; n)$-graph with e edges. Let $\mathcal{R}(G, H)$, $\mathcal{R}(G, H; n)$ and $\mathcal{R}(G, H; n, e)$ denote the set of all (G, H)-graphs, $(G, H; n)$-graphs and $(G, H; n, e)$-graphs, respectively. The Ramsey number $R(G, H)$ is defined to be the least $n > 0$ such that there is no $(G, H; n)$-graph. Instead of a graph F of order n, one often considers an equivalent concept of a 2-coloring of edges of the complete graph K_n, where we identify F with the edges in the first color, and the complement $\overline{F}$ with the edges in the second color. Thus, for example, the Ramsey number $R(G, H)$ can be defined equivalently as the minimal n such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color.

A regularly updated survey by the first author [13] includes the most recent results on Ramsey numbers $R(G, H)$, for different graphs G and H. This paper considers a special case when G is a cycle C_4 (quadrilateral) and H is a complete graph K_n. The cycle-complete pair of graphs forms perhaps the second most studied case in Ramsey theory after the classical case, in which both G and H are complete.

In Section 2, we overview known results for cycle-complete Ramsey numbers. Section 3 presents the results of our computations: full enumeration of all (C_4, K_n)-graphs for $n < 7$, and of all $(C_4, K_7; m)$-graphs for $m \geq 19$. Based on these enumerations and further computations, we conclude that $R(C_4, K_7) = 22$ and $R(C_4, K_8) = 26$. In Section 4 we describe the algorithms and computations performed. For each task, two separate implementations of each algorithm were prepared by the two authors, the results compared, and no discrepancies were found.

2. Bigger Picture

Known asymptotic upper bounds on cycle-complete Ramsey numbers for fixed cycle are shown in (1) through (5) below, where c_i's are some positive constants.

$$R(C_3, K_n) = \Theta\left(\frac{n^2}{\log n}\right) \tag{1}$$

$$R(C_4, K_n) \leq c_4 \left(\frac{n}{\log n}\right)^2 \tag{2}$$

$$R(C_5, K_n) \leq c_5 \frac{n^{3/2}}{\sqrt{\log n}} \tag{3}$$

$$R(C_{2m}, K_n) \leq c_{2m} \left(\frac{n}{\log n}\right)^{\frac{m}{m-1}} \tag{4}$$

$$R(C_{2m-1}, K_n) \leq c_{2m-1}\, n^{\frac{m}{m-1}} \tag{5}$$

Notice that, in the general case, we have different expressions for the best known bounds for even and odd fixed cycle. For $m = 2$, (4) is the same as (2), but for $m = 3$, (5) is weaker than (3). The exact asymptotics in (1) is the 1995 breakthrough result obtained by Kim [9] for the classical Ramsey numbers; i.e., for $C_3 = K_3$. Caro, Li, Rousseau and Zhang [3] in a recent paper established (3) and (4), and they give credit for (2) to an unpublished result by Erdős and Szemerédi [5]. The bound (5) was derived in an earlier work by Erdős, Faudree, Rousseau and Schelp [4].

Spencer [16] using probabilistic method obtained a lower bound

$$\hat{c}_m \left(\frac{n}{\log n}\right)^{\frac{m-1}{m-2}} \leq R(C_m, K_n),$$

which holds even if all cycles of lengths up to m are forbidden (instead of only C_m). An explicit general construction for the lower bound still remains to be seen.

The situation for a fixed complete graph and a growing cycle length seems to be somewhat easier. The following amazingly simple conjecture was posed in 1974, and to date its various parts (for different ranges of n and m) have been confirmed by several authors.

Conjecture (Erdős, Faudree, Rousseau and Schelp [4]).

For all $n \geq m \geq 3$, except $(3,3)$,

$$R(C_n, K_m) = (m-1)(n-1) + 1 \tag{6}$$

Equation (6) was initially known to be true for all $n \geq m^2 - 2$ [2]. In addition, it has been proved for all $n \geq m$, for $m \leq 6$; namely for $m = 3$ [6], $m = 4$ [17], $m = 5$ [1], $m = 6$ [15], and recently also for all $m \geq 7$ with $n \geq m^2 - 2m$ [15]. An exception for $(n,m) = (3,3)$ must be made since $R(C_3, K_3) = R(K_3, K_3) = 6$. In Table I below, we have collected known and conjectured (marked with a *c*) small values of $R(C_n, K_m)$. Further detailed references to papers establishing specific values are listed in [13].

Three recent papers by Jayawardene and Rousseau contain results involving a quadrilateral C_4: the exact values of $R(C_4, G)$ for $G = K_6$ [14], and later for all graphs G on at most 6 vertices [8], and the bounds $21 \leq R(C_4, K_7) \leq 22$ [7]. The main contribution of our paper is the computation of two more exact values of $R(C_4, K_m)$, shown in Table I in boldface. We hope that the latter and the knowledge of some families of graphs $\mathcal{R}(C_4, K_m)$, for small m, can provide foundation for a general construction establishing a good lower bound for $R(C_4, K_m)$.

	C_3	C_4	C_5	C_6	C_7	C_8	...	C_n
K_3	6	7	9	11	13	15		$2n-1$
K_4	9	10	13	16	19	22		$3n-2$
K_5	14	14	17	21	25	29		$4n-3$
K_6	18	18	21	26	31	36		$5n-4$
K_7	23	**22**	25	?	37^c	43^c		$6n-5^c$
K_8	28	**26**	?	?	?	50^c		$7n-6^c$

Table I. Known and conjectured small values of $R(C_n, K_m)$.

e \ n	1	2	3	4	5	6	total
0	1	1					2
1		1	1				2
2			1	1			2
3			1	2			3
4				1	1		2
5					2		2
6					1	1	2
7						1	1
total	1	2	3	4	4	2	16

Table II. Statistics for $(C_4, K_3; n, e)$-graphs.

e \ n	1	2	3	4	5	6	7	8	9	total
0	1	1	1							3
1		1	1	1						3
2			1	2	1					4
3			1	3	3	1				8
4				1	4	2				7
5					4	6	1			11
6					1	9	4			14
7						4	9	1		14
8							11	3		14
9							5	6	1	12
10								9	1	10
11								3	2	5
12									3	3
13									1	1
total	1	2	4	7	13	22	30	22	8	109

Table III. Statistics for $(C_4, K_4; n, e)$-graphs.

3. Enumerations and Results

We present here statistics from enumeration of various families $\mathcal{R}(C_4, K_m)$ obtained by algorithms and computations outlined in Section 4. The Tables II, III, IV and V give the number of nonisomorphic $(C_4, K_m; n, e)$-graphs for $m = 3$, 4, 5 and 6, respectively, for all possible values of n and e (the columns for $n < 6$ in Table V are omitted, since they are the same as the corresponding ones in Table IV, except that the empty graph on 5 vertices is a $(C_4, K_6; 5, 0)$-graph). This detailed data may be useful in future work towards deriving bounds on the minimum and maximum number of edges in general $(C_4, K_m; n)$-graphs, which in turn may lead to better bounds for

e \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	total
0	1	1	1	1										4
1		1	1	1	1									4
2			1	2	2	1								6
3			1	3	4	4	1							13
4				1	5	7	3	1						17
5					4	11	10	2						27
6					1	11	22	9	1					44
7						4	27	27	4					62
8							17	53	16	1				87
9							5	62	50	5				122
10								31	108	18	1			158
11								5	130	55	3			193
12									66	138	10	1		215
13									10	200	32	1		243
14										126	75	3		204
15										29	129	9		167
16										2	139	15		156
17											59	22		81
18											9	33		42
19												25		25
20												14		14
21												3		3
22														0
23														0
24													1	1
total	1	2	4	8	17	38	85	190	385	574	457	126	1	1888

Table IV. Statistics for $(C_4, K_5; n, e)$-graphs.

general Ramsey numbers of the form $R(C_4, K_m)$. We note that a similar approach worked for the classical Ramsey numbers $R(K_3, K_m)$ and $R(K_4, K_m)$ [12]; cf. [13].

We found an agreement between the results of our computations and all data presented in [7], with an exception of the number of graphs in $\mathcal{R}(C_4, K_6; 17)$: there are only 5 such graphs, not 6. The authors of [7] missed that the two bottom graphs in their Lemma 2 on page 17 are isomorphic.

It was computationally infeasible to generate all of $\mathcal{R}(C_4, K_7)$, but starting from the full enumeration of $\mathcal{R}(C_4, K_6)$ we managed to enumerate all $(C_4, K_7; n)$-graphs for $n \geq 19$, and their statistics is presented in Table VI. These are the graphs which were used for further computation of the exact value of $R(C_4, K_8) = 26$.

e \ n	6	7	8	9	10	11	12	13	14	15	16	17
1	1											
2	2	1										
3	5	4	1									
4	8	8	4	1								
5	12	17	13	3	1							
6	11	28	33	13	2							
7	4	30	63	45	9	1						
8		17	85	117	37	4						
9		5	72	222	135	19	1					
10			31	274	380	82	5					
11			5	197	757	316	25	1				
12				74	944	1005	115	5				
13				10	649	2299	483	22	1			
14					221	3237	1753	97	3			
15					34	2484	4859	425	11	1		
16					2	931	8783	1624	47	1		
17						146	8847	5166	188	3		
18						11	4402	12436	703	11		
19							946	19102	2280	36		
20							82	15468	6151	112		
21							3	5618	13091	330		
22								785	19290	823		
23								38	15181	1815		
24								2	4933	3522		
25									565	5487		
26									24	5294	1	
27									1	2275	12	
28										338	68	
29										20	166	
30										2	204	
31											97	
32											11	
33											2	
34												1
35												4
	43	110	307	956	3171	10535	30304	60789	62469	20070	561	5

Table V. Statistics for $(C_4, K_6; n, e)$-graphs, $|\mathcal{R}(C_4, K_6)| = 189353$.

Theorem 1. $R(C_4, K_7) = 22$.

Proof. Jayawardene and Rousseau [7] proved that $R(C_4, K_7) \leq 22$. Independently, our computations described in Section 4 showed the nonexistence of $(C_4, K_7; 22)$-graphs, and thus confirmed this upper bound. Figure 1 presents an adjacency matrix of a $(C_4, K_7; 21, 45)$-graph G establishing the lower bound. Actually, we claim that

e \ n	19	20	21
29	1		
30	18		
31	233		
32	2399		
33	17474		
34	83786		
35	261093		
36	520551		
37	605219	1	
38	328849	12	
39	64919	126	
40	4132	999	
41	107	3611	
42	4	3762	
43		897	
44		53	
45		2	1
46			2
total	1888785	9463	3

Table VI. Statistics for $(C_4, K_7; n, e)$-graphs, for $n \geq 19$.

there are exactly 3 nonisomorphic $(C_4, K_7; 21)$-graphs (see Table VI). ∎

Graph G, presented in Figure 1, has four orbits of vertices. For a reference we give its automorphism group. Define

$$g_1 = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21),$$
$$g_2 = (1\ 2)(3\ 6)(4\ 5)(7\ 8)(9\ 12)(10\ 11)(13)(14)(15\ 18)(16\ 17)(19)(20\ 21),$$
$$g_3 = (1\ 3\ 6\ 2\ 4\ 5)(7\ 10\ 11\ 8\ 9\ 12)(13\ 16\ 18\ 14\ 15\ 17)(19\ 20\ 21).$$

Then the full automorphism group of the graph G is defined by $\mathrm{Aut}(G) = \langle g_1, g_2, g_3 \rangle$, a group of order 12. The full automorphism groups of the other two $(C_4, K_7; 21)$-graphs have orders 2 and 4, respectively.

Theorem 2. $R(C_4, K_8) = 26$.

Proof. Figure 2 presents an adjacency matrix of a $(C_4, K_8; 25, 60)$-graph H establishing the lower bound. The nonexistence of $(C_4, K_8; 26)$-graphs, implying the upper bound, follows from the computations described in Section 4. ∎

1	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0
2	0	0	0	1	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0
3	1	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
4	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0
5	1	0	0	1	0	0	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0
6	0	1	1	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0
7	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0
8	0	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0
9	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0
10	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0
11	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1
12	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	1
13	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0
14	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
15	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
16	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0
17	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
18	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
19	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0

Figure 1. Adjacency matrix of a $(C_4, K_7; 21, 45)$-graph G.

The graph H, presented in Figure 2, has two orbits of vertices. The first 10 vertices are of degree 6, they induce the Petersen graph, and let us denote by H_{10} the subgraph induced by them. The other 15 vertices are of degree 4, and they induce $5K_3$, i.e., five vertex-disjoint triangles. The graph H has a large automorphism group, isomorphic to that of the Petersen graph, since each automorphism of H_{10} (out of 120 automorphisms of the Petersen graph) extends uniquely to an automorphism of H.

We have found 36 $(C_4, K_8; 25)$-graphs, with the number of edges ranging from 58 to 61, and having automorphism group orders not exceeding 10, except for the graph H, for which $|\mathrm{Aut}(H)| = 120$. We don't claim that we have obtained a full enumeration of $\mathcal{R}(C_4, K_8; 25)$, but it is likely that no other $(C_4, K_8; 25)$-graphs exist.

Our computations led also to the construction of several $(C_4, K_9; 29)$-graphs and $(C_4, K_{10}; 33)$-graphs, which establish the lower bounds listed in the next theorem. We don't present these graphs, since they were not very difficult to find, and it is quite possible that larger graphs for the same parameters can be constructed.

Theorem 3. $R(C_4, K_9) \geq 30$ and $R(C_4, K_{10}) \geq 34$.

1	0	1	0	0	0	0	1	0	0	1	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	1	1	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0
4	0	0	1	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0
5	0	0	0	0	0	1	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
6	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0
7	1	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0
8	0	1	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0
9	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0
10	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1
11	1	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	1	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0
15	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
16	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0
17	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
18	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
19	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
20	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
21	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
22	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
23	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0
24	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
25	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0

Figure 2. Adjacency matrix of a $(C_4, K_8; 25, 60)$-graph H.

4. Algorithms and Computations

We will find it convenient to adopt the following notational conventions. If G is a graph, then VG and EG are its vertex set and edge set, respectively. If $v \in VG$, then $N_G(v) = \{w \in VG \mid vw \in EG\}$, and let $deg_G(v) = |N_G(v)|$. The subgraph of G induced by W will be denoted by $G[W]$. Also, for $v \in VG$, define the induced subgraphs $G_v^+ = G[N_G(v)]$ and $G_v^- = G[VG - N_G(v) - \{v\}]$.

Note that if $G \in \mathcal{R}(C_4, K_m; n)$ and $v \in VG$, then $G_v^+ \in \mathcal{R}(P_3, K_m; d)$, where $d = deg_G(v)$, and $G_v^- \in \mathcal{R}(C_4, K_{m-1}; n-d-1)$. Hence, G_v^+ must be simply a disjoint union of isolated edges and vertices, and G_v^- is of the same type as G, but for $m-1$. These properties formed the basis for one of our algorithms to enumerate graphs in $\mathcal{R}(C_4, K_m; n)$.

Lemma. *If a C_4-free graph G with n vertices has minimum degree d, then*

$$d^2 - d + 1 \leq n \tag{7}$$

Proof. Let v be a vertex of minimum degree d, so $|VG_v^+| = d$. No two distinct vertices in VG_v^+ may have a common neighbor in VG_v^-, and thus their neighborhoods cover disjoint subsets of VG_v^-. G_v^+ is P_3-free, so for each $x \in VG_v^+$ at least $d-2$ edges join x to VG_v^-. Hence $d(d-2) \leq n - d - 1 = |VG^-|$, and the lemma follows. ∎

Suppose we have a particular $X \in \mathcal{R}(P_3, K_m; s)$ and $Y \in \mathcal{R}(C_4, K_{m-1}; t)$, and we wish to build them into a graph $G \in \mathcal{R}(C_4, K_m; s+t+1)$, by adding a new vertex v of degree s, so that $X = G_v^+$ and $Y = G_v^-$. We need to choose the edges between X and Y. A *feasible cone* is a subset of VY that does not cover both endpoints of any P_3 in Y. To avoid C_4, the neighborhood in Y of each vertex in X must be a feasible cone.

Two algorithms were implemented to generate various subfamilies of $\mathcal{R}(C_4, K_m; n)$.

Algorithm 1: Given graph $G \in \mathcal{R}(C_4, K_m; n)$ generate all one-vertex extensions of G which are in $\mathcal{R}(C_4, K_m; n+1)$.

Algorithm 2: Given $X \in \mathcal{R}(P_3, K_m; s)$ and $Y \in \mathcal{R}(C_4, K_{m-1}; t)$ generate all graphs $G \in \mathcal{R}(C_4, K_m; s+t+1)$ such that $X = G_v^+$ and $Y = G_v^-$.

Algorithm 1 is a standard procedure in graph theoretical computations, here with the performance enhanced by the degree restriction of (7), and by other obvious conditions to avoid C_4 and $\overline{K}_m$ containing the new vertex. This algorithm was sufficient to generate all (C_4, K_m)-graphs for $m \leq 6$, and the results are reported in Tables II through V. There are too many $(C_4, K_7; \leq 18)$-graphs to generate them all by any reasonable approach.

Algorithm 2, significantly more sophisticated than Algorithm 1, assigns in all possible ways feasible cones to vertices in G_v^+, so that C_4 and $\overline{K}_m$ are avoided in G. In particular, no two cones assigned to distinct vertices in G_v^+ may have nonempty intersection. The method by which this and other rules were built into a search procedure was very similar to that of [11] and [12], so we will not repeat its details.

Algorithm 2 was first tested by using it to generate several subfamilies of (C_4, K_m)-graphs for $m \leq 6$, and it agreed with Algorithm 1. Next, $(C_4, K_7; n)$-graphs for all $n \geq 19$ were generated, and the results are reported in Table VI. For example, all $(C_4, K_7; 21)$-graphs were obtained as follows. By (7) and $R(C_4, K_6) = 18$ the minimum degree d must be 3, 4 or 5. Applying Algorithm 2 to $X \in \mathcal{R}(P_3, K_7; 3)$

(there are two P_3-free graphs on 3 vertices) and $Y \in \mathcal{R}(C_4, K_6; 17)$ produced no graphs, and applying Algorithm 2 to $X \in \mathcal{R}(P_3, K_7; 4)$ (there are three P_3-free graphs on 4 vertices) and $Y \in \mathcal{R}(C_4, K_6; 16)$ produced three $(C_4, K_7; 21)$-graphs with minimum degree 4. Similarly, Algorithm 2 was used to show that no such graph may have minimum degree 5.

By the Lemma above and $R(C_4, K_7) = 22$, any $(C_4, K_8; 26)$-graph must have minimum degree 4 or 5. To compute all such graphs, Algorithm 2 was used again with $Y \in \mathcal{R}(C_4, K_7; 21)$ and $Y \in \mathcal{R}(C_4, K_7; 20)$, respectively. No such graphs were found, and thus $R(C_4, K_8) \leq 26$. Note that, using (7) and $R(C_4, K_8) = 26$, one can conclude easy upper bounds $R(C_4, K_9) \leq 33$ and $R(C_4, K_{10}) \leq 40$ without any computations.

The computational effort of this project was moderate — all computations could now be repeated overnight on a local departmental network. A general utility program for graph isomorph rejection, *nauty* [10], written by Brendan McKay, was used extensively. The graphs themselves are available from the first author.

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