On a New Formulation of Xinrong for the Embedding of Catalan Numbers in Series Forms of the Sine Function

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Abstract

In developing an observation made by the author concerning a class of expansions of the sine function, M. Xinrong has recently analysed the question of a generalised form through a succinct use of linear operator theory. This paper constitutes an extension of his work, in which the current problem is solved completely by examining that generating function of a finite sequence central to the formulation.

Introduction

In 1988 the scientific historian Luo published a paper [1]—the first of a group of related works—accrediting Chinese scholar Antu Ming with the discovery of the Catalan numbers through some expansions of the trigonometric function $\sin(m\alpha)$ in odd powers of $\sin(\alpha)$ (m integer). For m odd the resulting expressions are finite, whilst m even gives rise to an infinite series representation wherein the Catalan numbers appear in a prescribed manner that is both surprising and interesting. This subject was discussed in an article [2] by the author, who gave proofs of the non-terminating series when $m = 2, 4$ and further deduced those for $m = 6, 8, 10$. Denoting the $(n+1)$th

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term of the Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$ as

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

it was evident in [2] that the result

$$\sin(2\alpha) = 2\left\{\sin(\alpha) - \sum_{n=1}^{\infty}\left[\frac{c_{n-1}}{2^{2n-1}}\right]\sin^{2n+1}(\alpha)\right\} \tag{2}$$

is the critical one, for it serves as the foundation for all others; on the basis of this, expansions for $\sin(4\alpha), \sin(6\alpha), \sin(8\alpha), \ldots$, can be developed sequentially using the simple recursion

$$\sin[(m+2)\alpha] = 2\left\{[1 - 2\sin^2(\alpha)]\sin(m\alpha) - (1/2)\sin[(m-2)\alpha]\right\} \tag{3}$$

applied respectively for $m = 2, 4, 6$, *etc.*, and the embedding of Catalan numbers in each is dictated by their initial occurrence in (2) as shown (such is the significance of this that the realisation of (2), via the theory of hypergeometric functions, has been dealt with in [3] as a separate topic[1]).

Those cases treated in [2] allowed the following generalised statement to be written down by Larcombe for integer $p \geq 1$:

$$\begin{aligned}\sin(2p\alpha) \;=\; & 2\left\{\sum_{n=1}^{p}\alpha_n^{(p)}\sin^{2n-1}(\alpha)\right. \\ & \left. + \sum_{n=1}^{\infty} f_p(n) g_p(c_{n-1}, \ldots, c_{n+p-2})\sin^{2(n+p)-1}(\alpha)\right\}, \end{aligned} \tag{4}$$

where $\alpha_n^{(p)}$ is a constant and f_p, g_p are functions. Note that the r.h.s. is convergent for $|\alpha| < \frac{\pi}{2}$. More importantly, from the methodology in [2] it became apparent that g_p is always a *linear* sum of the given p Catalan elements—a fact that has since provided motivation for a subsequent article by Xinrong [4] where the author's analysis of [2] is extended. We shall summarise briefly some of the work in [4] directly related to (4), to which addition will be made. Specifically, the general coefficient in the abovementioned sum is found, and then simplified to a final closed form by appeal to hypergeometric function theory. Computational aspects of its calculation are also discussed, and to finish the complete (p-variable) function g_p is determined as a reduced univariate one. All of this is done in the context of a slightly re-cast problem, as will now be explained.

[1] For clarity we emphasise, as done in [3, see Footnote 1], that in [2] $c_n = \frac{1}{n}\binom{2(n-1)}{n-1}$ is the nth Catalan number with $\{1, 1, 2, 5, 14, \ldots\} = \{c_1, c_2, c_3, c_4, c_5, \ldots\}$; this should prevent any confusion on the part of the reader.

The Analysis of Xinrong

Upon close inspection, the systematic build up of results in [2] is seen to rapidly become less and less algebraically tractable with increasing p (it is worth mentioning that neither, if at all even possible, can the manipulations involved be easily automated symbolically), and the procedure taken, whilst quite natural, does not yield any further insight into the structure of g_p other than suggesting its general linearity for any chosen p. Very recently, Xinrong [4] has adopted the modified version

$$\sin(2p\alpha) = 2\left\{\sum_{n=1}^{p} \alpha_n^{(p)} \sin^{2n-1}(\alpha) + \sum_{n=1}^{\infty} \frac{h_p(c_{n-1},\ldots,c_{n+p-2})}{2^{2(n+p)-3}} \sin^{2(n+p)-1}(\alpha)\right\} \tag{5}$$

of (4) and derived analytically a means to obtain h_p computationally, terming (4) Ming's Theorem and the question of finding f_p, g_p in closed form Ming's Problem; in effect Xinrong standardises (4) to (5), leaving Ming's Problem as the formulation of h_p.[2] Writing the linear function h_p as

$$h_p(c_{n-1},\ldots,c_{n+p-2}) = \beta_0^{(p)} c_{n-1} + \beta_1^{(p)} c_n + \cdots + \beta_{p-1}^{(p)} c_{n+p-2}, \tag{6}$$

a degree $p-1$ polynomial (with *matching* coefficients)

$$H_p(x) = \beta_0^{(p)} + \beta_1^{(p)} x + \cdots + \beta_{p-1}^{(p)} x^{p-1} \tag{7}$$

is associated with h_p so that (in standard combinatorics notation)

$$\beta_i^{(p)} = [x^i]\{H_p(x)\}, \qquad i = 0,\ldots,p-1. \tag{8}$$

Moreover, underpinned by operator theory (taken from the authoritative paper on umbral calculus by Roman and Rota [5], and applied to the assumed form (5) of $\sin(2p\alpha)$) $H_p(x)$ is identified as

$$H_p(x) = (-1)^{p+1} \frac{(2-\sqrt{4-x})^{2p} - (2+\sqrt{4-x})^{2p}}{8\sqrt{4-x}}, \qquad p \geq 1, \tag{9}$$

acting as the (ordinary) generating function for the finite sequence of coefficients $\{\beta_0^{(p)},\ldots,\beta_{p-1}^{(p)}\}$ (the multiplier $(-1)^{p+1}$ is omitted in error in

[2]This appellation is perhaps a little curious. Ming produced but eight expansions of $\sin(m\alpha)$—six for m even, two for m odd (see [2])—and as far as the author is aware made no conjecture of an extrapolated representation such as (4).

Theorem 1.2 of [4]). Comparing (4) and (5) we observe that if, for p fixed, $f_p(n) = 2^{3-2(n+p)}$ then g_p and h_p clearly coincide. Noting that (2) gives $h_1(c_{n-1}) = -c_{n-1}$, by way of further example we list the other explicit series detailed in [2] for $p = 2, 3, 4, 5$, from which the corresponding function h_p is equally immediate in each instance. We have

$$\sin(4\alpha) = 2\left\{2\sin(\alpha) - 5\sin^3(\alpha) + \sum_{n=1}^{\infty}\left[\frac{8c_{n-1} - c_n}{4^n}\right]\sin^{2n+3}(\alpha)\right\}, \quad (10)$$

so

$$h_2(c_{n-1}, c_n) = 2(8c_{n-1} - c_n), \quad (11)$$

together with

$$\begin{aligned}
\sin(6\alpha) &= 2\left\{3\sin(\alpha) - (35/2)\sin^3(\alpha) + (189/8)\sin^5(\alpha)\right. \\
&\quad \left. - \sum_{n=1}^{\infty}\left[\frac{256c_{n-1} - 64c_n + 3c_{n+1}}{2^{2n+3}}\right]\sin^{2n+5}(\alpha)\right\}, \\
\sin(8\alpha) &= 2\left\{4\sin(\alpha) - 42\sin^3(\alpha) + (231/2)\sin^5(\alpha) - (429/4)\sin^7(\alpha)\right. \\
&\quad \left. + \sum_{n=1}^{\infty}\left[\frac{1024c_{n-1} - 384c_n + 40c_{n+1} - c_{n+2}}{2^{2n+3}}\right]\sin^{2n+7}(\alpha)\right\}, \\
\sin(10\alpha) &= 2\left\{5\sin(\alpha) - (165/2)\sin^3(\alpha) + (3003/8)\sin^5(\alpha)\right. \\
&\quad - (10725/16)\sin^7(\alpha) + (60775/128)\sin^9(\alpha) \\
&\quad \left. - \sum_{n=1}^{\infty}\left[\frac{g(c_{n-1}, c_n, c_{n+1}, c_{n+2}, c_{n+3})}{2^{2n+7}}\right]\sin^{2n+9}(\alpha)\right\} \quad (12)
\end{aligned}$$

(with linear function $g(c_{n-1}, \ldots, c_{n+3}) = 65536c_{n-1} - 32768c_n + 5376c_{n+1} - 320c_{n+2} + 5c_{n+3}$), whence

$$\begin{aligned}
h_3(c_{n-1}, c_n, c_{n+1}) &= -(256c_{n-1} - 64c_n + 3c_{n+1}), \\
h_4(c_{n-1}, c_n, c_{n+1}, c_{n+2}) &= 4(1024c_{n-1} - 384c_n + 40c_{n+1} - c_{n+2}), \\
h_5(c_{n-1}, c_n, c_{n+1}, c_{n+2}, c_{n+3}) &= \\
&\quad -(65536c_{n-1} - 32768c_n + 5376c_{n+1} - 320c_{n+2} + 5c_{n+3}). \quad (13)
\end{aligned}$$

With reference to (6) (and (7)), the coefficients in $h_1, \ldots, h_5$ above are readily checked by calculating the polynomial $H_p(x)$ (9) (for the appropriate value of p, in which case it is found that $H_1(x) = -1$, $H_2(x) = 16 - 2x$, $H_3(x) = -256+64x-3x^2$, and so on). In general, the latter is accomplished realistically using any one of the several mainstream computer algebra packages currently available commercially.

Further Results

This section constitutes the part of the paper in which, based on Xinrong's standardised format for $\sin(2p\alpha)$ as described, new results of interest are presented and an analytic representation of the function h_p established which is dependent only on one Catalan number.

We first state, and prove, an explicit form for the general coefficient of the polynomial $H_p(x)$ (7) (and of the sum h_p (6)) using (9).

<u>Lemma 1</u> For $p \geq 1$, $0 \leq n \leq p-1$,

$$\beta_n^{(p)} = (-1)^{n+p} 2^{2(p-n)-3} \sum_{i=n}^{p-1} \binom{i}{n} \binom{2p}{2i+1}.$$

<u>Proof</u> Rather than expand $H_p(x)$ (9) as a power series in $4 - x$, it is rewritten more conveniently as

$$H_p(x) = (-1)^{p+1} 4^{p-2} \frac{(1-\sqrt{1-\frac{x}{4}})^{2p} - (1+\sqrt{1-\frac{x}{4}})^{2p}}{\sqrt{1-\frac{x}{4}}}, \qquad p \geq 1.$$

Denoting $\sqrt{1-\frac{x}{4}}$ as $f(x)$, consider

$$\begin{aligned}\left(1-\sqrt{1-\frac{x}{4}}\right)^{2p} - \left(1+\sqrt{1-\frac{x}{4}}\right)^{2p} &= [1-f(x)]^{2p} - [1+f(x)]^{2p} \\ &= \sum_{i=0}^{2p}[(-1)^i - 1]\binom{2p}{i} f^i(x).\end{aligned}$$

Because the summand is non-zero only for odd values $i = 1, 3, 5, \ldots, 2p-1$, we have

$$\begin{aligned}\left(1-\sqrt{1-\frac{x}{4}}\right)^{2p} - \left(1+\sqrt{1-\frac{x}{4}}\right)^{2p} &= -2\sum_{i=1}^{p}\binom{2p}{2i-1} f^{2i-1}(x) \\ &= -2\sum_{i=0}^{p-1}\binom{2p}{2i+1} f^{2i+1}(x),\end{aligned}$$

with a small shift in the summing index. Hence,

$$\begin{aligned} H_p(x) &= (-1)^p 2^{2p-3} \sum_{i=0}^{p-1} \binom{2p}{2i+1} f^{2i}(x) \\ &= (-1)^p 2^{2p-3} \sum_{i=0}^{p-1} \binom{2p}{2i+1} \left(1 - \frac{x}{4}\right)^i, \end{aligned}$$

and since $[x^n]\{(1-\frac{x}{4})^i\} = (-\frac{1}{4})^n \binom{i}{n}$ $(0 \le n \le i)$, then for $0 \le n \le p-1$, by definition (8)

$$\beta_n^{(p)} = [x^n]\{H_p(x)\} = (-1)^{n+p} 2^{2(p-n)-3} \sum_{i=n}^{p-1} \binom{i}{n} \binom{2p}{2i+1}. \square$$

Extensive computations have verified this result.[3]

Remark 1 The integrality of $\beta_n^{(p)}$ is easily deduced as a corollary to Lemma 1. Noting that the sum of binomial coefficient products therein is always a whole number, it suffices to show that the exponent $e(p;n) = 2(p-n)-3 \ge 0$ $(0 \le n \le p-1)$. When $0 \le n \le p-2$ then $e(p;n) \ge 1$ (and odd). However, $e(p;p-1) = -1$ so for this case we consider the full coefficient $\beta_{p-1}^{(p)} = -\frac{1}{2}\binom{2p}{2p-1} = -p$, which is integer (and, as required, concurs with that of the relevant term of those functions $h_1, \ldots, h_5$ given earlier).

We are now in a position to simplify $\beta_n^{(p)}$ to a better (*i.e.*, single binomial coefficient) closed form without too much difficulty. Consider, from Lemma 1, the sum

$$\begin{aligned} S(p;n) &= \sum_{i=n}^{p-1} \binom{i}{n} \binom{2p}{2i+1} \\ &= \sum_{i=0}^{p-n-1} \binom{i+n}{n} \binom{2p}{2(i+n)+1} \\ &= \binom{2p}{2n+1} {}_2F_1\left(\left. \begin{matrix} -(p-n-1), -(p-n-\frac{1}{2}) \\ n+\frac{3}{2} \end{matrix} \right| 1 \right), \quad (14) \end{aligned}$$

having first modified the summation range so as to start at zero and then converted $S(p;n)$ to hypergeometric form (with standard notation employed) by means of undergraduate level theory (see, for example, either of

[3]Thanks are due to Dr. Kate Woodham for performing these calculations, checking that (9) produces coefficients of this general form using the Symbolic Toolbox of MATLAB; results also show agreement with those given at the end of [4] in Table 3.2.

the classic texts [6,7]); briefly, the hypergeometric delineation of an infinite series

$$\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_s)_k}{(b_1)_k (b_2)_k \cdots (b_t)_k} \cdot \frac{z^k}{k!} \tag{15}$$

is

$${}_sF_t \left(\begin{matrix} a_1, a_2, \ldots, a_s \\ b_1, b_2, \ldots, b_t \end{matrix} \middle| \, z \right), \tag{16}$$

with $a_1, a_2, \ldots, a_s$ and $b_1, b_2, \ldots, b_t$ its respective upper and lower parameters, and z the argument—all possibly complex variables—and where we write

$$(u)_k = u(u+1)(u+2)(u+3)\cdots(u+k-1) \tag{17}$$

to denote the rising factorial function which is defined for integer $k \geq 0$ ($(u)_0 = 1$). Since it applies here, we mention that a negative integer upper parameter reflects the series represented hypergeometrically as being a finite one. Gauss' Theorem—a long established result—gives that a hypergeometric series

$${}_2F_1 \left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} \middle| \, z \right) \tag{18}$$

with unity argument and real a_1, a_2, b_1 (b_1 also non-integer or a positive integer) evaluates to

$$\frac{\Gamma(b_1)\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)} \tag{19}$$

iff $b_1 - (a_1 + a_2) > 0$. The corresponding condition to be satisfied by F in (14) is $2p - n > 0$, which clearly holds since $n \leq p - 1 < p < 2p$ for $p \geq 1$. Thus, in terms of the Gamma function,

$${}_2F_1 \left(\begin{matrix} -(p-n-1), -(p-n-\frac{1}{2}) \\ n + \frac{3}{2} \end{matrix} \middle| \, 1 \right) = \frac{\Gamma(n+\frac{3}{2})\Gamma(2p-n)}{\Gamma(p+\frac{1}{2})\Gamma(p+1)}. \tag{20}$$

Replacing $\Gamma(n+\frac{3}{2})$ with $(n+\frac{1}{2})\Gamma(n+\frac{1}{2})$ and using another known result, namely,

$$\Gamma\left(s + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{4^s} \frac{(2s)!}{s!}, \qquad s \geq 0, \tag{21}$$

it is found after some algebraic manipulation that $S(p; n)$ (14) can be simplified to

$$S(p; n) = 2^{2(p-n)-1} \left(\begin{matrix} 2p - (n+1) \\ n \end{matrix} \right), \tag{22}$$

yielding the desired representation (below) of $\beta_n^{(p)}$ from Lemma 1.

Lemma 2 For $p \geq 1$, $0 \leq n \leq p-1$,

$$\beta_n^{(p)} = (-1)^{n+p} 16^{p-(n+1)} \binom{2p-(n+1)}{n}.$$

Remark 2 The type of identity (22) (for the sum of binomial coefficient terms of which $S(p;n)$ consists) does not appear in Gould's listing of the 1970s [8] which is still consulted today as a useful resource for such results.

The Lemma 2 form of $\beta_n^{(p)}$ is quite obviously more computationally efficient than that of Lemma 1. Improvement can still be made in this regard, however, by applying Zeilberger's algorithm by computer (its algebraic implementation here is due to Prof. Dr. Wolfram Koepf [9], to whom thanks are expressed by the author) which offers up a recursive formula for the coefficient. Writing

$$ {}_2F_1\left(\begin{matrix} -(p-n-1), -(p-n-\frac{1}{2}) \\ n+\frac{3}{2} \end{matrix} \middle| 1 \right) = F(p;n), \tag{23}$$

then by (14)

$$S(p;n) = u(p;n)F(p;n) \tag{24}$$

where

$$u(p;n) = \binom{2p}{2n+1}, \tag{25}$$

and symbolic output gives a relation

$$(2n+3)F(p;n) - 2[2p-(n+1)]F(p;n+1) = 0. \tag{26}$$

In other words,

$$[p-(n+1)][2(p-n)-1]S(p;n) = 2(n+1)[2p-(n+1)]S(p;n+1), \tag{27}$$

using (24),(25). In view of Lemma 1, we thus arrive at an equation linking successive coefficients of interest—we state this as Lemma 3.

Lemma 3

$$\beta_{n-1}^{(p)} = -8\frac{n(2p-n)}{(p-n)[2(p-n)+1]}\beta_n^{(p)}, \qquad p \geq 2.$$

For fixed $p \geq 2$ (when $p = 1$, $\beta_0^{(1)} = -1$ trivially from either Lemma 1 or Lemma 2) the result can be used for descending values of $n = p-1, \ldots, 1$, given that $\beta_{p-1}^{(p)} = -p$ (see Remark 1[4]), or equally for ascending values of n

[4]Lemma 2 gives $\beta_{p-1}^{(p)} = -\binom{p}{p-1} = -p$ as an easy check.

with initial value (by Lemma 2) $\beta_0^{(p)} = (-1)^p 16^{p-1}$. Note that the reason for the alternating sign in neighbouring coefficients is self-evident here, as indeed it is from Lemmas 1,2.

Remark 3 Zeilberger's algorithm has been executed for illustrative purposes, (26) demonstrating the power of modern combinatorial software that deals with hypergeometric functions in this way. The reader may, of course, care to derive Lemma 3 directly from Lemma 2. We have not considered the counterpart recursion satisfied by $F(p;n)$ w.r.t. p; this is, for completeness, $2(2p-n)(2p-n+1)F(p;n)-(p+1)(2p+1)F(p+1;n) = 0$ (generated similarly by computer), which would relate $\beta_n^{(p)}$ and $\beta_n^{(p+1)}$.

We now determine the entire function $h_p(c_{n-1},\ldots,c_{n+p-2})$ (6) in compact form as the main result of the article. A routine hand calculation yields, from Lemma 3 and (1),

$$\begin{aligned} h_p &= \sum_{i=0}^{p-1} \beta_i^{(p)} c_{n+i-1} \\ &= {}_3F_2\left(\begin{matrix} -(p-1), -(p-\frac{1}{2}), n-\frac{1}{2} \\ -(2p-1), n+1 \end{matrix} \,\middle|\, 1\right) \beta_0^{(p)} c_{n-1}, \end{aligned} \tag{28}$$

with, further,

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} -(p-1), -(p-\frac{1}{2}), n-\frac{1}{2} \\ -(2p-1), n+1 \end{matrix} \,\middle|\, 1\right) \\ &\qquad = \frac{\Gamma(n+1)\Gamma(p+1)\Gamma(p+\frac{1}{2})\Gamma(n+2p-\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(n+p)\Gamma(2p)\Gamma(n+p+\frac{1}{2})} \end{aligned} \tag{29}$$

from simultaneous application of the identity

$${}_3F_2\left(\begin{matrix} a, b, -t \\ c, a+b-c-t+1 \end{matrix} \,\middle|\, 1\right) = \frac{(c-a)_t (c-b)_t}{(c)_t (c-a-b)_t} \tag{30}$$

of Pfaff-Saalschütz (with $a = -(p-\frac{1}{2})$, $b = n-\frac{1}{2}$, $c = n+1$ and $t = p-1$) and conversion of all rising factorials to Gamma function terms using the general result $(s)_t = \Gamma(s+t)/\Gamma(s)$.[5] Employing only (21) eventually leads to a final form for h_p (28) as follows.

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[5]Potential terms of the form $\frac{0}{0}$ in the series (29) (caused by the appearance of the parameters $-(p-1)$, $-(2p-1)$ which, for $p \geq 1$, are either zero or a negative integer) are avoided by a simple limiting argument that justifies the use of (30). Replacing $-(2p-1)$ with $-(2p-1)+\varepsilon$ in (29), and likewise $n-\frac{1}{2}$ with $n-\frac{1}{2}+\varepsilon$ ($|\varepsilon| \ll 1$), Pfaff-Saalschütz still applies. Both sides of the resulting equation are rational functions of ε, which can then be set to zero as a limit.

Theorem For $p, n \geq 1$,

$$h_p(c_{n-1}) = (-1)^p p(n+p)\frac{[2(n+2p-1)]!n!}{(n+2p-1)![2(n+p)]!}c_{n-1}.$$

The result, as it arises naturally, expresses h_p as a (functional) multiple of but the nth Catalan number c_{n-1}. The possibility of such a representation is anticipated because successive Catalan numbers are related through the recursion

$$c_{n+1} = 2\frac{(2n+1)}{(n+2)}c_n \tag{31}$$

(valid for $n \geq 0$, given $c_0 = 1$) established originally by Euler; this could, in principle, be applied repeatedly in a 'cascading' manner to transform the linear function $h_p(c_{n-1}, \ldots, c_{n+p-2})$ to that above. The correctness of the Theorem is readily demonstrated by a few (algebraically) low level cases. Firstly, for $p = 1$ we have

$$h_1(c_{n-1}) = -(n+1)\frac{[2(n+1)]!n!}{(n+1)![2(n+1)]!}c_{n-1} = -c_{n-1} \tag{32}$$

as noted in the previous section. When $p = 2$ the Theorem reads

$$h_2(c_{n-1}) = 2(n+2)\frac{[2(n+3)]!n!}{(n+3)![2(n+2)]!}c_{n-1} = 4\frac{(2n+5)}{(n+1)}c_{n-1}, \tag{33}$$

whilst (11) gives, in agreement,

$$\begin{aligned} h_2(c_{n-1}) &= h_2(c_{n-1}, c_n(c_{n-1})) \\ &= 2[8c_{n-1} - c_n(c_{n-1})] \\ &= 2\left[8c_{n-1} - 2\frac{(2n-1)}{(n+1)}c_{n-1}\right] \\ &= 4\frac{(2n+5)}{(n+1)}c_{n-1}, \end{aligned} \tag{34}$$

having used (31). We emphasise the point by considering the case $p = 3$, for which, by (31) again, (13) gives this time

$$\begin{aligned} h_3(c_{n-1}) &= h_3(c_{n-1}, c_n(c_{n-1}), c_{n+1}(c_{n-1})) \\ &= -[256c_{n-1} - 64c_n(c_{n-1}) + 3c_{n+1}(c_{n-1})] \\ &= -\Bigg[256c_{n-1} - 64\cdot 2\frac{(2n-1)}{(n+1)}c_{n-1} \\ &\qquad + 3\cdot 2\frac{(2n+1)}{(n+2)}\cdot 2\frac{(2n-1)}{(n+1)}c_{n-1}\Bigg] \\ &= -12\frac{(2n+7)(2n+9)}{(n+1)(n+2)}c_{n-1}, \end{aligned} \tag{35}$$

once more consistent with the Theorem (confirmation being left as a short reader exercise). Further examples are dealt with in a similar fashion.

Remark 4 The function h_p formulated as the Theorem assumes a more pleasing form to the eye than Xinrong's multiple product of Theorem 2.1 in [4], and is a considerable improvement on it. In contrast to our result, that of Xinrong is an easy deduction made by equating (5) with an expansion of the sine function that was known to Euler and has been referred to in some detail elsewhere [2,3].

Remark 5 As an aside, we remark that the asymptotic behaviour of h_p is inferred directly from the Theorem. Consider the factorial function

$$\begin{aligned} B_p(n) &= \frac{[2(n+2p-1)]!n!}{(n+2p-1)![2(n+p)]!} \\ &= \frac{(2n+2p+1)(2n+2p+2)\cdots(2n+4p-2)}{(n+1)(n+2)\cdots(n+2p-1)}. \end{aligned} \tag{36}$$

For large $n \gg p \geq 1$ $B_p(n) \sim (2n)^{2p-2}/n^{2p-1} = 4^{p-1}/n$, from which it is straightforward to write down that

$$\begin{aligned} h_p(c_{n-1}) &\sim (-1)^p 4^{p-1} p c_{n-1} \\ &= (-1)^p 4^{p-1} p \frac{1}{n} \begin{pmatrix} 2(n-1) \\ n-1 \end{pmatrix} \\ &\sim \frac{(-1)^p 4^{n+p-2} p}{\sqrt{\pi}} \frac{1}{n\sqrt{n}} \end{aligned} \tag{37}$$

using Stirling's approximation (where $K = \sqrt{2\pi}$)

$$n! \sim K n^{n+\frac{1}{2}} e^{-n}. \tag{38}$$

For prescribed p, at large powers of $\sin(\alpha)$ in the expansion of $\sin(2p\alpha)$ (5), the *full* coefficient $\sim h_p/4^n \sim (-1)^p 4^{p-2} p/(n\sqrt{n\pi}) = O(1/n\sqrt{n})$.

Remark 6 It is interesting to relate the multi-variate function h_p to a corresponding entity of Luo, to whom reference was made at the beginning of the Introduction; we do so formally here. In [2] it was noted that Luo, in his appreciation of the work of Antu Ming (published as a textbook in 1998), assumed a generic expansion

$$\sin(m\alpha) = m\sin(\alpha) + \sum_{n=1}^{\infty} \frac{A_n^{(m)}}{4^{n-1}} \sin^{2n+1}(\alpha) \tag{39}$$

for integer m (even) ≥ 2 (he then went on to develop a recurrence w.r.t. n in $A_n^{(m)}$). Putting $m = 2p$ ($p \geq 1$), this becomes

$$\begin{aligned}
\sin(2p\alpha) &= 2\left\{p\sin(\alpha) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{A_n^{(2p)}}{4^{n-1}}\sin^{2n+1}(\alpha)\right\} \\
&= 2\left\{p\sin(\alpha) + \sum_{n=1}^{\infty}\frac{A_n^{(2p)}}{2^{2n-1}}\sin^{2n+1}(\alpha)\right\} \\
&= 2\left\{p\sin(\alpha) + \sum_{n=1}^{p-1}\frac{A_n^{(2p)}}{2^{2n-1}}\sin^{2n+1}(\alpha)\right. \\
&\qquad\qquad \left. + \sum_{n=p}^{\infty}\frac{A_n^{(2p)}}{2^{2n-1}}\sin^{2n+1}(\alpha)\right\}, \qquad (40)
\end{aligned}$$

comparison of which with (5) gives by inspection that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{h_p}{2^{2(n+p)-3}}\sin^{2(n+p)-1}(\alpha) &= \sum_{n=p}^{\infty}\frac{A_n^{(2p)}}{2^{2n-1}}\sin^{2n+1}(\alpha) \\
&= \sum_{n=1}^{\infty}\frac{A_{n+p-1}^{(2p)}}{2^{2(n+p)-3}}\sin^{2(n+p)-1}(\alpha); \qquad (41)
\end{aligned}$$

that is to say,

$$h_p(c_{n-1}, \ldots, c_{n+p-2}) = A_{n+p-1}^{(2p)}, \qquad p, n \geq 1. \qquad (42)$$

A glance at the Appendix of [2] shows that (42) is correct for $p = 1, 2, 3$.

Remark 7 Not surprisingly, the identity evaluating $h_p = \sum_{i=0}^{p-1}\beta_i^{(p)}c_{n+i-1}$ to the closed form of the Theorem cannot be located in [8]. It is typical of the sort of result available from hypergeometric function theory, and in this instance has been achieved with no need for any of the computer algebra tools designed over the past decade or so to support (and go well beyond) manual analysis of this nature (see [9,10] for more information on the recent provision of software routines/suites in existence).

Summary

The structure to certain series forms of the sine function—examined in [2,3] and studied subsequently in [4] also—has been re-visited in this paper and progress made. In particular, the problem of explicitly determining those linear combinations of Catalan numbers indigenous to such expansions is solved in the general case through the identification of associated coefficients. The complete linear function thus described may then be condensed to a form dependent only on a single Catalan number. Work continues on this topic.

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