On blocking sets in almost balanced path designs *

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Dedicated to "una farfalla senza tempo", the Italian artist Ernesto Treccani on the occasion of his 80th birthday.

Abstract

Let $r(a)$ be the replication number of the vertex a of a path design $P(v,k,1)$, $k \geq 3$. Let $\bar{r}(v,k) = \min\{\max_{a \in V} r(a) \mid (V, \mathcal{B})$ is a $P(v,k,1)\}$. A path design $P(v,k,1)$, $(W, \mathcal{D})$, is said to be *almost balanced* if $\bar{r}(v,k) - 1 \leq r(y) \leq \bar{r}(v,k)$ for each $y \in W$. Let $v \equiv 0$ or $1 \pmod{2(k-1)}$ (for each odd k, $k \geq 3$) and let $v \equiv 0$ or $1 \pmod{k-1}$ (for each even k, $k \geq 4$). In this note we determine the spectrum $\mathcal{BSABP}(v,k,1)$ of integers x such that there exists an almost balanced path design $P(v,k,1)$ with a blocking set of cardinality x.

1 Introduction

Let G be a subgraph of K_v, the complete undirected graph on v vertices. A *G-design* of K_v is a pair $(V, \mathcal{B})$, where V is the vertex set of K_v and $\mathcal{B}$ is an edge-disjoint decomposition of K_v into copies of the graph G. Usually we say that b is a block of the G-design if $b \in \mathcal{B}$, and $\mathcal{B}$ is called the block-set. A G-design of K_v is also called a *G-design of order v*.

A *balanced G-design* [4, 5] is a G-design such that each vertex belongs to the same number of copies of G. Obviously not every G-design is balanced.

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A *(balanced) path design* $P(v,k,1)$ [4] is a (balanced) P_k-design of K_v, where P_k is the simple path with $k-1$ edges (k vertices) $(a_1, a_2, \ldots, a_k) = \{\{a_1,a_2\},\{a_2,a_3\},\ldots,\{a_{k-1},a_k\}\}$.

M. Tarsi [9] proved that the necessary conditions for the existence of a $P(v,k,1)$, $v \geq k$ (if $v > 1$) and $v(v-1) \equiv 0 \pmod{2(k-1)}$, are also sufficient.

S. H. Y. Hung and N. S. Mendelsohn [5] proved that a balanced $P(v, 2h+1, 1)$ ($h \geq 1$) exists if and only if $v \equiv 1 \pmod{4h}$, and a balanced $P(v, 2h, 1)$ ($h \geq 2$) exists if and only if $v \equiv 1 \pmod{2h-1}$.

Let $(V, \mathcal{B})$ be a $P(v,k,1)$. A subset X of V is called a *blocking set* of $\mathcal{B}$ if for each $b \in B$, $b \cap X \neq \emptyset$, and $b \cap (V-X) \neq \emptyset$. A $P(v,k,1)$ with a blocking set X is said to be *2-colorable*, and the partition $(X, V-X)$ is called a *2-coloring*.

Numerous articles have been written on the existence of blocking sets in projective spaces, in t-designs and in G-designs [1, 2, 6, 7].

The spectrum $\mathcal{BSH}(v,k,1)$ of integers x such that there exists a balanced $P(v,k,1)$ with a blocking set of cardinality x is determined in [8].

Theorem 1 *Let $v \equiv 1 \pmod{2(k-1)}$ (for each odd k, $k \geq 3$), and let $v \equiv 1 \pmod{k-1}$ (for each even k, $k \geq 4$). Then it is*

$$\mathcal{BSH}(v,k,1) = \left\{ x \mid \frac{v-1}{k-1} \leq x \leq \frac{(k-2)v+1}{k-1} \right\}.$$

Let v be a positive integer such that $v(v-1) \equiv 0 \pmod{2(k-1)}$. When v not verifies the necessary (and sufficient) conditions for the existence of a balanced $P(v,k,1)$, the following problem is immediate: How *close* can we come to constructing a balanced $P(v,k,1)$? The most satisfying answer seems to be the following. Given a $P(v,k,1)$, $k \geq 3$, say $r(a)$ be the replication number of a vertex a (i.e. $r(a)$ is the number of paths of the decomposition having a as a vertex). Define $\bar{r}(v,k) = \min\{\max_{a \in V} r(a) \mid (V,\mathcal{B}) \text{ is a } P(v,k,1)\}$.

Definition. A path design $P(v,k,1)$ is said to be *almost balanced* if $\bar{r}(v,k) - 1 \leq r(a) \leq \bar{r}(v,k)$ for each vertex a.

Example 1. The following is an example of an almost balanced $P(8,5,1)$ (see Theorem 2). $V = Z_8$, $\mathcal{B} = \{(0,2,6,4,1), (0,7,4,3,2), (0,4,2,7,6), (5,3,0,1,7), (1,3,7,5,4), (2,1,5,6,3), (1,6,0,5,2)\}$.

Let $\mathcal{BSABP}(v,k,1)$ be the spectrum of integers x such that there is an almost balanced $P(v,k,1)$ with a blocking set of cardinality x. For each $v \equiv 1 \pmod{2(k-1)}$ (k odd, $k \geq 3$) or $v \equiv 1 \pmod{k-1}$ (k even, $k \geq 4$), a $P(v,k,1)$ is almost balanced if and only if it is balanced. Then it is $\mathcal{BSABP}(v,k,1) = \mathcal{BSH}(v,k,1)$.

The aim of this note is to determine $\mathcal{BSABP}(v,k,1)$ for each $v \equiv 0 \pmod{2(k-1)}$ if $k \geq 3$ is odd, and $v \equiv 1 \pmod{k-1}$ if $k \geq 4$ is even.

The following theorem gives a limitation to $\bar{r}(v,k)$.

Theorem 2 *Let $v \equiv 0 \pmod{2(k-1)}$ (for each odd k, $k \geq 3$) and let $v \equiv 0 \pmod{k-1}$ (for each even k, $k \geq 4$). Then it is $\bar{r}(v,k) \geq \frac{kv}{2(k-1)}$.*

Proof. Let $\sigma(a)$ be the number of paths of a $P(v,k,1)$ having a as an endpoint. Define $\bar{\sigma}(v,k) = \min\{\max_{a\in V}\sigma(a) \mid (V,\mathcal{B}) \text{ is a } P(v,k,1)\}$. Suppose at first $k=2h$ and $v=(2h-1)(2t-1)$, $h \geq 2$ and $t \geq 2$. Clearly it is

$$v\max_{a\in V}\sigma(a) \geq \sum_{a\in V}\sigma(a) = \frac{v(v-1)}{2h-1}.$$

Hence $\max_{a\in V}\sigma(a) \geq 2t-1-\frac{1}{2h-1}$. The fact that $\max_{a\in V}\sigma(a)$ is an even positive integer implies the following inequality $\max_{a\in V}\sigma(a) \geq 2t = \frac{v}{k-1}+1$. To complete the proof it is sufficient to note that by $r(a) = \frac{v-1-\sigma(a)}{2}+\sigma(a)$, it is $\bar{r}(v,k) = \frac{v-1+\bar{\sigma}(v,k)}{2}$. The remaining cases can be treated in a similar way. □

Theorem 3 (Necessary condition). *Let v and k be given by Theorem 2. Suppose that $\bar{r}(v,k) = \frac{kv}{2(k-1)}$. If $x \in \mathcal{BSABP}(v,k,1)$, then*

$$\frac{v}{k-1} \leq x \leq \frac{(k-2)v}{k-1} \quad \text{for all } k \geq 4$$

$$\frac{v-2}{2} \leq x \leq \frac{v+2}{2} \quad \text{for } k=3.$$

Proof. Put $k=2h+1$, $v=4ht$. Let X be a blocking set in an almost balanced $P(4ht,2h+1,1)$ $(V,\mathcal{B})$, $|X|=x$. Then

$$\sum_{a\in V} r(a) - \frac{x(x-1)}{2} \geq \frac{x(x-1)}{4h}.$$

By $r(a) \leq t(2h+1)$, we obtain

$$\sum_{a\in V} r(a) \leq xt(2h+1).$$

Hence,

$$x \geq \tau = \left\lceil \frac{2t(2h+1)+1-\sqrt{16t^2h^2+4t^2-16ht^2+8th+12t+1}}{2} \right\rceil.$$

It is $2t-2<\tau\leq 2t-1$ if $h=1$ and $2t-1<\tau\leq 2t$ if $h\geq 2$. These inequalities and the fact that $V-X$ is a blocking set imply the proof when k is odd.

The proof for even k is left for the reader. □

Theorem 4 *Suppose that* $\left\{x \mid \frac{v}{k-1} \leq x \leq \lfloor \frac{v}{2} \rfloor\right\} \subseteq \mathcal{BSABP}(v,k,1)$ *for all* $k\geq 4$, *and* $\left\{x \mid \frac{v-2}{2} \leq x \leq \frac{v}{2}\right\} \subseteq \mathcal{BSABP}(v,3,1)$. *Then the necessary condition is also sufficient.*

Proof. Let X be a blocking set of a path design $(V,\mathcal{B})$. Then $V-X$ is also a blocking set. □

2 $\mathcal{BSABP}(v,2h+1,1)$ for $v\equiv 0 \pmod{4h}$, $h\geq 1$.

In this section we determine the set $\mathcal{BSABP}(v,2h+1,1)$ for $v\equiv 0 \pmod{4h}$, $h\geq 1$. By Theorem 4, it suffices to show that

$$\left\{x \mid \frac{v-2}{2} \leq x \leq \frac{v}{2}\right\} \subseteq \mathcal{BSABP}(v,3,1)$$

and

$$\left\{x \mid \frac{v}{2h} \leq x \leq \frac{v}{2}\right\} \subseteq \mathcal{BSABP}(v,2h+1,1) \text{ for } h\geq 2.$$

Case $h=1$ is settled by Theorem 5. For $h\geq 2$, we need to construct an almost balanced $P(4h,2h+1,1)$ with a blocking set of minimum cardinality (Lemmas 1 and 2). These constructions are founded on *trade-off* method [3] and Tarsi's construction [9].

Suppose that a $P(v,k,1)$, $(V,\mathcal{B})$, contains a set of s paths T_1. Suppose also that there exists another set of paths P_k, T_2, based on the same set V such that $T_1\cap T_2=\emptyset$ and both sets contain the same edges. Clearly, if we remove T_1 from $\mathcal{B}$ and replace it by T_2, then we obtain a new $P(v,k,1)$. We will say that the pair (T_1,T_2) forms a trade of volume s and the process of replacing T_1 by T_2 a trade-off.

In order to construct a path design, Tarsi [9] constructed at first a collection $\mathcal{D}$ of paths P_{2h+1}. When these paths are deleted from the completed graph K_v, what remains is Eulerian and Tarsi produced an Eulerian walk in which any two occurrences of a particular vertex are separated by at least $k-1$ distinct vertices all different from it. This walk can be *broken* into copies of P_{2h+1}. We prove Lemmas 1 and 2 by *breaking* opportunely this walk and by applying the trade-off method [3] (with $T_1\subseteq\mathcal{D}$) so as to construct an almost balanced path design with a blocking set of minimum cardinality.

Theorem 5 *Let $v \equiv 0 \pmod 4$, $v \geq 4$. Then*

$$\mathcal{BSABP}(v,3,1) = \left\{ x \mid \frac{v-2}{2} \leq x \leq \frac{v+2}{2} \right\}.$$

Proof. Let $V_1 = \{a_0^1, a_1^1, a_2^1, a_3^1\}$ and $\mathcal{B}_1 = \{(a_0^1, a_1^1, a_2^1), (a_0^1, a_2^1, a_3^1), (a_0^1, a_3^1, a_1^1)\}$. Then $(V_1, \mathcal{B}_1)$ is an almost balanced $P(4,3,1)$ with blocking sets $X_1 = \{a_0^1\}$ and $Y_1 = \{a_1^1, a_2^1\}$.

Let $V_2 = V_1 \cup \{a_0^2, a_1^2, a_2^2, a_3^2\}$ and $\mathcal{B}_2 = \mathcal{B}_1 \cup \{(a_0^2, a_1^2, a_2^2), (a_2^1, a_2^2, a_3^2), (a_0^2, a_3^2, a_1^2), (a_0^2, a_0^1, a_2^2), (a_0^2, a_1^1, a_2^2), (a_1^2, a_2^1, a_3^2), (a_1^2, a_3^1, a_3^2), (a_2^1, a_0^2, a_3^1), (a_0^1, a_1^2, a_1^1), (a_0^2, a_2^2, a_3^1), (a_0^1, a_3^2, a_1^1)\}$. Then $(V_2, \mathcal{B}_2)$ is an almost balanced $P(8,3,1)$ with blocking sets $X_2 = X_1 \cup \{a_0^2, a_3^2\}$ and $Y_2 = Y_1 \cup \{a_1^2, a_2^2\}$.

Suppose there is an almost balanced $P(4t,3,1)$ $(V, \mathcal{B})$, $t \geq 2$, with two blocking sets X and Y, $|X| = 2t-1$, $|Y| = 2t$. Put $V = \cup_{i=1}^t \{a_0^i, a_1^i, a_2^i, a_3^i\}$, $X = \{a_0^1\} \cup (\cup_{i=2}^t \{a_0^i, a_3^i\})$, $Y = \cup_{i=1}^t \{a_1^i, a_2^i\}$, $W = V \cup \{a_0^{t+1}, a_1^{t+1}, a_2^{t+1}, a_3^{t+1}\}$, $\bar{X} = X \cup \{a_0^{t+1}, a_3^{t+1}\}$, $\bar{Y} = Y \cup \{a_1^{t+1}, a_2^{t+1}\}$, $\mathcal{D} = \{(a_0^{t+1}, a_1^{t+1}, a_2^{t+1}), (a_2^1, a_2^{t+1}, a_3^{t+1}), (a_0^{t+1}, a_3^{t+1}, a_1^{t+1}), (a_0^{t+1}, a_0^1, a_2^{t+1}), (a_0^{t+1}, a_1^1, a_2^{t+1}), (a_1^{t+1}, a_2^1, a_3^{t+1}), (a_1^{t+1}, a_3^1, a_3^{t+1}), (a_2^1, a_0^{t+1}, a_3^1), (a_0^1, a_1^{t+1}, a_1^1), (a_0^{t+1}, a_2^{t+1}, a_3^1), (a_0^1, a_3^{t+1}, a_1^1)\}$ and, for $i = 2, 3, \ldots, t$, $\mathcal{E}_i = \{(a_0^i, a_0^{t+1}, a_2^i), (a_0^i, a_1^{t+1}, a_2^i), (a_1^i, a_2^{t+1}, a_3^i), (a_1^i, a_3^{t+1}, a_3^i), (a_2^{t+1}, a_0^i, a_3^{t+1}), (a_0^{t+1}, a_1^i, a_1^{t+1}), (a_2^{t+1}, a_2^i, a_3^{t+1}), (a_0^{t+1}, a_3^i, a_1^{t+1})\}$. Let $\mathcal{E} = \cup_{i=1}^t \mathcal{E}_i$. It is easy to see that $(W, \mathcal{B} \cup \mathcal{D} \cup \mathcal{E})$ is an almost balanced $P(4t+4,3,1)$ with blocking sets $\bar{X}$, $|\bar{X}| = 2t+1$, and $\bar{Y}$, $|\bar{Y}| = 2t+2$. □

Lemma 1 *For each $\mu \geq 1$ there is an almost balanced $P(8\mu, 4\mu+1, 1)$ with a blocking set of cardinality* 2.

Proof. Let $V = Z_8$ and $\mathcal{B} = \{(6,0,1,7,2), (5,1,2,0,3), (6,2,3,1,4), (7,3,4,2,5), (4,5,3,6,1), (5,6,4,7,0), (6,7,5,0,4)\}$.

Then $(V, \mathcal{B})$ is an almost balanced $P(8,5,1)$ with blocking set $X = \{3,7\}$.

Let $V = Z_{16}$. For $i = 0, 1, \ldots, 7$, put $b_i = (3+i, 15+i, 2+i, i, 9+i, 8+i, 10+i, 7+i, 11+i)$ (the sum is $\pmod{16}$). Let $\mathcal{D} = (\{b_i \mid i = 0, 1, \ldots, 7\}$ and $\mathcal{E} = \{(0,1,2,3,4,5,6,7,8), (0,8,1,9,2,10,3,11,4), (4,12,5,13,6,14,7,15,8), (3,9,4,10,5,11,6,12,7), (7,13,8,14,9,15,10,0,11), (11,1,12,2,13,3,14,4,15), (15,5,0,6,1,7,2,8,3)\}$. It is easy to see that $(V, \mathcal{D} \cup \mathcal{E})$ is an almost balanced $P(16,9,1)$. Now we construct a trade (T_1, T_2) of volume 2 as follows: $T_1 = \{b_2, b_4\}$, $T_2 = \{(5,1,4,2,11,14,12,9,13), (7,3,6,4,13,12,10,11,15\}$. By replacing T_1 by T_2 we obtain an almost balanced $P(16,9,1)$ with blocking set $X = \{3,14\}$.

Let $V = Z_{8\mu}$, $\mu \geq 3$. For each path $b = (y_0, y_1, \ldots, y_{4\mu})$ denote by $b+i$, $i \in V$, the following path $b+i = (y_0+i, y_1+i, \ldots, y_{4\mu}+i)$ (the sum is $\pmod{8\mu}$).

Define the following paths:

(1) $d_1 = (0, 1, \ldots, 4\mu)$.
(2) $d_2 = (2\mu, 6\mu, 2\mu + 1, 6\mu + 1, \ldots, 2\mu + 2\mu - 1, 6\mu + 2\mu - 1, 4\mu)$.
(3) $d_3 = (0, 4\mu, 1, 4\mu + 1, 2, 4\mu + 2, \ldots, 2\mu - 1, 4\mu + 2\mu - 1, 2\mu)$.
(4) For $i = 0, 1, \ldots, 4\mu - 1$, let $b_i = (a_0, a_1, \ldots, a_{4\mu}) + i$,

$$a_{2\sigma} = \begin{cases} \mu + 1 - \sigma & \sigma = 0, 1, \ldots, \mu - 1 \\ 3\mu + 1 + \sigma & \sigma = \mu + 1, \mu + 2, \ldots, 2\mu - 1 \end{cases}$$

$$a_{2\sigma+1} = \begin{cases} 7\mu + 1 + \sigma & \sigma = 0, 1, \ldots, \mu - 1 \\ 5\mu - \sigma & \sigma = \mu + 1, \mu + 2, \ldots, 2\mu - 1 \end{cases}$$

and

$$a_{2\mu} = 4\mu + 1,\ a_{2\mu+1} = 4\mu \text{ and } a_{4\mu} = 5\mu + 1.$$

(5) For $\alpha = 1, 2, \ldots, \mu - 1$ let $c_\alpha = (y_0^\alpha, y_1^\alpha, \ldots, y_{4\mu}^\alpha)$, $y_{2\sigma}^\alpha = 2 + \alpha + \sigma$, $y_{2\sigma+1}^\alpha = 4\mu + 2 - \alpha + \sigma$, $\sigma = 0, 1, \ldots, 2\mu - 1$ and $y_{4\mu}^\alpha = 2\mu + 2 + \alpha$.

Put $\mathcal{B} = \{d_1, d_2, d_3\} \cup \{b_0, b_1, \ldots, b_{4\mu-1}\} \cup (\cup_{\alpha=1}^{\mu-1} \{c_\alpha, c_\alpha + 2\mu, c_\alpha + 4\mu, c_\alpha + 6\mu\})$. Then $(V, \mathcal{B})$ is a $P(8\mu, 4\mu + 1, 1)$ (see [9]).

It is easy to verify that each vertex of V is an endpoint of either 1 or 3 paths of $\mathcal{B}$. Therefore $(V, \mathcal{B})$ is almost balanced.

Put
$\bar{b}_0 = (\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \ldots, \bar{a}_{4\mu}) = (\mu + 1, 7\mu + 1, \mu, \mu + 2, \mu - 1, \ldots, 5\mu + 1)$,
$\bar{b}_\mu = (\bar{a}_0, \ldots, \bar{a}_{2\mu-1}, \bar{a}_{2\mu}, \bar{a}_{2\mu+1}, \ldots, \bar{a}_{4\mu}) =$
$(2\mu + 1, \ldots, \mu - 1, 7\mu + 2, \mu, \ldots, 46\mu + 1)$.

By (4), we have
$b_0 = (a_0, a_1, a_2, a_3, a_4, \ldots, a_{4\mu}) = (\mu + 1, 7\mu + 1, \mu, 7\mu + 2, \mu - 1, \ldots, 5\mu + 1)$,
$b_\mu = (a_0, \ldots, a_{2\mu-1}, a_{2\mu}, a_{2\mu+1}, \ldots, a_{4\mu}) =$
$(2\mu + 1, \ldots, \mu - 1, \mu + 2, \mu, \ldots, 6\mu + 1)$.

Then $T_1 = \{b_0, b_1\}$ and $T_2 = \{\bar{b}_0, \bar{b}_1\}$ form a trade of volume 2.

By replacing T_1 by T_2 we obtain an almost balanced $P(8\mu, 4\mu + 1, 1)$.

At last note that $\mu + 1$ is a vertex of c_α, $c_\alpha + 4\mu$, d_1, d_3, $\bar{b}_0$ and b_j, $j \in \{1, 2, \ldots, 2\mu\} - \{\mu\}$, while $7\mu + 2$ is a vertex of $c_\alpha + 2\mu$, $c_\alpha + 6\mu$, d_2, $\bar{b}_\mu$ and b_j, $j \in \{2\mu + 1, 2\mu + 2, \ldots, 4\mu - 1\}$. Then $X = \{\mu + 1, 7\mu + 1\}$ is a blocking set. □

Lemma 2 *For each $\mu \geq 1$ there is an almost balanced $P(8\mu + 4, 4\mu + 3, 1)$ with a blocking set of cardinality 2.*

Proof. Let $V = Z_{8\mu+4}$. Define the following paths:
(1) $d_1 = (4\mu + 2, 4\mu + 3, \ldots, 8\mu + 3, 0)$.
(2) For $i = 0, 1, \ldots, 4\mu + 1$, let $b_i = (a_0, a_1, \ldots, a_{4\mu+2}) + i$,

$$a_{2\sigma+2} = \begin{cases} 1 + \sigma & \sigma = 0, 1, \ldots, \mu - 1 \\ 6\mu + 3 - \sigma & \sigma = \mu + 1, \mu + 2, \ldots, 2\mu - 1 \end{cases}$$

$$a_{2\sigma+3} = \begin{cases} 8\mu + 3 - \sigma & \sigma = 0, 1, \ldots, \mu - 1 \\ 2\mu + 2 + \sigma & \sigma = \mu + 1, \mu + 2, \ldots, 2\mu - 1 \end{cases}$$

and

$$a_0 = 1,\ a_1 = 0,\ a_{2\mu+2} = 3\mu + 2,\ a_{2\mu+3} = 5\mu + 3 \text{ and } a_{4\mu+2} = 4\mu + 2.$$

(3) Let $c_\alpha = (y_0^\alpha, y_1^\alpha, \dots, y_{4\mu+2}^\alpha)$, $y_{2\sigma}^0 = 4\mu+3+\sigma$, $y_{2\sigma+1}^0 = 2\mu+1+\sigma$, $\sigma = 0,1,\dots,2\mu$, $y_{4\mu+2}^0 = 6\mu+4$ and, for $\alpha = 1,2,\dots,\mu-1$, $y_{2\sigma}^\alpha = 8\mu+4-2\alpha+\sigma$, $y_{2\sigma+1}^\alpha = 2\mu+3+\sigma$, $\sigma = 0,1,\dots,2\mu$, $y_{4\mu+2}^\alpha = 2\mu+1-2\alpha$.

Put $\mathcal{B} = \{d_1\} \cup \{b_0, b_1, \dots, b_{4\mu+1}\} \cup (\cup_{\alpha=0}^{\mu-1}\{c_\alpha, c_\alpha + 2\mu + 1, c_\alpha + 4\mu + 2, c_\alpha + 6\mu + 3\})$. Then $(V, \mathcal{B})$ is a $P(8\mu, 4\mu+1, 1)$ (see [9]).

It is easy to verify that each vertex of V is an endpoint of either 1 or 3 paths of $\mathcal{B}$. Therefore $(V, \mathcal{B})$ is almost balanced. Moreover $2\mu+2$ is a vertex of c_0, $c_0 + 4\mu + 2$, $c_\alpha + 2\mu + 1$, $c_\alpha + 6\mu + 3$, $\alpha \in \{1, 2, \dots, \mu-1\}$, and b_i, $i \in \{\mu+1, \mu+2, \dots, 3\mu\}$, while $4\mu+2$ is a vertex of d_1, $c_0 + 2\mu + 1$, $c_0 + 6\mu + 3$, c_α, $c_\alpha + 4\mu + 2$ $\alpha \in \{1, 2, \dots, \mu-1\}$, and b_i, $i \in \{0, 1, \dots, \mu\} \cup \{3\mu+1, 3\mu+2, \dots, 4\mu+1\}$. Then $X = \{2\mu+2, 4\mu+2\}$ is a blocking set.
□

Lemma 3 *For each $h \geq 2$ it is possible to decompose the bipartite graph $K_{4h,4h}$ into copies of P_{2h+1} in such a way that: 1) each element is an endpoint of exactly two paths; 2) there is a $\Omega \subset K_{4h,4h}$ such that each path meets Ω (i.e. Ω is a blocking set of the decomposition); 3)* $|\Omega| = 4$.

Proof. Let $V(K_{4h,4h}) = \{a_0, a_1, \dots, a_{4h-1}\} \cup \{y_0, y_1, \dots, y_{4h-1}\}$.
For $i = 0, 1, \dots, 4h-1$, put:

$$b_i = (y_i, a_i, y_{1+i}, a_{4h-1+i}, y_{2+i}, a_{4h-2+i}, \dots, y_{h-1+i}, a_{3h+1+i}, y_{h+i})$$

and

$$c_i = (a_i, y_{2h+i}, a_{4h-1+i}, y_{2h+1+i}, a_{4h-2+i}, \dots$$
$$\dots, y_{3h-2+i}, a_{3h+1+i}, y_{3h-1+i}, a_{3h+i}).$$

Let $\mathcal{B} = \{b_0, b_1, \dots, b_{4h-1}\} \cup \{c_0, c_1, \dots, c_{4h-1}\}$. It is easy to see that $(V, \mathcal{B})$ is an edge-disjoint decomposition of $K_{4h,4h}$ into P_{2h+1} and that each vertex of V appears as endpoint of two paths.

If $h = 2$, then each path of $\mathcal{B}$ meets $\Omega = \{a_0, a_4, y_4, y_7\}$. Suppose $h \geq 3$. Let $\Omega = \{a_0, a_{2h}, y_{4h-1}, y_{2h+1}\}$. It is easy to see that $b_i \cap X \neq \emptyset$, $i \in \{0, 1, \dots, 4h-i\} - \{h\}$, and $c_i \cap X \neq \emptyset$, $i \in \{0, 1, \dots, 4h-i\} - \{3h+1\}$.

Define:

$$\bar{b}_h = (y_h, a_h, y_{h+1}, a_{h+1}, \dots, y_{h-3}, a_4, y_{h-2}, a_3, y_{2h+3}, a_{2h}, y_{2h+2}, a_1, y_{2h}),$$

$$\bar{b}_{2h+1} = (y_{2h+1}, a_{2h+1}, y_{2h+2}, a_2, y_{2h+3}, a_{2h-1}, \dots, y_{3h}, a_{h+2}, y_{3h+1}),$$

$$\bar{c}_0 = (a_{2h+1}, y_{2h}, a_{4h-1}, y_{2h+1}, a_{4h-2}, y_{2h+2}, \dots, a_{3h+1}, y_{3h-1}, a_{3h}),$$

$$\bar{c}_2 = (a_2, y_{2h-1}, a_1, y_{2h+3}, a_0, y_{2h+4}, \dots, a_{3h+3}, y_{3h+1}, a_{3h+2}),$$

$\bar{c}_3 = (a_3, y_{2h-2}, a_2, y_{2h+4}, a_1, y_{2h+5}, \dots, a_{3h+4}, y_{3h+2}, a_{3h+3}),$

$\bar{c}_{3h+1} = (a_{3h+1}, y_{h+1}, a_{3h}, y_{h+2}, a_{3h-1}, y_{h+3}, \dots, a_{2h+2}, y_{2h}, a_0).$

Then $T_1 = \{b_h, b_{2h+1}, c_0, c_2, c_3, c_{3h+1}\}$ and $T_2 = \{\bar{b}_h, \bar{b}_{2h+1}, \bar{c}_0, \bar{c}_2, \bar{c}_3, \bar{c}_{3h+1}\}$ form a trade of volume 4. By replacing T_1 by T_2, we obtain the required decomposition. □

Theorem 6 *Let $v \equiv 0 \pmod{4h}$, $v \geq 4h$, $h \geq 2$. Then*

$$\mathcal{BSABP}(v, 2h+1, 1) = \left\{ x \mid \frac{v}{2h} \leq x \leq \frac{(2h-1)v}{2h} \right\}.$$

Proof. By Lemmas 1 and 2 construct an almost balanced $P(4h, 2h+1, 1)$ $(V, \mathcal{B})$ with a blocking set X of cardinality 2. Let $\bar{X}$ be a subset of V such that $X \subset \bar{X}$ and $|\bar{X}| \leq 2h$. Clearly $\bar{X}$ is a blocking set. So by Theorem 4 the theorem is proved when $v = 4h$.

Let $v = 4ht$, $t \geq 2$. Let V_i be t mutually disjoint v-sets. For each i, $i = 1, 2, \dots, t$, let $(V_i, \mathcal{B}_i)$ be an almost balanced $P(4h, 2h+1, 1)$ with a blocking set X_i, $|X_i| \in \{2, 3, \dots, 2h\}$. Let $(V_i \cup V_j, \mathcal{D}_{ij})$, $i, j \in \{1, 2, \dots, t\}$, $i \neq j$, be a decomposition of $K_{4h,4h}$ into P_{2h+1} with a blocking set Ω_{ij} such that $\Omega_{ij} \subseteq X_i \cup X_j$. This is possible by Lemma 3. Put $W = \cup_{i=1}^{t} V_i$, $\mathcal{E} = (\cup_{i=1}^{t} \mathcal{B}_i) \cup (\cup_{i,j=1}^{t} \mathcal{D}_{ij})$ and $X = \cup_{i=1}^{t} X_i$. It is easy to verify that $(W, \mathcal{E})$ is an almost balanced $P(4ht, 2h+1, 1)$ with the blocking set X, with $2t \leq |X| \leq t(4h-2)$. □

3 $\mathcal{BSABP}(v, 2h, 1)$ for $v \equiv 0 \pmod{2h-1}$, $h \geq 2$.

In this section we determine the set $\mathcal{BSABP}(v, 2h, 1)$ for $v \equiv 0 \pmod{2h-1}$, $h \geq 1$ and $v \geq 4h-2$. By Theorem 4 it suffices to show that $\left\{ x \mid \frac{v}{2h-1} \leq x \leq \lfloor \frac{v}{2} \rfloor \right\} \subseteq \mathcal{BSABP}(v, 2h, 1)$ for all $h \geq 2$. The first step is to construct an almost balanced $P(4h-2, 2h, 1)$ with a blocking set of minimum cardinality (see Lemmas 4 and 5). To do this we use the trade-off method and Tarsi's construction as described at the beginning of above section.

Lemma 4 *For each $\mu \geq 1$ there is an almost balanced $P(8\mu+2, 4\mu+2, 1)$ with a blocking set of cardinality 2.*

Proof. Let $V = Z_{8\mu+2}$. Define the following paths:
(1) For $i = 0, 1, \dots, 4\mu$, let $b_i = (a_0, a_1, \dots, a_{4\mu}) + i$, $a_0 = 0$,

$$a_{2\sigma+2} = \begin{cases} 8\mu + 1 - \sigma & \sigma = 0, 1, \dots, \mu - 1 \\ 6\mu + 1 - \sigma & \sigma = \mu, \mu + 1, \dots, 2\mu - 1 \end{cases}$$

$$a_{2\sigma+1} = \begin{cases} 1+\sigma & \sigma = 0,1,\ldots,\mu-1 \\ 2\mu+1+\sigma & \sigma = \mu,\mu+1,\ldots,2\mu \end{cases}$$

(2) For $\alpha = 1,2,\ldots,\mu-1$ let $c_\alpha = (y_0^\alpha, y_1^\alpha, \ldots, y_{4\mu}^\alpha)$ and $d_\alpha = (z_0^\alpha, z_1^\alpha, \ldots, z_{4\mu}^\alpha)$,

$$y_{2\sigma}^\alpha = 2\mu - \alpha + \sigma, \;\; y_{2\sigma+1}^\alpha = 4\mu + 2 + \alpha + \sigma, \;\; \sigma = 0,1,\ldots,2\mu,$$

$$z_{2\sigma}^\alpha = 2\mu + 1 + \alpha + \sigma, \;\; y_{2\sigma+1}^\alpha = \sigma - \alpha, \;\; \sigma = 0,1,\ldots,2\mu.$$

Put $\mathcal{B} = \{b_0, b_1, \ldots, b_\mu\} \cup (\cup_{\alpha=0}^{\mu-1}\{c_\alpha, c_\alpha + 4\mu+1, d_\alpha, d_\alpha + 4\mu+1\})$. Then $(V,\mathcal{B})$ is an almost balanced $P(8\mu+2, 4\mu+2, 1)$ (see [9]).

It is easy to verify that each vertex of V is an endpoint of either 1 or 3 paths of $\mathcal{B}$. Therefore $(V,\mathcal{B})$ is almost balanced. Moreover 0 is a vertex of d_α, $d_\alpha + 4\mu + 1$ and b_i, $i \in \{0,1,\ldots,\mu\} \cup \{3\mu+1, 3\mu+2, \ldots, 4\mu\}$, while 2μ is a vertex of c_α, $c_\alpha + 4\mu + 1$ and b_i, $i \in \{2\mu+1, 2\mu+2, \ldots, 3\mu\}$. Therefore $X = \{0, 2\mu\}$ is a blocking set. □

Lemma 5 *For each $\mu \geq 1$ there is an almost balanced $P(8\mu-2, 4\mu, 1)$ with a blocking set of cardinality* 2.

Proof. Let $V = Z_{8\mu-2}$ and let $X = \{0, 5\mu-1\}$.

Put

$\mathcal{B}_1 = \{(0,2,5,3),(1,3,0,4),(2,4,1,5),(0,1,2,3),(3,4,5,0)\}$;

$\mathcal{B}_2 = \{(0,2,13,3,10,6,9,7),(1,3,0,4,11,7,10,8),(2,4,1,5,12,8,11,9),$
$(3,5,2,6,13,9,12,10),(4,6,3,7,0,10,13,11),(5,7,4,8,1,11,0,12),$
$(6,8,5,9,2,12,1,13),(0,1,2,3,4,5,6,7),(7,8,9,10,11,12,13,0),$
$(2,8,3,9,4,10,5,11),(11,6,12,7,13,8,0,9),(9,1,10,2,11,3,12,4),$
$(4,13,5,0,6,1,7,2)\}$;

$\mathcal{B}_3 = \{(0,2,21,3,20,4,15,9,17,10,13,11),$
$(1,3,0,4,21,5,16,10,15,11,14,12),(2,4,1,5,0,6,17,11,16,12,15,13),$
$(3,5,2,6,1,7,18,12,17,13,16,14),(4,6,3,7,2,8,19,13,18,14,17,15),$
$(5,7,4,8,3,9,20,14,19,15,18,16),(6,8,5,9,14,10,21,15,20,16,19,17),$
$(7,9,6,10,5,11,0,16,21,17,20,18),(8,10,7,11,6,12,1,17,0,18,21,19),$
$(9,11,8,12,7,13,2,18,1,19,0,20),(10,12,9,13,8,14,3,19,2,20,1,21),$
$(0,1,2,3,4,5,6,7,8,9,10,11),(11,12,13,14,15,16,17,18,19,20,21,0),$
$(3,11,4,12,5,13,6,14,7,15,8,16),(16,9,4,10,18,11,19,12,20,13,21,14),$
$(14,0,15,1,16,2,17,3,18,4,19,5),(5,20,6,21,7,0,8,1,9,2,10,3),$
$(2,12,3,13,4,14,5,15,6,16,7,17),(17,8,18,9,19,10,20,11,21,12,0,13),$
$(13,1,14,2,15,3,16,4,17,5,18,6),(6,19,7,20,8,21,9,0,10,1,11,2)\}$.

Then $(V,\mathcal{B}_\mu)$, $\mu = 1,2,3$, is an almost balanced $P(8\mu-2, 4\mu, 1)$ with blocking set X.

Now we prove the theorem for $\mu \geq 4$. Define the following paths:

(1) $d_1 = (0, 1, \ldots, 4\mu-1)$,

(2) $d_2 = (4\mu-1, 4\mu, \ldots, 8\mu-3, 0)$.

(3) For $i = 0, 1, \dots, 4\mu - 2$, let $b_i = (a_0, a_1, \dots, a_{4\mu-1}) + i$,

$$a_{2\sigma} = \begin{cases} 8\mu - 2 - \sigma & \sigma = 0, 1, \dots, \mu - 1 \\ 6\mu - \sigma & \sigma = \mu, \mu + 1, \dots, 2\mu - 1 \end{cases}$$

$$a_{2\sigma+1} = \begin{cases} \sigma + 2 & \sigma = 0, 1, \dots, \mu - 1 \\ 2\mu + \sigma & \sigma = \mu, \mu + 1, \dots, 2\mu - 1 \end{cases}$$

(4) For $\alpha = 0, 1, \dots, \mu - 2$ let $c_\alpha = (y_0^\alpha, y_1^\alpha, \dots, y_{4\mu-1}^\alpha)$ and $\hat{c}_\alpha = (\hat{y}_0^\alpha, \hat{y}_1^\alpha, \dots, \hat{y}_{4\mu-1}^\alpha)$, where for each $\sigma = 0, 1, \dots, 2\mu - 1$, it is

$$y_{2\sigma}^\alpha = \begin{cases} \sigma - \alpha + \frac{3\mu-3}{2} & \mu \equiv 1 \pmod 2 \\ \sigma - \alpha + \frac{3\mu-2}{2} & \mu \equiv 0 \pmod 2 \end{cases}$$

$$y_{2\sigma+1}^\alpha = \begin{cases} \sigma + \alpha + \frac{7\mu+1}{2} & \mu \equiv 1 \pmod 2 \\ \sigma + \alpha + \frac{7\mu+2}{2} & \mu \equiv 0 \pmod 2 \end{cases}$$

$$\hat{y}_{2\sigma}^\alpha = \begin{cases} \sigma + \alpha + \frac{11\mu-1}{2} & \mu \equiv 1 \pmod 2 \\ \sigma + \alpha + \frac{11\mu}{2} & \mu \equiv 0 \pmod 2 \end{cases}$$

$$\hat{y}_{2\sigma+1}^\alpha = \begin{cases} \sigma - \alpha + \frac{7\mu-3}{2} & \mu \equiv 1 \pmod 2 \\ \sigma - \alpha + \frac{7\mu-2}{2} & \mu \equiv 0 \pmod 2 \end{cases}$$

Put $\mathcal{B} = \{d_1, d_2\} \cup \{b_0, b_1, \dots, b_{4\mu-2}\} \cup (\cup_{\alpha=1}^{\mu-2}\{c_\alpha, c_\alpha + 4\mu - 1, \hat{c}_\alpha, \hat{c}_\alpha + 4\mu - 1\})$. Then $(V, \mathcal{B})$ is a $P(8\mu, 4\mu + 1, 1)$ (see [9]). It is easy to verify that each vertex of V is an endpoint of either 1 or 3 paths of $\mathcal{B}$. Therefore $(V, \mathcal{B})$ is almost balanced.

Let $X = \{0, 5\mu - 1\}$. It is not difficult to prove that $b_i \cap X = \emptyset$, $i = 2\mu, 2\mu + 1, \dots, 3\mu - 3$, while the remaining paths of $\mathcal{B}$ meet X as we show in the following.

(I) $0 \in d_1 \cap X$;

(II) $5\mu - 1 \in d_2 \cap X$;

(III) $a_{2i} + i = 0 \in b_i \cap X$, $i = 0, 1, \dots, \mu - 1$;

(IV) $a_{2(3\mu-1-i)} + i = 5\mu - 1 \in b_i \cap X$, $i = \mu, \mu + 1, \dots, 2\mu - 1$;

(V) $a_{2(i-2\mu+2)} + i = 0 \in b_i \cap X$, $i = 3\mu - 2, 3\mu - 1, \dots, 4\mu - 3$;

(VI) $a_{2(\mu-1)+1} + 4\mu - 2 = 5\mu - 1 \in b_i \cap X$.

(VII) Let $\mu \equiv 1 \pmod 2$. Then

(VII.a) $y_{2\sigma+1}^\alpha = 5\mu - 1 \in c_\alpha \cap X$, $\sigma = \frac{3\mu-3}{2} - \alpha$, $\alpha = 0, 1, \dots, \mu - 2$;

(VII.b) $y_{2\sigma+1}^\alpha + 4\mu - 1 = 0 \in (c_\alpha + 4\mu - 1) \cap X$, $\sigma = \frac{\mu-3}{2} - \alpha$, $\alpha = 0, 1, \dots, \frac{\mu-3}{2}$;

$y_{2\sigma+1}^\alpha + 4\mu - 1 = 5\mu - 1 \in (c_\alpha + 4\mu - 1) \cap X$, $\sigma = \mu - \frac{\mu-3}{2} + \alpha$, $\alpha = \frac{\mu-3}{2} + 1, \frac{\mu-3}{2} + 2, \dots, \mu - 2$;

(VII.c) $\hat{y}_{2\sigma+1}^\alpha = 5\mu - 1 \in \hat{c}_\alpha \cap X$, $\sigma = \frac{3\mu+1}{2} + \alpha$, $\alpha = 0, 1, \dots, \frac{\mu-3}{2}$;

$\hat{y}_{2\sigma}^\alpha = 0 \in \hat{c}_\alpha \cap X$, $\sigma = \frac{5\mu-3}{2} - \alpha$, $\alpha = \frac{\mu-3}{2} + 1, \frac{\mu-3}{2} + 2, \dots, \mu - 2$;

(VII.d) $\hat{y}_{2\sigma+1}^\alpha + 4\mu - 1 = 0 \in (\hat{c}_\alpha + 4\mu - 1) \cap X$, $\sigma = \frac{\mu+1}{2} + \alpha$, $\alpha = 0, 1, \dots, \mu - 2$.

(VIII) Let $\mu \equiv 0 \pmod 2$. Then it is

(VIII.a) $y^{\alpha}_{2\sigma+1} = 5\mu - 1 \in c_\alpha \cap X$, $\sigma = \frac{3\mu-4}{2} - \alpha$, $\alpha = 0,1,\ldots,\mu-2$;

(VIII.b) $y^{\alpha}_{2\sigma+1} + 4\mu - 1 = 0 \in (c_\alpha + 4\mu - 1) \cap X$, $\sigma = \frac{\mu-4}{2} - \alpha$, $\alpha = 0,1,\ldots,\frac{\mu-4}{2}$;

$y^{\alpha}_{2\sigma+1} + 4\mu - 1 = 5\mu - 1 \in (c_\alpha + 4\mu - 1) \cap X$, $\sigma = \alpha - \frac{\mu-2}{2}$, $\alpha = \frac{\mu-4}{2}+1, \frac{\mu-4}{2}+2,\ldots,\mu-2$;

(VIII.c) $\hat{y}^{\alpha}_{2\sigma+1} = 5\mu - 1 \in \hat{c}_\alpha \cap X$, $\sigma = \frac{3\mu}{2} + \alpha$, $\alpha = 0,1,\ldots,\frac{\mu-2}{2}$;
$\hat{y}^{\alpha}_{2\sigma} = 0 \in \hat{c}_\alpha \cap X$, $\sigma = \frac{5\mu-4}{2} - \alpha$, $\alpha = \frac{\mu-2}{2}+1, \frac{\mu-2}{2}+2,\ldots,\mu-2$;

(VIII.d) $\hat{y}^{\alpha}_{2\sigma+1} + 4\mu - 1 = 0 \in (\hat{c}_\alpha + 4\mu - 1) \cap X$, $\sigma = \frac{\mu}{2} + \alpha$, $\alpha = 0,1,\ldots,\mu-2$.

In order to construct an almost balanced $P(8\mu-2, 4\mu, 1)$ with blocking set X, we use the trade-off method.

At first suppose $\mu \equiv 1 \pmod 2$, $\mu \geq 5$. Let $\sigma = \frac{3\mu-3}{2} - \rho$, $\rho = 0,1,\ldots,\frac{\mu-3}{2}$. Then $a_{2\sigma+1} + \frac{3\mu-3}{2} - \rho = 4\mu - 3 - 2\rho$, $a_{2\sigma+2} + \frac{\mu-3}{2} - \rho = 5\mu - 1$, $a_{2\sigma+3} + \frac{\mu-3}{2} - \rho = 4\mu - 2 - 2\rho$, $a_{2\sigma} + \frac{3\mu-3}{2} - \rho = 0$. Therefore the paths $b_{\frac{\mu-3}{2}-\rho}$ meet both vertices 0 and $5\mu-1$ and edges $\{4\mu-3-\rho, 5\mu-1\}$, $\{5\mu-1, 4\mu-2-2\rho\}$. Moreover it is easy to verify that $2\mu-2$ is not a vertex of $b_{\frac{\mu-3}{2}-\rho}$.

Let $\sigma = \mu-\rho-2$, $\rho = 0,1,\ldots,\frac{\mu-3}{2}$. Then $a_{2\sigma+1} + 3\mu-3-\rho = 4\mu-3-2\rho$, $a_{2\sigma+2} + 3\mu-3-\rho = 2\mu-2$, $a_{2\sigma+3} + 3\mu-3-\rho = 4\mu-2-2\rho$. Therefore the paths $b_{3\mu-3-\rho}$ meet both edges $\{4\mu-3-2\rho, 2\mu-2\}$, $\{2\mu-2, 4\mu-2-2\rho\}$.

Put $\bar{b}_{\frac{\mu-3}{2}-\rho} = (\gamma_0, \gamma_1, \ldots, \gamma_{4\mu-1})$ and $\bar{b}_{3\mu-3-\rho} = (\tau_0, \tau_1, \ldots, \tau_{4\mu-1})$, where for $\sigma = 0,1,\ldots,2\mu-1$, it is: $\gamma_{2\sigma+1} = a_{2\sigma+1} + \frac{\mu-3}{2} - \rho$, $\gamma_{2\sigma} = a_{2\sigma}$ if $\sigma \neq \frac{3\mu-3}{2} - \rho$, $\gamma_{\frac{3\mu-3}{2}-\rho} = 2\mu-2$, $\tau_{2\sigma+1} = a_{2\sigma+1} + 3\mu - 3 - \rho$, $\tau_{2\sigma} = a_{2\sigma} + 3\mu - 3 - \rho$ if $\sigma \neq \mu - 2 - \rho$, $\tau_{\mu-2-\rho} = 5\mu - 1$.

Then $T'_1 = \{b_{\frac{\mu-3}{2}-\rho}, b_{3\mu-3-\rho} \mid \rho = 0,1,\ldots,\frac{\mu-3}{2}\}$ and $T'_2 = \{\bar{b}_{\frac{\mu-3}{2}-\rho}, \bar{b}_{3\mu-3-\rho} \mid \rho = 0,1,\ldots,\frac{\mu-3}{2}\}$ form a trade of volume $\mu-1$.

Let $\sigma = \frac{3\mu-7}{2} - \rho$, $\rho = \frac{\mu-3}{2}+1, \frac{\mu-3}{2}+2,\ldots,\mu-3$. Then $a_{2\sigma+1} + \frac{3\mu-5}{2} - \rho = 3\mu - 4 - 2\rho$, $a_{2\sigma+2} + \frac{3\mu-5}{2} - \rho = 0$, $a_{2\sigma+3} + \frac{3\mu-5}{2} - \rho = 3\mu - 3 - 2\rho$.

Let $\sigma = \frac{5\mu-3}{2} - \rho$, $\rho = \frac{\mu-3}{2}+1, \frac{\mu-3}{2}+2,\ldots,\mu-3$. Then $a_{2\sigma} + \frac{3\mu-5}{2} - \rho = 5\mu - 1$. Therefore the paths $b_{\frac{3\mu-5}{2}-\rho}$ meet both vertices 0 and $5\mu-1$ and edges $\{3\mu-4-2\rho, 0\}$, $\{0, 3\mu-3-2\rho\}$. Moreover it is easy to verify that $3\mu-1$ is not a vertex of $b_{\frac{3\mu-5}{2}-\rho}$.

Let $\sigma = \rho$, $\rho = \frac{\mu-3}{2}+1, \frac{\mu-3}{2}+2,\ldots,\mu-3$. Then $a_{2\sigma} + 3\mu - 3 - \rho = 3\mu - 3 - 2\rho$, $a_{2\sigma+1} + 3\mu - 3 - \rho = 3\mu - 1$, $a_{2\sigma+2} + 3\mu - 3 - \rho = 3\mu - 4 - 2\rho$. Therefore the paths $b_{3\mu-3-\rho}$ meet both edges $\{3\mu-3-2\rho, 3\mu-1\}$, $\{3\mu-1, 3\mu-4-2\rho\}$.

Put $\bar{b}_{\frac{3\mu-5}{2}-\rho} = (\gamma_0, \gamma_1, \ldots, \gamma_{4\mu-1})$ and $\bar{b}_{3\mu-3-\rho} = (\tau_0, \tau_1, \ldots, \tau_{4\mu-1})$, where for $\sigma = 0,1,\ldots,2\mu-1$, it is: $\gamma_{2\sigma+1} = a_{2\sigma+1} + \frac{3\mu-5}{2} - \rho$,

$\gamma_{2\sigma} = a_{2\sigma} + \frac{3\mu-5}{2} - \rho$ if $\sigma \neq \frac{3\mu-7}{2} - \rho$, $\gamma_{2(\frac{3\mu-7}{2}-\rho)} = 3\mu - 1$,
$\tau_{2\sigma} = a_{2\sigma} + 3\mu - 3 - \rho$, $\tau_{2\sigma+1} = a_{2\sigma+1} + 3\mu - 3 - 2\rho$ if $\sigma \neq \rho$, $\tau_{2\rho+1} = 0$.

Then $T"_1 = \{b_{\frac{3\mu-5}{2}-\rho}, b_{3\mu-3-\rho} \mid \rho = \frac{\mu-3}{2}+1, \frac{\mu-3}{2}+2, \dots, \mu-3\}$ and $T"_2 = \{\bar{b}_{\frac{3\mu-5}{2}-\rho}, \bar{b}_{3\mu-3-\rho} \mid \rho = \frac{\mu-3}{2}+1, \frac{\mu-3}{2}+2, \dots, \mu-3\}$ form a trade of volume $\mu - 3$.

By replacing $T'_1 \cup T'_2$ by $T"_1 \cup T"_2$, we obtain (for each odd $\mu \geq 5$) an almost balanced $P(8\mu-2, 4\mu, 1)$ with blocking set X.

For $\mu \equiv 0 \pmod 2$, $\mu \geq 4$ the proof is similar.We leave it for the reader. □

Lemma 6 *Let $v \equiv 0 \pmod{(2h-1)}$, $v \geq 4h-2$. Suppose there exists an almost balanced $P(v, 2h, 1)$ with a blocking set of cardinality x. Then there exists an almost balanced $P(v+2h-1, 2h, 1)$ with a blocking set of cardinality x+1.*

Proof. Let $(V, \mathcal{B}), V = Z_v \times \{1\}$, be a $P(v, 2h, 1)$ with a blocking set X, $|X| = x$. Put $v = t(2h-1)$, $t \geq 2$. Suppose $\{(\sigma(2h-1), 1) \mid \sigma = 0, 1, \dots, t-1\} \subseteq X$. Put $W = Z_{2h-1} \times \{2\}$. In the following we will suppose that each pair $(y, 1)$ [$(y, 2)$, respectively] is taken $\pmod{(v, -)}$ [$\pmod{(2h-1, -)}$, respectively].

Let

$$b_i = (y_0^i, a_0^i, y_1^i, a_1^i, \dots, y_{h-1}^i, a_{h-1}^i) \quad i \in Z_v,$$

$$y_\rho^i = (i+\rho, 1), \; a_\rho^i = (i-\rho, 2), \; \rho = 0, 1, \dots, h-1.$$

It is easy to verify that $\{b_i \mid i \in Z_v\}$ is a P_{2h}-decomposition of the complete bipartite graph $K_{v,2h-1}$ on vertex set $V \cup W$.

At first we settle the case $h = 2$. Let $\bar{b}_0 = ((1,2), (0,2), (1,1), (2,2))$, $\bar{b}_i = b_i$, $i = 1, 2, \dots, v-1$ and $\bar{c}_0 = ((0,1), (0,2), (2,2), (1,2))$. Put $\mathcal{D} = \{\bar{c}_0\} \cup \{\bar{b}_i \mid i = 0, 1, \dots, v-1\}$. Then $(V, \mathcal{D})$ is an almost balanced $P(v, 4, 1)$ with blocking set $X \cup \{(0,2)\}$.

Now let $h \geq 3$. Let

$$c_j = (p_0^j, q_0^j, p_1^j, q_1^j, \dots, p_{h-2}^j, q_{h-2}^j, p_{h-1}^j) \quad j = 0, 1, \dots, h-2,$$

$$p_\sigma^j = (\sigma + j, 2), \; \sigma = 0, 1, \dots, h-1,$$

$$q_\sigma^j = (2h-2-\sigma+j, 2), \; \sigma = 0, 1, \dots, h-2.$$

Note that each path c_j has exactly $2h-1$ vertices and the set $\{c_j \mid j = 0, 1, \dots, h-2\}$ covers all the edges of K_{2h-1} on W except the following ones: $\{(j, 2), (2h-3-j, 2)\}$. Let

$$\alpha = (\beta_0 \;\; \gamma_0)(\beta_1 \;\; \gamma_1) \cdots (\beta_{\mu(h)} \;\; \gamma_{\mu(h)})$$

be the permutation of W so defined:
$\beta_\tau = (h-1+\tau, 2)$, $\gamma_\tau = (2h-3-\tau, 2)$, $\tau = 0, 1, \ldots, \mu(h)$,

$$\mu(h) = \begin{cases} \frac{h-3}{2} & \text{if } h \equiv 1 \pmod 2 \\ \frac{h-4}{2} & \text{if } h \equiv 0 \pmod 2 \end{cases}$$

Let
$\mathcal{C} = \{\bar{c}_j = (\bar{q}^j_{-1}, \alpha p^j_0, \alpha q^j_0, \alpha p^j_1, \alpha q^j_1, \ldots$
$\ldots, \alpha p^j_{h-2}, \alpha q^j_{h-2}, \alpha p^j_{h-1}) \mid j = 0, 1, \ldots, h-2\}$, $\bar{q}^j_{-1} = (j, 1)$.

Note that $\mathcal{C}$ covers the edges $\{(j,1),(j,2)\}$, $j = 0, 1, \ldots, h-2$, and all the edges of K_{2h-1} on W except the following ones $\{\alpha(j,2), \alpha(2h-3-j,2)\} = \{(j,2),(h-1+j,2)\}$.

Let $\mathcal{E} = \{\bar{b}_i \mid i \in Z_v\}$,

$$\bar{b}_i = \begin{cases} b_i & i = h-1, h, \ldots, t(2h-1)-1 \\ (\bar{y}^i_0, \bar{a}^i_0, \bar{y}^i_1, \bar{a}^i_1, \ldots, \bar{y}^i_{h-1}, \bar{a}^i_{h-1}) & i = 0, 1, \ldots, h-2 \end{cases}$$

$\bar{y}^i_0 = (h-1+i, 2)$, $\bar{y}^i_\rho = y^i_\rho$, $\bar{a}^i_\rho = a^i_\rho$, $\rho = 1, 2, \ldots, h-1$.

Note that $\mathcal{E}$ covers the edges (missing in $\mathcal{C}$) $\{\alpha(j,2), \alpha(2h-3-j,2)\} = \{(j,2),(h-1+j,2)\}$ and all the edges of the complete bipartite graph $K_{v,2h-1}$ on vertex set $V \cup W$ except the following ones $\{(j,1),(j,2)\}$ (these edges are in $\mathcal{C}$).

Therefore $(V \cup W, \mathcal{B} \cup \mathcal{C} \cup \mathcal{E})$ is a $P(v+2h-1, 2h, 1)$. It is easy to check that each vertex of V meets exactly h paths of $\mathcal{C} \cup \mathcal{E}$, each vertex of W meets either $h-1$ or h paths of $\mathcal{E}$ and also each path of $\mathcal{C}$. Therefore $(V \cup W, \mathcal{B} \cup \mathcal{C} \cup \mathcal{E})$ is almost balanced.

To prove that $\Omega = X \cup \{(0,2)\}$ is a blocking set note that:
(1) $(0,1) \in \bar{b}_i$, $i = t(2h-1)-h+1, t(2h-1)-h+2, \ldots, t(2h-1)-1$;
(2) $(0,2) \in \bar{b}_i$, $i = \sigma(2h-1), \sigma(2h-1)+1, \ldots, \sigma(2h-1)+h-1$, $\sigma = 0, 1, \ldots, t-1$;
(3) $((\sigma+1)(2h-1), 1) \in \bar{b}_i$, $i = \sigma(2h-1)+h, \sigma(2h-1)+h+1, \ldots, \sigma(2h-1)+2h-1$, $\sigma = 0, 1, \ldots, t-2$;
(4) $(0,2) \in \bar{c}_j$, $j \in Z_{2h-1}$. □

Theorem 7 *Let $v \equiv 0 \pmod{2h-1}$, $v \geq 4h-2$, $h \geq 2$. Then*

$$\mathcal{BSABP}(v, 2h, 1) = \left\{x \mid \frac{v}{2h-1} \leq x \leq \frac{(2h-2)v}{2h-1}\right\}.$$

Proof. Let $(V, \mathcal{B})$ be an almost balanced $P(4h-2, 2h, 1)$ with a blocking set X, $|X| = 2$ (see Lemmas 4 and 5). For each x, $3 \leq x \leq 2h-1$, say Y be a subset of V such that $|Y| = x-2$ and $|Y \cap X| = 0$. Then $X \cup Y$ is a blocking set. Therefore $\{2, 3, \ldots, 2h-1\} \subset \mathcal{BSABP}(4h-2, 2h, 1)$. By Theorem 4, we obtain the proof for $v = 4h-2$.

Now let $\mathcal{BSABP}(v,2h,1) = \left\{x \mid \frac{v}{2h-1} \le x \le \frac{(2h-2)v}{2h-1}\right\}$. By Lemma 6, it is
$\left\{x \mid \frac{v}{2h-1} + 1 \le x \le \frac{(2h-2)v}{2h-1} + 1\right\} \subseteq \mathcal{BSABP}(v+2h-1,2h,1)$.

Hence by Theorem 4 it follows
$\mathcal{BSABP}(v+2h-1,2h,1) = \left\{x \mid \frac{v+2h-1}{2h-1} \le x \le \frac{(2h-2)(v+2h-1)}{2h-1}\right\}$. □

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Addendum to
"Pairwise Balanced Designs on 4s+4 Points"

With reference to paper [2}, two comments are necessary.

First, there is an error in the title; the title refers to "Pairwise Balanced Designs on 4s+4 Blocks"; this should read "Pairwise Balanced Designs on 4s+4 Points".

Secondly, the purpose of the paper was to give a self-contained account of the "case of first failure" for 4s+4 points. The actual result that

$$g - 1 = 2s^2 + 4s + 1$$

is not new; it is a special case of the much more general result given by Rolf Rees in Theorem 4.3 (i) of [1], namely, that

$$cp(K_{m+2} \vee K_m{}^c) = (m^2 + 2m - 1)/2 \text{ for all odd } m \geq 5.$$

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[2] R.G. Stanton, Pairwise Balanced Designs on 4s+4 Blocks with Longest Block of Cardinality 2s+1, JCMCC 40 (2002), 161-165.