q-polynomial identities obtained from finite classical polar spaces.

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Abstract

Several q-polynomial identities are derived from a consideration of classical finite polar spaces. One class of identities is obtained by sorting maximal singular spaces with respect to a given one. Another class is derived from sorting sesquilinear and quadratic forms according to their radicals.

1 Introduction

The idea of a q-polynomial identity, while suffering a certain nebulosity, at a minimum, includes specific known pairs of functions (f, g) satisfying one of the equivalent relations

$$g(n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q f(k), \tag{1}$$

$$f(n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q g(k)(-1)^{n-k} q^{\binom{n-k}{2}} \tag{2}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} g(n-k). \tag{3}$$

The q-polynomial identities are derived from several sources. Sometimes they arise from equating powers of x in known factorizations in the ring of q-series in x; at other times they arise from brute-force synthesis of q-analogues of combinatorial identities upon inserting appropriate powers of q at strategic points[1].

Occasionally we find them derived from vector spaces over $GF(q)$. Here are two classical examples:

An example due to Rota.

$$z^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (z-1)(z-q)\cdots(z-q^{k-1}) \qquad (4)$$

results from sorting the z^n linear mappings $GF(q)^{(n)} \to Z$ into a $GF(q)$-space Z with z vectors according to their kernels. Möbius inversion then gives:

$$\prod_{k=0}^{n-1}(z-q^k) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} z^{n-k}. \qquad (5)$$

Putting $z = -y/x$ in (5) yields

$$\prod_{k=0}^{n-1}(y+xq^k) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} y^{n-k}x^k. \qquad (6)$$

This is called the "q-binomial theorem", since it gives the binomial theorem when $q=1$. (See page 218 of [2].)

A second example.

Suppose V is an n-dimensional vector space over $GF(q)$ and U is a fixed d-subspace of V. Then the collection $\mathcal{V}_e$ of all e-dimensional subspaces can be sorted according to how they intersect U. To start with, the number of elements $\mathcal{V}_e$ which meet U trivially is

$$q^d \begin{bmatrix} n-d \\ e \end{bmatrix}_q. \qquad (7)$$

Fix a k-subspace W of U. Then there is a bijection between the set of subspaces X_W of $\mathcal{V}_e$ which intersect U at W, and the

[1]As in Example 2.2.5, p. 70, *Enumerative Combinatorics, vol. I* by R. Stanley [3].

set of $(e-k)$-spaces of V/W which meets U/W trivially. The cardinality of the latter set can be calculated from equation 7 with the parameter change $(n,d,e) \rightarrow (n-k, d-k, e-k)$ as

$$q^{d-k} \begin{bmatrix} n-d \\ e-k \end{bmatrix}_q .$$

Thus

$$\begin{bmatrix} n \\ e \end{bmatrix}_q = \sum_{k=0}^{d} \begin{bmatrix} d \\ k \end{bmatrix}_q \begin{bmatrix} n-d \\ e-k \end{bmatrix}_q q^{d-k}, \qquad (8)$$

the q-analogue of the Chu-VanderMonde identity (see eq (3.3.10), p. 37 of Andrews [1]).

These two examples used subspaces and mappings of a finite vector space. This raises the question whether something more could be learned from the more elaborate context of classical finite polar spaces. This talk merely records a few simple experiments probing this question.

2 The finite non-degenerate polar spaces.

2.1 Beginning concepts

A **point-line geometry** is an incidence system $(\mathcal{P}, \mathcal{L})$ of points ($\mathcal{P}$, just some set) and lines ($\mathcal{L}$, certain (pairwise distinct) subsets of $\mathcal{P}$ of size at least two). A subset X of $\mathcal{P}$ is called a **subspace** if every line $L \in \mathcal{L}$ which has at least two of its points in X has all of its points in X. The subspace X is said to be **singular** if any two of its points are **collinear** – that is, are members of a common line. (Graphs, grids, block designs, projective spaces are all examples of point-line geometries.)

A (thick, non-degenerate) **polar space** is a point-line geometry $(\mathcal{P}, \mathcal{L})$ with these properties:

(P1) Every line has at least three points: no point is collinear with all remaining points.

(P2) Given a non-incident point-line pair (p, L), either (1) p is collinear with exactly one point of L, or (2) p is collinear with all points of L.

A (thick, non-degenerate) **generalized quadrangle** is just a polar space in which alternative (2) of (P2) does not occur.[2]

Lemma 1 *Any (nondegenerate) polar space (with thick lines) is a partial linear space (that is two distinct points are together collinear with* at most *one line) and all singular subspaces are projective spaces. Moreover, if a maximal singular subspace has finite projective rank* $r-1$*, then all maximal singular spaces have this rank and* r *is called the* **rank of the polar space.**

Thus a generalized quadrangle is just a polar space of rank 2.

2.2 Finite polar spaces

Except for non-square grids, all finite generalized quadrangles have $s+1$ points on each line, and $t+1$ lines on each point, and the parameter-pair (s, t) is called the **order of the generalized quadrangle.**

Of course from Lemma 1 a finite polar space has a finite rank r. If A is a singular subspace of projective rank $r-3$ (that is, it has codimension 2 in some maximal singular subspace), then the system of singular subspaces containing A is the incidence system of a generalised quadrangle of order (s, t). It can be shown that these parameters are independant of the particular choice of A, and so (s, t) is called the **order of the residual quadrangle of the polar space.** This is an important invariant of finite polar spaces. All lines of the polar space have $s+1$

[2]The adjectives "thick" and "nondegenerate" are in parenthesese because a good half of the literature would be compelled to add these terms to make the definitions here compatible. They are also there because I soon intend to drop these adjectives altogether, asking the reader to rely internally on the definitions just as I have given them. It will save time.

points: all subspaces of codimension 1 in some maximal singular subspace lie in exactly $t + 1$ maximal singular subspaces.

2.3 The finite classical polar spaces

There are two very important examples of finite polar spaces:

1. Let $s : V \times V \to GF(q)$ be a reflexive sesquilinear form on vector space V over $GF(q)$. The **radical of** V (with respect to s) is the sub-vector space Rad $(V) := \{v \in V | s(v, V) = 0\}$. The form s is said to be **non-degenerate** if Rad $(V) = 0$. A subspace of U of V is said to be **totally isotropic** if and only if $s(U, U) = 0$. Now let $\mathcal{P}$ and $\mathcal{L}$ be the full collections of totally isotropic 1- and 2-dimensional subspaces, respectively. Then (viewing each 2-space as the set of 1-spaces within it) $(\mathcal{P}, \mathcal{L})$ is a polar space.

2. Now let $Q : V \to GF(q)$ be a quadratic form on the $GF(q)$-space V with associated bilinear form B. A subspace U of V is **totally singular** if and only if, $Q(U) = 0$. The **radical of the form** Q is the subspace of all totally singular 1-subspaces of the radical of V with respect to the form B. The form Q is a **non-degenerate quadratic form** if and only if Rad $(Q) = 0$. The geometry $(\mathcal{P}, \mathcal{L})$ of totally singular 1- and 2 subspaces of a space V with a non-degenerate quadratic form is a polar space **of orthogonal type.**

Then the two examples above are called **the finite classical polar spaces.** In addition to the classical generalized quadrangles, there are, amazingly, several further infinite families of non-classical finite generalized quadrangles and it is not clear that they can be classified. In contrast, all finite polar spaces of rank at least three are one of the known classical examples.

There are just six types of finite classical polar spaces of rank r; they are listed by the name of their associated classical group in the first column of Table 1. (One should note that the names

Type	Ovoid no.	(s,t)
$Sp(2n,q)$	q^n+1	(q,q)
$O(2n+1,q)$	q^n+1	(q,q)
$O^+(2n,q)$	$q^{n-1}+1$	$(q,1)$
$O^-(2n+2,q)$	$q^{n+1}+1$	(q,q^2)
$U(2n,q^2)$	$q^{2n-1}+1$	(q^2,q)
$U(2n+1,q^2)$	$q^{2n+1}+1$	(q^2,q^3)

Table 1: The ovoid numbers and the orders of the residual quadrangles of the finite classical polar spaces of rank n.

of the classical groups utilizes two parameters: the field size q, and the dimension of its natural module V). This dimension has been expressed here in terms of the polar rank r – so that we have a consistent look at rank r polar spaces. Each posseses a residual quadrangle of order (s,t), and these parameters are listed in the third column of Table 1.

With each finite polar space Δ of polar rank r whose residual quadrangle has order (s,t), there is associated a rather peculiar number which we call the **ovoid number of** Δ, which is denoted and calculated by the expression: $O_{st}(r) := s^{r-1}t+1$,. It is a useful number because both the number of points, and the number of maximal singular subspaces of Δ can be expressed by means of this number:

Lemma 2 *Let Δ be a finite polar space of rank r whose residual quadrangle has order (s,t).*

1. *The number of points of the polar space Δ is given by*

$$|\mathcal{P}| = (1+s+\cdots+s^{r-1})\cdot(s^{r-1}t+1).$$

2. *The number of maximal singular subspaces of Δ is*

$$\begin{aligned} m_{st}(r): &= O_{st}(1)O_{st}(2)\cdots O_{st}(r) && (9)\\ &= (t+1)(st+1)\cdots(s^{r-1}t+1). && (10)\end{aligned}$$

[The reader might note that $m_{st}(r)$ is in fact the function $(t : r)_s$ used so pervasively throughout Andrews' book on partitions ([1]). Is this made transparent by some connection?]

To sum up the situation we record the following:

Corollary 3 *Let P be one of the six types of non-degenerate polar spaces of rank n over a field $GF(q)$ (or $GF(q^2)$ in the Hermitian case).*

1. *Then the number of points of P is given below:*

Type of geometry	*Number of polar points*	*No.opposite one*
$Sp(2n,q)$	$(1+q^n)(1+q+\cdots q^{n-1})$	q^{2n-1}
$O(2n+1,q)$	$(1+q^n)(1+q+\cdots q^{n-1})$	q^{2n-1}
$O^+(2n,q)$	$(1+q^{n-1})(1+q+\cdots q^{n-1})$	q^{2n-2}
$O^-(2n+2,q)$	$(1+q^{n+1})(1+q+\cdots q^{n-1})$	q^{2n}
$U(2n,q^2)$	$(1+q^{2n-1})(1+q^2+\cdots q^{2(n-1)})$	q^{4n-3}
$U(2n+1,q^2)$	$(1+q^{2n+1})(1+q^2+\cdots q^{2(n-1)})$	q^{4n-1}

2. *The number of points opposite a given point p (that is, points not collinear with p) is $|\mathcal{P}| - |p^\perp|$, and this is the highest power of q to be discovered when the expression in the middle column of the above table is written out as a polynomial in q. The answer is given in the third column.*

3. *The number of maximal singular subspaces is given in the following table*

Type of geometry	*No. of max.singular subspaces*	*No. opposite one*
$Sp(2n,q)$	$(1+q)(1+q^2)\cdots(1+q^n)$	$q^{n(n+1)/2}$
$O(2n+1,q)$	$(1+q)\cdots(1+q^n)$	$q^{n(n+1)/2}$
$O^+(2n,q)$	$2\cdot(1+q)\cdots(1+q^{n-1})$	$q^{n(n-1)/2}$
$O^-(2n+2,q)$	$(1+q^2)\cdots(1+q^{n+1})$	$q^{n(n+3)/2}$
$U(2n,q^2)$	$(1+q)(1+q^3)\cdots(1+q^{2n-1})$	q^{n^2}
$U(2n+1,q^2)$	$(1+q^3)(1+q^5)\cdots(q^{2n+1})$	q^{n^2+2n}

4. *The number of maximal singular subspaces opposite a given one is the highest power of q appearing when the expression appearing in the middle column of the above table is expressed as a polynomial in q. This power is recorded in the third column. (The definition of "opposite" will be clear in the next subsection.)*

2.4 The curious oriflame phenomenon

In the polar space of type $O^+(2r,q)$, the number of maximal singular subspaces would be written $(1:r)_q$ in Andrews' notation, which I hope adjusts us to the funny "2" in front of the factorized expression in the third line of the table presented in part three of Corollary 3. The "2" is prophetic here. In fact there are exactly two classes of maximal singular subspaces of the $O^+(2r,q)$ polar space, say $\mathcal{M}_1$ and $\mathcal{M}_2$ defined by this property: *Two maximal singular subspaces belong to the same class if and only if their intersection has even codimension in each of the spaces.* In other words, we are saying that this "even codimension" relation is in fact an equivalence relation.[3]

2.5 Counting opposite subspaces.

Let us return, for a moment to a reflexive sesquilinear form $s : V \times V \to GF(q)$. For any subspace U of V, we set

$$U^\perp := \{v \in V | s(v,u) = 0 \text{ for all } u \in U\}.$$

Comparing previous definitions, $V^\perp = \text{Rad }(V)$. Let A and B be singular subspaces of V with respect to s. We say that A is **opposite** B if and only if

$$A^\perp \cap B = 0 = B^\perp \cap A.$$

[3] A lucky accident that works in finite rank geometries with the infinitly many points as well. Without it, we should have no diagram geometries of type D_r, and no half-spin geometries.

This condition forces s to induce two injective linear mappings

$$A \to B^* \text{ and } B \to A^*.$$

Since dimensions are finite, we have $\dim A = \dim B$. (Thus as an alternative, we could have defined "opposite subspaces" by saying that $A^\perp \cap B = 0$ and A and B have the same finite dimension.)

Elementary arguments yield the following

Lemma 4 *Let Δ be a polar space over $GF(q)$ of polar rank r and residual quadrangle of order (s,t). Let d be a non-negative integer at most r. Then the number of d-subspaces opposite a given one is*

$$s^{2d(r-d)+d(d-1)/2}t^d.$$

In the special case that $d = r$, the formula yields the number of maximal singular subspaces having trivial intersection with a given one, and that number is in fact the highest degree monomial in s and t in the formula for $m_{st}(r)$ – that is,

$$s^{r(r-1)/2}t^r.$$

2.6 Counting forms.

The sesquilinear forms we have encountered are of three types: alternating (giving rise to the symplectic groups when non-degenerate), Hermitian (giving rise to the unitary groups when non-degenerate) and the symmetric forms (giving rise to three species of orthogonal groups when non-degenerate). We can count the total number of forms of these basic types (non-degenerate or not) by simply counting the possible Grammian matrices of each of the three types.

We record this count of forms in dimension n:

Δ	g
$Sp(2r,q)$	$(q-1)(q^3-1)\cdots(q^{2r-1}-1)q^{r(r-1)}$
$O^+(2r,q)$	$(q^r+1)[(q-1)(q^3-1)\cdots(q^{2r-1}-1)]q^{r^2}/2$
$O^-(2r+2,q)$	$(q^{r+1}-1)[(q-1)(q^3-1)\cdots(q^{2r+1}-1)]q^{(r+1)^2}/2$
$O(2r+1,q)$	$(q-1)(q^3-1)\cdots(q^{2r+1}-1)q^{r(r+1)}$
$U(2r,q^2)$	$(q-1)[(q^2+1)(q^3+1)\cdots(q^{2r}+1)]q^{r(2r-1)}$
$U(2r+1,q^2)$	$(q-1)[(q^2+1)(q^3+1)\cdots(q^{2r+1}+1)]q^{r(2r+1)}$

Table 2: The number of non-degenerate sesquilinear or quadratic forms on a vector space V of a given classical type.

Type of reflexivity	value of s	no. of forms
Alternating	q	$q^{n(n-1)/2}$
Symmetric (q odd)	q	$q^{(n+1)n/2}$
Hermitian	q^2	q^{n^2}

But how many non-degenerate forms are there of each type?

Lemma 5 *The total number of non-degenerate forms yielding a polar space of rank r of one of the six classical types is given by the function g explicitly computed in the second column of Table 2.*

This is about all that we need to know about polar spaces in order to derive the subsequent results.

3 The identities from polar spaces.

3.1 Polynomial identities from maximal singular subspaces of polar spaces.

Here suppose Δ is one of the six types of classical polar spaces of rank $r \geq 2$. Let $\mathcal{M}(\Delta)$ denote its full class of maximal singular subspaces (these are r-dimensional subspaces of V). If we fix one of these maximal singular subspaces - say M - we may sort the remaining maximal singular subspaces according to how they intersect M. Given a k-subspace U of M, the number of maximal singular subspaces in $\mathcal{M}$ intersecting M at subspace U, is exactly the number of maximal singular subspaces of the non-degenerate rank $r - k$ polar space induced on $U^{\perp}/U$ which are opposite M/U. This number is the highest power of q appearing in the polynomial in q representing number of maximal singular subspaces of $U^{\perp}/U$ - a number depending on Δ and rank $r - \dim U$ only.

Thus if $m_{st}(r)$ denotes the number of maximal singular subspaces of a classical polar space with residual quadrangle of (s, t) and rank r, and a_{st} is the number of maximal singular subspaces opposite a given one in this rank and type, we have

$$\begin{aligned} m_{st}(r) &= \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_s a_{st}(r-k) \\ &= \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_s a_{st}(k), \end{aligned}$$

by the duality. Since $a_{st}(r) = s^{r(r-1)/2}t^r$, this means

$$m_{st}(r) = (t+1)(st+1)\cdots(s^{r-1}t+1) \tag{11}$$

$$= \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_s s^{k(k-1)/2}t^k. \tag{12}$$

The classical polar spaces produce five identities from equation (11), for $(s, t) = (q, q), (q, 1), (q, q^2), (q^2, q), (q^2, q^3)$.

On the "up side", an induction argument shows that this is a polynomial identity in arbitrary s and t. But actually the identity can be obtained from the q-binomial theorem (equation 4) upon setting $y = 1$. So not much new seems to be happening.

However in one case there is an unforeseen breakup in summands.

When $(s,t) = (q,1)$, there are two parities to n, and in the polar spaces (which are type $O^+(2n,q)$) the oriflame phenomenon causes us to have two classes of maximal singular subspaces to choose to decompose. Thus altogether four distinct identities appear.

$$(1) : \prod_{i=1}^{2n-2}(q^i+1) = \sum_{i=0}^{n-1} \begin{bmatrix} 2n-1 \\ 2i \end{bmatrix}_q q^{2i(2i-1)/2}.$$

$$(2) : \prod_{i=1}^{2n-1}(q^i+1) = \sum_{i=0}^{n} \begin{bmatrix} 2n \\ 2i \end{bmatrix}_q q^{2i(2i-1)/2}.$$

$$(3) : \prod_{i=1}^{2n-2}(q^i+1) = \sum_{i=0}^{n-1} \begin{bmatrix} 2n-1 \\ 2i+1 \end{bmatrix}_q q^{i(2i+1)}.$$

$$(4) : \prod_{i=1}^{2n-1}(q^i+1) = \sum_{i=0}^{n-1} \begin{bmatrix} 2n \\ 2i+1 \end{bmatrix}_q q^{i(2i+1)}$$

3.2 Polynomial identities from quadratic and sesquilinear forms.

Here we can present just three identities:

$$q^{r(2r-1)} = \sum_{k=0}^{r} \begin{bmatrix} 2r \\ 2k \end{bmatrix}_q (q-1)(q^3-1)\cdots(q^{2k-1}-1)q^{k(k-1)}$$

$$q^{n^2}-1 = \frac{q-1}{q+1}\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} [(q+1)(q^2+1)\cdots(q^k+1)]q^{k(k-1)/}$$

$$q^{m(2m+1)} = \sum_{k=0}^{m} \begin{bmatrix} 2m \\ 2k \end{bmatrix}_q q^{k(k+1)}[(q-1)(q^3-1)\cdots(q^{2k-1}-1)]$$
$$+\sum_{k=1}^{m} \begin{bmatrix} 2m \\ 2k-1 \end{bmatrix}_q q^{k(k-1)}[(q-1)(q^3-1)\cdots(q^{2k-1} -$$

REMARK: The product in brackets on the right side of the first and third lines above is understood to be 1 when $k=0$.

Here, the paradigm works as follows: We consider the $GF(q)$-vector space of all forms on an n-dimensional vector space V which are either (1) symplectic bilinear forms over $GF(q)$, (2) Hermitian forms over $GF(q^2)$ or (3) quadratic forms $Q : V \to GF(q)$. It is not difficult to see that the total number of forms is a power of q – even when we are counting quadratic forms in characteristic 2.

But of course, these forms can be sorted according to their radicals. Now for each of these three cases, there is clearly a bijection between the collection of forms on V of the given type having radical U, and the collection of $h_\Delta(d)$ non-degenerate forms on a d-dimensional vector space such as V/U of the same type. Thus, according to our paradigm,

$$\begin{aligned} q^{(\text{appropriate } x)} &= \text{no. forms in a class ((1)-(3)) on } V \\ &= \sum_{k=0}^{n=\dim V} \begin{bmatrix} n \\ k \end{bmatrix}_q h_\Delta(n-k). \end{aligned}$$

So three seperate Möbius-invertible identities appear. We examine these cases separately.

As one can see, the formulae depend on the number $h_\Delta(n)$ of non-degenerate forms of a given type on a space of fixed dimension n. It's dependence on the dimension of the ambient space V distinguishes it from the function $g_\Delta(r)$ of non-degenerate forms of a certain type Δ having *rank* r, which was more convenient for inductive counting arguments in the last section. (In the case of g the "type" Δ even distinguished between elliptic and hyperbolic forms. The present case division does not.)

In the symplectic option for Δ there is not much to say. The function h is as follows

$$h_\Delta(n) = \begin{cases} (q-1)(q^3-1)\cdots(q^{n/2}-1)q^{n(n-2)/4} & \text{if } n \text{ is even.} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Plugging in, we obtain the first sum when $\dim V$ is even. The odd-dimensional case is just a paraphrase of the first.

The case (2) of Hermitian forms is also relatively simple. As one sees, the two cases of even and odd dimension in Table 2 coalesce to a one-parameter formula, and the second identity results.

The case of orthogonal forms is a little more compicated for two reasons: (i) there are two different geometries with two different formulaes and ranks when the dimension $n = \dim V$ is even (summing to a nice formula), and (ii) the fact that this does not coalesce well with the formula for the odd dimensional cases (not to mention the different Gaussian coefficients which come into play in the sum upon changing parities).

Consider the even dimension $n = 2m$. We must add the number of non-degenerate forms of type $O^+(2m, q)$ to the number of type $O^-(2m, q)$. From our table, this is

$$q^{m(m+1)}[(q-1)(q^3-1)\cdots(q^{2m-1}-1)].$$

For the odd dimension, $n = 2m-1$ the number of non-degenerate quadratic forms is

$$q^{m(m-1)}[(q-1)(q^3-1)\cdots(q^{2m-1}-1).$$

In the case that $n = 2m-1$, the value of h is given by Table 2. Assembling these incompatible cases causes two seperate sums to appear on the right in the third identity.

REMARK: One might think of the example of Rota at the beginning of this note as being about 1-linear forms, $f : V \to Y$. The results of this last section might be regarded as a disussion about sesquilinear forms $V \times V \to W$ specialized to the case $W = GF(q)$. Could these experiments be extended to multilinear forms $V \times \cdots \times V \to W$?

[N.B. All of the unproved counting results in Section 2 on non-degenerate polar spaces and forms explicitly proved in the author's unpublished monograph "A manual of enumerative combinatorics of the finite classical geometries". Of course there are many other scattered sources for these results.]

References

[1] Andrews, George, *The Theory of Partitions*, The Encyclopedia of Mathematics: vol. 2, G-C. Rota, ed. Addison-Wesley Publishing Company, 1976, Reading Massachusetts.

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[3] Stanley, Richard, *Enumerative Combinatorics: Volume I.* Cambridge Studies in Advanced Mathematics 49. Cambridge University Press, 1996, Cambridge, England.