On The Integer -Magic Spectra of Graphs*

by

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" A teacher can never truly teach unless he is still learning himself"
------------ Rabindranath Tagore

Abstract: For k>0, we call a graph G=(V,E) as <u>Z_k-magic</u> if there exists a labeling l:E(G) $\rightarrow$ Z_K^* such that the induced vertex set labeling l^+: V(G) $\rightarrow$ Z_K

$$l^+(v)=\Sigma \{l(u,v) : (u,v) \text{ in } E(G)\}$$

is a constant map. We denote the set of all k such that G is k-magic by IM(G). We call this set as the **integer-magic spectrum** of G. We investigate these sets for general graphs.

1. **<u>Introduction</u>.** For any abelian group A, written additively we denote A* = A−{0}. Any mapping l: E(G) $\rightarrow$ A* is called a labeling. Given a labeling on edge set of G we can induced a vertex set labeling l^+: V(G) $\rightarrow$ A as follows:

$$l^+(v)=\Sigma \{l(u,v) : (u,v) \text{ in } E(G)\}$$

A graph G is known as <u>A-magic</u> if there is a labeling l: E(G) $\rightarrow$ A* such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; i.e., $l^+(v)$ = c for some fixed c in **A**. We will called <G,l> a A-magic graph. In general, a graph G may admits more than one labeling to become a A-magic graph.

We denote the class of all graphs (either simple or multiple graphs) by **Gph**. The class of all abelian groups by **Ab**. For each A in **Ab** we denote the class of all A-magic graphs by $_A$**MGp.**

When A = Z, the Z-magic graphs were considered in Stanley[17]; he pointed out that the theory of magic labelings can be put into the more general context of linear homogeneous diophantine equations [23]. When the group is Z_k, we shall refer to the Z_k –magic graph as *k-magic*. Graphs which are k-magic had been studied in [12,15].

*Paper presented at the Fourteenth Midwestern Conference on Combinatorics, Crytography and Computing, Wichita, October 2000

JCMCC 42 (2002), pp. 77-86

Doob [1,2,3] also considered A-magic graphs where A is an abelian group. Given the graph G , the problem of deciding whether G admits a magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equation has a solution [22]. At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs.

The original concept of A-magic graph is due to J. Sedlacek [19,20], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

In this paper we use **N** to denote the set {1,2,3,...} and for each k>0, we write the set {kx: x in N} by **kN** and {k+x: x in N} by **k+N**. We will define the graph G with a magic labeling l: E(G) → N as **N-magic.** It is well-known that a graph G is N-magic if and only if each edge of G is contained in a 1-factor (a perfect matching) or a {1,2}-factor (see [9, 17,26]). Reader refer to [5,6,7,8,10, 12, 24, 25] for N-magic graphs. The Z-magic is weaker than N-magic. Figure 1 shows a graph which is Z-magic but not N-magic.

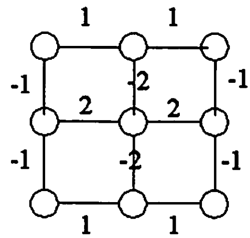


Figure 1.

For simplicity, we will consider Z-magic as 1-magic. Given a graph G, we denote the set of all k> 0 such that G is k-magic by IM(G). We call this set as **integer-magic spectrum** of G. We investigate these sets for general graphs [15].

Note the magic valuation considered in [4, 11] do not relate to our concept. Papers [13,14,18,21] deal with more general concept of k-magic graphs.

2. Graphs whose IM(G)= ∅.

Definition 1. A graph G is called **nonmagic** if it is not A-magic for any group A in **Ab**.

It is obvious that IM(G)= ∅, for nonmagic graph G.

In [15], we show that any graph G is an induced subgraph of a non-magic graph $G^{\#}$. We can extend G to $G^{\#}$ by glue the end vertex of P_3 to any vertex u of G and the resulting graph (G,u) o P_3 = G# is nonmagic . Thus we have

Theorem 1. Any graph G which has a tail P_3 has IM(G)= ∅.

In general, it is difficult for trees to be k-magic. By Theorem 1 we see that trees with tail P_3 are non-magic. There are abundance trees with no tail of P_3 but with IM(T)=_∅. For integers m,n ≥2, we consider a tree of diameter 3 with m+n+2 vertices such that one side has m vertices of leaves and other side with n vertices of leaves. We denote this tree by ST(m,n). (Figure 2)

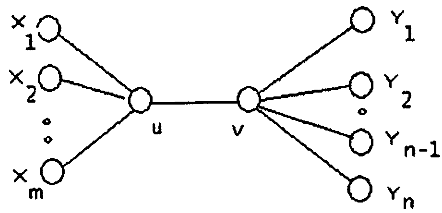


Figure 2

Theorem 2. The tree ST(m,n) has empty integer-magic spectrum if |m-n|=1.

3. **Graphs whose IM(G)= N .**

It is obvious that for any regular graph G, we have IM(G)= N and if **G** is N-magic then it is k-magic for all k>2 and hence if it is 2-magic then IM(G)= N. In this section we want to show that there are abundant non-regular graphs which are k-magic for all k>0.

Theorem 3. The tree ST(m,m) has integer-magic spectrum N if m is even.

Remark. When m is odd, the situation is quite difference. See Theorem 7.

For n≥ 3, the join of C_n and K_1, i.e. C_n+K_1 is called the **wheel with n spokes.** We denote it by W_n.

Theorem 4. The wheel W_n has $IM(W_n)$= N if n is odd.

Proof. Since W_n has degree set {3,n} It is 2-magic. It suffices to show that W_n is N-magic. Suppose n=2k+1. We label all the spokes by 1 and all the rim edges by k. We see that it has sum n.

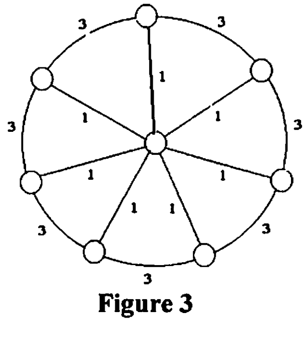


Figure 3

We want to consider a family of graphs which are constructed from cycles. For any m,n $\geq$ 3. We denote C_m @ C_n the graph depicit as follows (Figure 4):

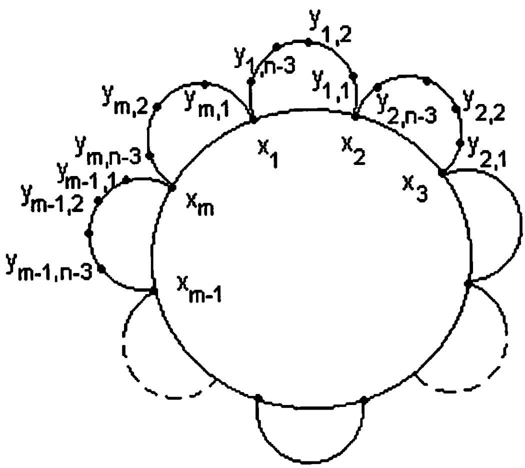


Figure 4.

We will call these graphs as ***flower*** graphs.We have the following general result.

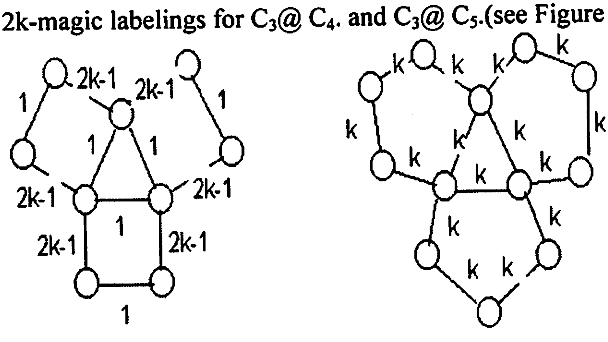
Theorem 5. For all m, n $\geq$ 3, the flower graphs C_m @ C_{2t} and C_{2t} @ C_n are A-magic for all non-trivial abelian groups A.

Proof. Every flower graphs are eulerian therefore they are 2-magic.

Let A be any abelian group with |A| >2. For the flower graph C_m @ C_{2t}. Pick x in A*, we label the edges in C_m by x, and all the petals of the flower graph C_m @ C_{2t} by -x,x,-x,x,..., consecutively. We see that each vertex in the C_m @ C_{2t} has sum 0.

For the flower graph C_{2t} @ C_m. Pick x in A*, we label the edges in C_{2t} by x, -x,x, -x,... consecutively and all the petals of the flower graph C_{2t}@C_m by -x,x,-x,x,..., consecutively. We see that each vertex in the C_{2t} @ C_n has sum 0.

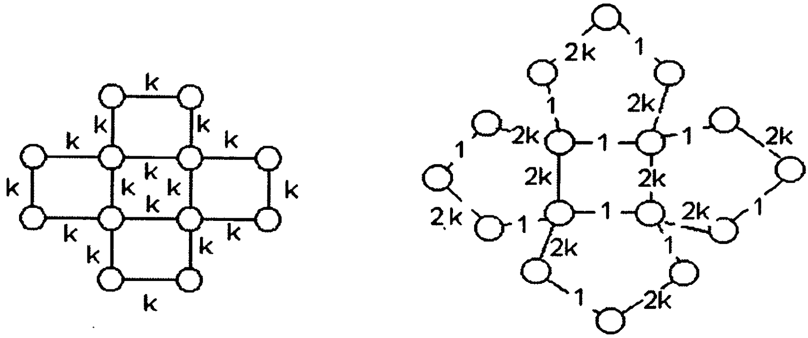
Example 3. 2k-magic labelings for C_3@ C_4. and C_3@ C_5.(see Figure 5)



C_3@ C_4 is 2k-magic C_3@ C_5 is 2k-magic

Figure 5.

Example 4. A 2k-magic labeling for $C_4@C_4$ and 2k+1-magic labeling for $C_4@C_5$. (see Figure 6).



$C_4@C_4$ is 2k-magic $\quad$ $C_4@C_5$ is 2k+1 -magic

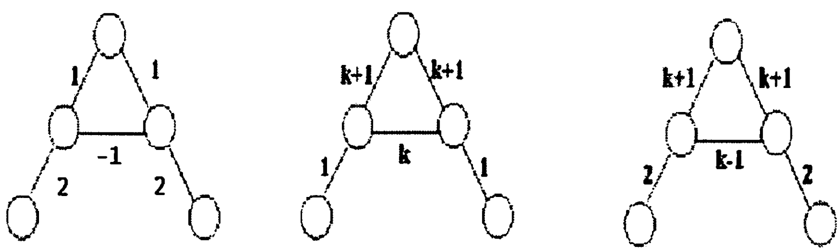
Figure 6.

Corollary 6. $IM(C_m @ C_{2t})$ and $IM(C_{2t} @ C_n)$ =N for all m,n ≥ 3 and $t \geq 2$.

3. Graphs whose IM spectra are of the form N-S where S is a finite set.
(a)Graphs whose IM(G)= N-{2}

Theorem 7 The integer-magic spectrum of tree ST(m,m) is N-{2} for all odd integer $m \geq 3$.

Theorem 8. The unicyclic graph A of order 5 has N-{2} as integer-magic index set.



Z-magic $\quad$ Z_{2k+1}- magic $\quad$ Z_{2k}- magic

Figure 7.

We can construct an infinite many graphs G with IM(G)=N-{2} as follows: Take two copies of C_3 with two specify vertices u and v respectively. Between u and v connect with a path of length k. We will denote the resulting graph by Y(u,v,k).

Theorem 9. The graph Y(u,v,2k+1) has integer magic spectrum N-{2}.

Proof. The degree set of Y(u,v,2k+1) is {2,3}. Thus it is not 2-magic. However, we can show that it is N-magic by labeling the edges between u and v by 1,2,1,2,....consecutively and the edge of C_3 by 1,1,2.

We illustrate the above labeling by the following example.

Example 5. The graph Y(u,v,3) is k-magic for all $k \geq 3$. Its integer magic spectrum is N-{2}.

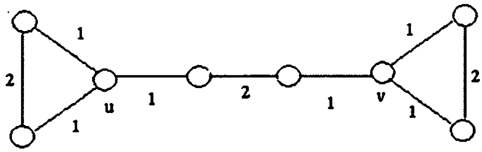


Figure 8.

Theorem 10 For $t \geq 2$, the corona of the cycle C_{2t}© K_1 has IM(C_{2t}© K_1) = **N-{2}.**

Proof. If we label all the append edges by 1.and all the edges of the cycle by x,-x,x,-x,... consecutively. We see that C_{2t}© K_1 is Z-magic. Since the degree set of C_{2t}© K_1 is {1,2,3}, it is not 2-magic. Figure 6 shows that it is k-magic for all k>2.

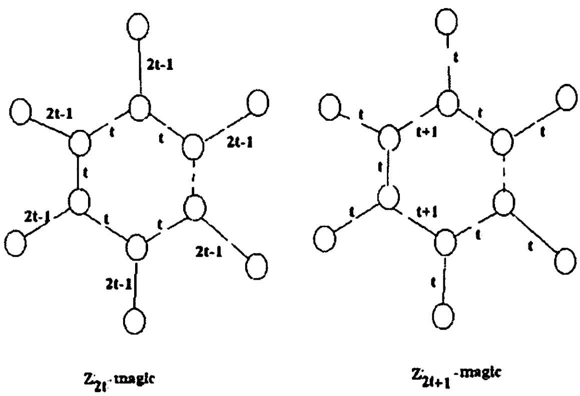


Figure 9

Theorem 11. All the grid graph P_mxP_n has IM(P_mxP_n) = **N-{2}, except** P_2xP_2 which is IM(P_2xP_2) = **N.**

Theorem 12. The fan graph $F_n=P_n+K_1$ has IM(F_n) = **N-{2} for all n >2.**

Theorem 13. The wheel W_{2k} has IM(W_{2k})= N-{2} for all k>1.

Proof. Since W_n has degree set {3,2k} It is not 2-magic. We want to show that W_n is N-magic. We label all the spokes by 1 and all the rim edges by 1, 2k-2, 1, 2k-2,... consecutively. We see that it has sum 2k.

(b) Graphs G with IM(G)=N-{2,3,4}.

Theorem 14. The following graph G has IM(G) = **N-{2,3, 4}.**
Proof. G is Z-magic. However, it is not 2,3,4-magic.

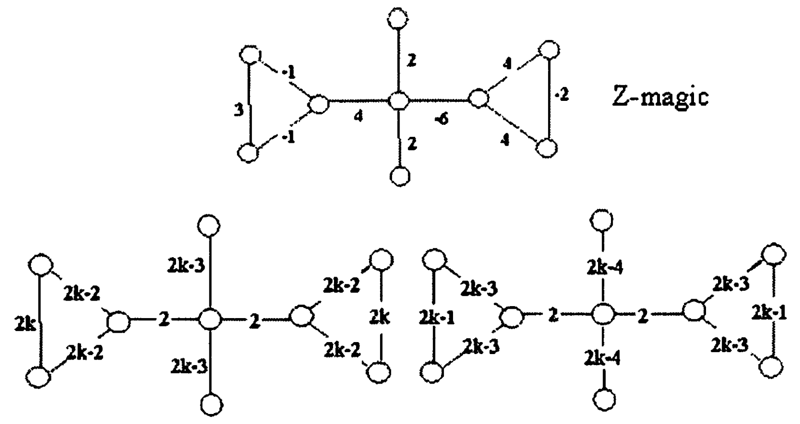


Figure 10.

4. Graphs G with IM(G)= {1}U {4+2k: k=1,2,3,....}

Theorem 15. The following graph G has IM(G) = **{1}U {4+2k: k=1,2,3...}.**
Proof. It is not 4-magic and it is also not 2k+1-magic for all k≥1.

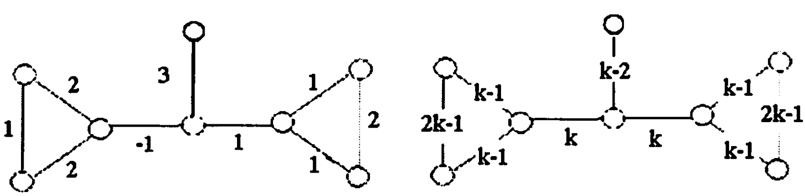


Figure 11.

5. Graphs whose IM(G) of the form a+bN

(a) Graphs with IM(G)=kN for some k>1..

Theorem 16. For t≥1, the corona of the cycle C_{2t+1}© K_1 has IM= **2N**

Theorem 17. The flower graph C_m @ C_n has IM(C_m @ C_n) =2N if m.n are both odd

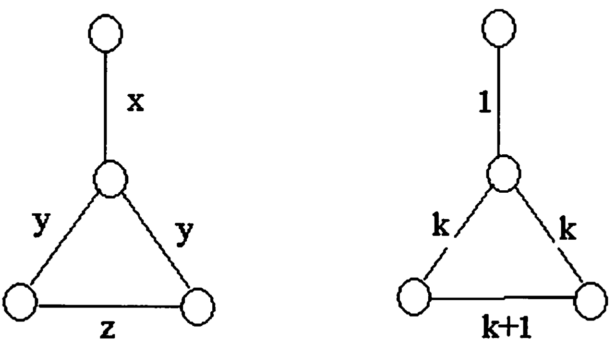
Theorem 18. All stars K(1,n) has the integer-magic spectrum of the form IM(K(1,k+1)) = **kN** **for all k>2.**

Remark. IM(Star K(1,2)) = **∅.**

(b) Graphs whose IM(G)=k+ mN for some k,m>0..

Theorem 19. The following unicyclic graph G has IM(G) = 2+2N={4,6,8,10,... }

Proof. If G is Z-magic then it must has the labeling as follows:



2k-magic for k≥2

Figure 12.

Thus 2 y =0, i.e. y =0. Hence G is not Z-magic.

Clearly it is not 2-magic and 2k+1-magic for all k> 1. The labeling in Figure 12 show that it is 2k-magic for all k≥2.

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