# On Super Edge-Magic n-Stars

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**ABSTRACT.** A (p,q) graph G is *total edge-magic* if there exits a bijection f: V $\cup$ E $\rightarrow$ {1, 2, ..., p+q} such that $\forall$ e = (u,v) $\in$ E, f(u) + f(e) + f(v) = constant. A total edge-magic graph is a *super edge-magic graph* if f(V(G))= {1,2,...,p}. For $n \geq 2$, let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of increasing non-negative integers. A n-star $St(a_1,a_2,a_3,\ldots,a_n)$ is a disjoint union of n stars $St(a_1)$, $St(a_2)$,..., $St(a_n)$. In this paper we investigate several classes of n-stars that are super edge-magic.

## 1. Introduction.

In this paper we consider graphs with no loops. For undefined concepts we refer the reader to [1]. A (p,q)-graph G= (V, E) with p vertices and q edges is called ***edge-magic*** if there is a bijection f : E $\rightarrow$ {1,2,...,q} such that the induced mapping $f^+:V \rightarrow \mathbf{Z}_p$, given by $f^+(u) = \Sigma\{f(u,v): (u,v) \text{ in } E\} \pmod p$ is a constant mapping. This concept of edge-magic graphs was introduced by Lee, Seah and Tan in 1992 [12]. An example of a (6,9)-graph with two different edge-magic labelings is shown in Figure 1.

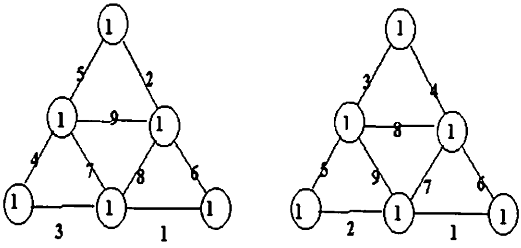


**Figure 1**

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A necessary condition of edge-magicness of G is given in [12]:

$$q(q+1) \equiv 0 \pmod{p}.$$

Given a graph G, the problem of deciding whether G admits an edge-magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equations has a solution. No polynomial time bounded algorithm is known for determining whether a graph is edge-magic. It was shown, however, that no trees (except $P_2$) and unicyclic graphs are edge-magic [12]. For more general results and some conjectures on edge-magic graphs , the reader is referred to [11,12,13,15,16].

Recently, another labeling problem was considered by Enomoto et al [4] and Wallis et al [20]. A (p,q)-graph G = (V, E) with p vertices and q edges is called ***total edge-magic*** if there is a constant s and a bijection $f : V \cup E \to \{1, 2,..., p+q\}$ such that $\forall\ e = (u,v) \in E$, $f(u) + f(e) + f(v) = s$. A total edge-magic graph is called ***super edge-magic*** if $f(V(G)) = \{1,2,\ldots,p\}$. An example of a unicyclic graph with 6 vertices and its total edge-magic labeling is shown in Figure 2.

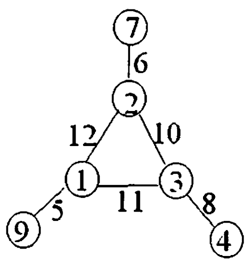



**Figure 2**

It has been shown that a wheel with n spokes has a total edge-magic labeling whenever $n \equiv 0$, 1, or 4 (mod 8); however, they are not edge-magic. Trees with more than 2 vertices, and all cycles, are not edge-magic. Kotzig and Rosa [10] showed that all cycles and caterpillars are total edge-magic. Thus the theory of edge-magic graphs and the theory of total edge-magic graphs are not related to each other.

The original concept of total edge-magic graph is due to Kotzig and Rosa [9]. They called it magic graph. They proved the following results:

(1) All cycles, complete bipartite graphs, and caterpillars are total edge-magic.
(2) A complete graph $K_n$ is total edge-magic if and only if $n \in \{1, 2, 3, 5, 6\}$.
(3) The disconnected graph $nK_2$ is total edge-magic if and only if n is odd.

They also showed caterpillars are super edge-magic. In [4], Enomoto et al gave a super edge-magic labeling for odd cycles. Several other classes of graphs have also been shown to be total edge-magic [2, 3, 4, 5, 7, 14].

We note here that our magic graphs are different from those investigated by Sedlacek et al [8, 17, 18, 19] and investigated in [8, 17]. In [6], Gallian presented an excellent survey article on magic graphs and graph labeling in general.

A subset S of integers is called ***consecutive*** if S consists of consecutive integers. Chen [2] showed that a graph G is super edge-magic if and only if there exists a vertex labeling f such that the two sets f(V(G)) and $\{f(u)+f(v) : (u,v) \in E(G)\}$ are both consecutive. We will apply this result of Chen to show that several classes of n-stars are super edge-magic.

For any $n \geq 2$, let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of increasing non-negative integers. We will use $St(a_1,a_2,a_3,\ldots,a_n)$ to denote a n-stars, which is a disjoint union of n stars $K(1,a_1)$, $K(1,a_2),\ldots, K(1,a_n)$. The graph $St(a_1,a_2,a_3,\ldots,a_n)$ is shown in Figure 3.

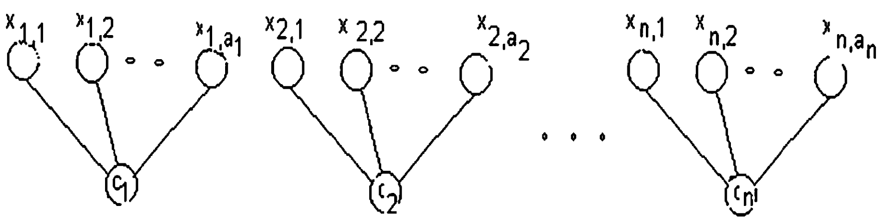



**Figure 3**

## 2. Super edge-magic 2-stars

Chen in [2] showed that if $f:V\to\{1,2,\ldots,p\}$ is a bijection and $f^+(\{u,v\})) = f(u)+f(v)$ for all (u,v) in E has the property that $f^+(E)$ is consecutive with $f^+(E)= \{c, c+1, \ldots,c+q-1\}$ then we can extend f to a total edge-magic labeling of G by define $f^*: V \cup E\to\{1,2,\ldots,p+q\}$ with $f^*(u)=f(u)$ and $f^*(\{u,v\})=p+q+c-f^+(\{u,v\})$. By applying the result of Chen, several classes of n-stars are shown to be super edge-magic.

**Theorem 1.** The 2-star St (n, n+1) is super edge-magic for all $n \geq 1$ .

***Proof.*** We will give two different super edge-magic labelings for St(n,n+1),

**Method 1.** We label the vertices by

$f(x_{1,j}) = 3 + 2j,\ 1\leq j \leq n,\qquad f(c_1) = 1,\quad f(x_{2,j}) = 2j,\ 1\leq j \leq n+1,$ $f(c_2) = 3.$

Then we see that the edges in K(1, n) has labels $\{6,8,\ldots,4+2n\}$ and the edges in K(1,n+1) has labels $\{5,7,\ldots,2n+5\}$. Thus St(n,n+1) is super edge-magic.

**Method 2.**

$g(x_{1,j}) = 2j-1,\ 1 \le j \le n,$
$g(c_1) = 2n+3,$
$g(x_{2,j}) = 2j,\ 1 \le j \le n+1,$
$g(c_2) = 2n+1.$

Then we see that the edges in K(1, n) has labels {2n+4, ...4n+2} and the edges in K(1,n+1) has labels {2n+3,2n+5, ...,4n+3}. Thus St(n,n+1) is super edge-magic

**Example 1.** Super edge-magic labelings for 2-stars St(1,2), St(2,3), and St(3,4) using the above two different methods.

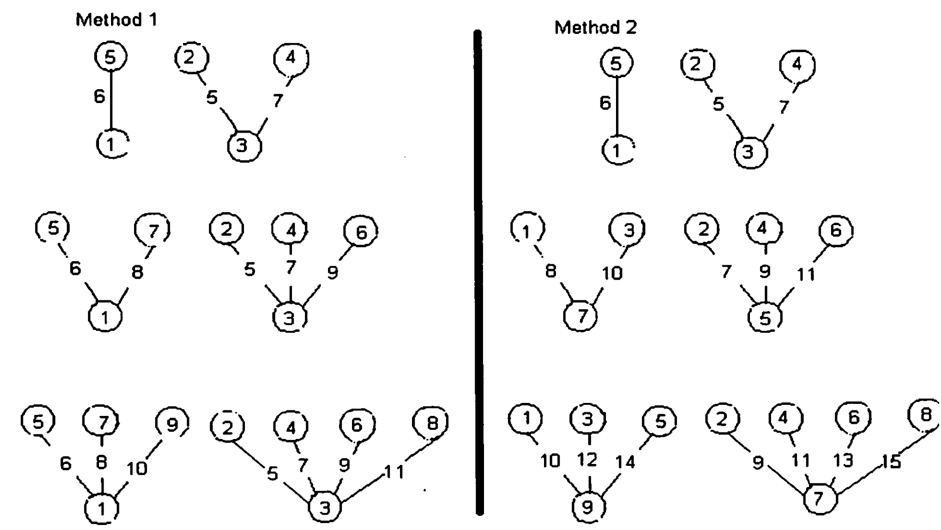



**Figure 4**

**Theorem 2.** The 2-star St(m,n) is super edge-magic for all $n \equiv 0 \pmod{m+1}$.
***Proof.*** Assume n = (m+1)k. The 2-star St(m,(m+1)k) has (m+1)(k+1)+1 vertices. We define a labeling

$$f: V(ST(m,(m+1)k) \to \{1, 2, \ldots, (m+1)(k+1) + 1\}$$

as follows:

$$\begin{aligned} f(c_1) &= (m+1)(k+1) + 1, \\ f(c_2) &= (m+1)(k+1) - k, \\ f(x_{1,j}) &= 1 + (j-1)(k+1),\ 1 \le j \le m, \\ f(x_{2,i}) &= 1+i,\ 1 \le i \le k, \\ f(x_{2,i}) &= i+2,\ k+1 \le i \le 2k. \end{aligned}$$

Hence, $f^{+}(E(St(1,2k)) = \{k+4, k+5, \ldots, 2k+4\}$ and f is a super edge-magic labeling.

**Corollary 1.** The 2-star St (1,n) is super edge-magic if n is even.

**Example 2.** A super edge-magic labeling of the 2-star St(1, n)

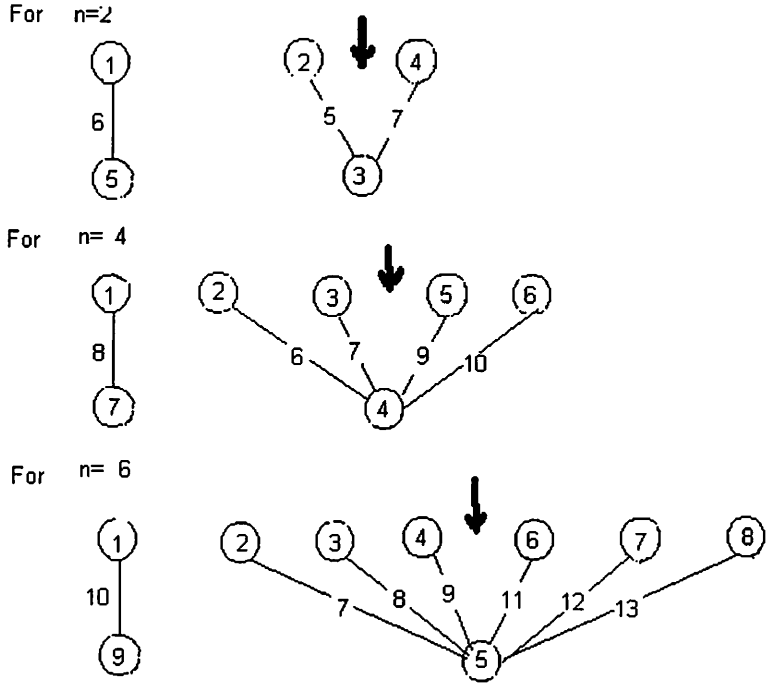



**Figure 5**

**Corollary 2.** The 2-star St (2,n) is super edge-magic if n is a multiple of 3.

**Example 3.** Super edge-magic labeling for 2-star St (2,n) n = 3 , 6.

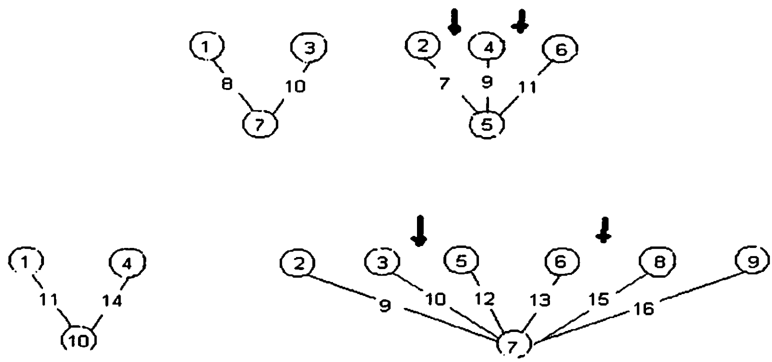



**Figure 6**

## 3. Super edge-magic 3-stars.

**Theorem 3.** The 3-star St (1,1,n) is super edge-magic, for all $n \geq 1$.

***Proof.*** A super edge-magic labeling of St(1,1,n) is given as follows:

Define f: V(St(1,1,n)) $\rightarrow$ {1, 2, ..., n+5} as follows:

$$f(c_1) = 1,$$
$$f(c_2) = 3,$$
$$f(c_3) = 2,$$
$$f(x_{1.1}) = 5,$$
$$f(x_{2.1}) = 4,$$
$$f(x_{3.i}) = 5 + i, \ 1 \leq i \leq n.$$

It can easily be verified that f induces a super edge-magic labeling.

**Example 4.** A super edge-magic labeling for 3-star St (1,1,n).

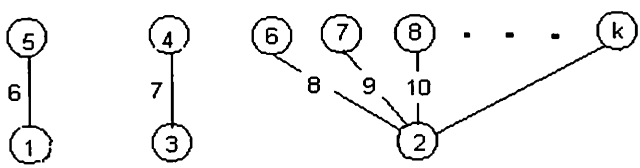



**Figure 7**

**Theorem 4.** The 3-star St (1,2,n) is super edge-magic for all $n \geq 2$.

***Proof.*** A super edge-magic labeling of St(1,2,n) is given in Figure 8.

For n=2,

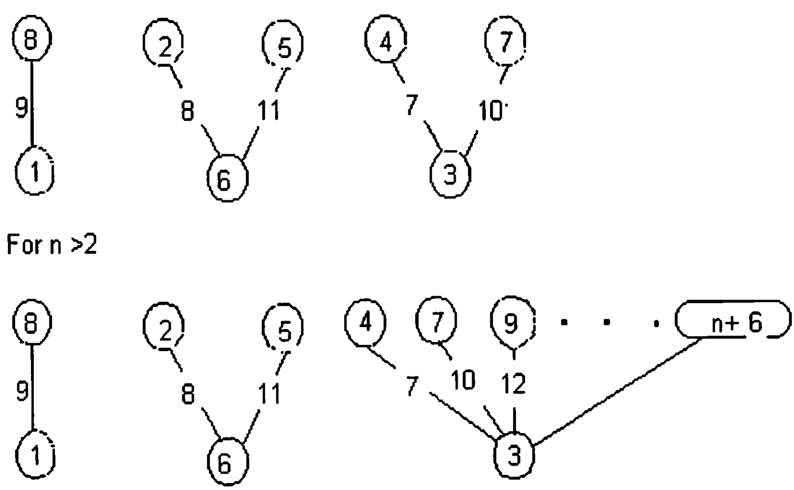



**Figure 8**

**Theorem 5.** The 3-star St (1,n,n) is super edge-magic for all n ≥ 1.
***Proof.*** A super edge-magic labeling of St(1,n,n) is given in Figure 9.

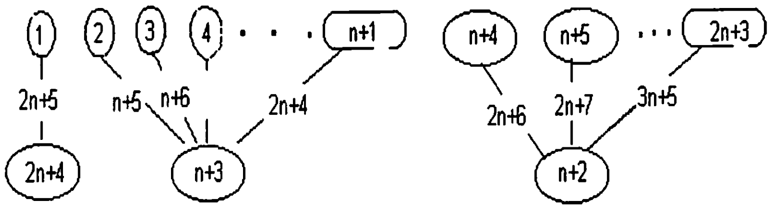
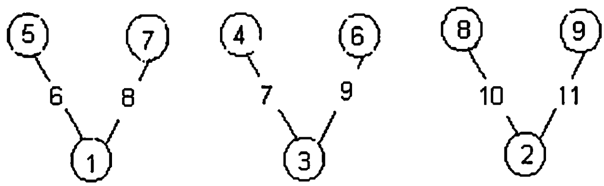



**Figure 9**

**Theorem** 6. The 3-star St (2,2,n) is super edge-magic for all n ≥ 2.
***Proof.*** A super edge-magic labeling of St(2,2,n) is given in Figure 10.
For n=2




For n= k

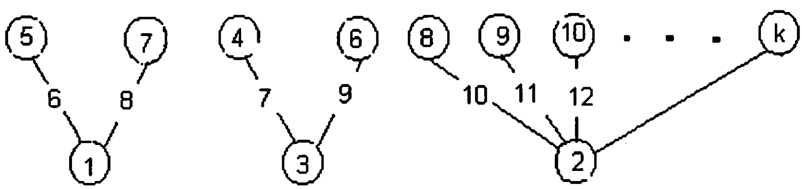



**Figure 10**

**Theorem 7.** The 3-star St(2,3,n) is super edge-magic for all n≥3
***Proof.*** A super edge-magic labeling of St(2,3,,n) is given in Figure 11.

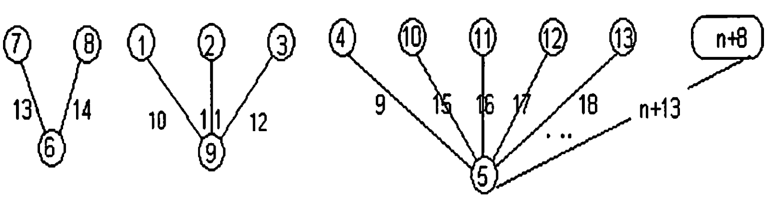



**Figure 11**

## 4. Super edge-magic 4-stars.

**Theorem 8.** The 4-star St(1,1,2,n) is super edge-magic for all $n \geq 2$.

***Proof.*** A super edge-magic labeling for St(1,1,2,n) for $n \geq 2$ is shown in Figure 12.

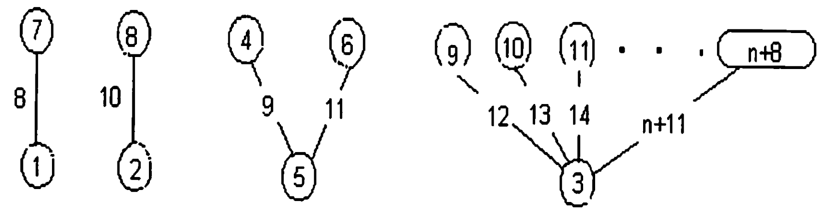



**Figure 12**

**Theorem 9.** The 4-star St(1,1,3,n) is super edge-magic for all $n \geq 3$.

***Proof.*** A super edge-magic labeling for St(1,1,3,n) for $n \geq 3$ is shown in Figure 13.

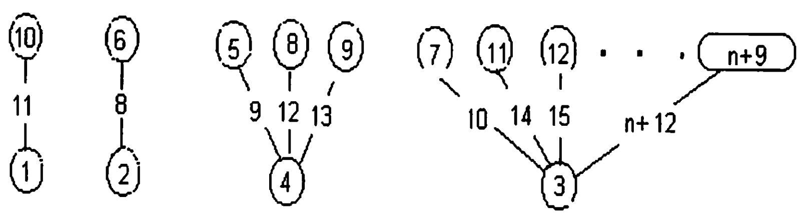



**Figure 13**

**Theorem 10.** The 4-star St(1,2,2,n) is super edge-magic for all $n \geq 2$.

***Proof.*** A super edge-magic labeling for St(1,2,2,n) is shown in Figure 14.

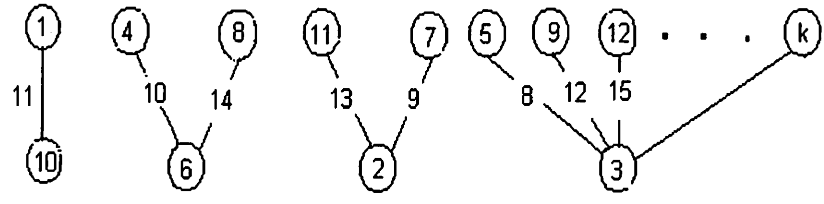



**Figure 14**

**Theorem 11.** The 4-star St(2,2,2,n) is super edge-magic for all $n \geq 2$.
***Proof.*** A super edge-magic labeling for St(2,2,2,2) and respectively St(2,2,2,n) for $n \geq 2$ is shown in Figure 15.

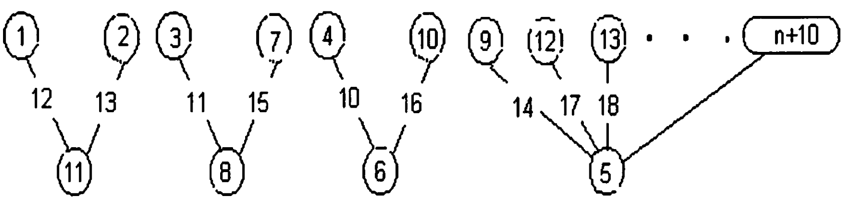



**Figure 15**

We propose the following
<u>**Conjecture.**</u> Given any odd integer $n \geq 2$. Let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of increasing non-negative integers, the n-star $St(a_1, a_2, a_3, \ldots, a_n)$ is super edge-magic.

# References

[1] G. Chartrand and L. Lesniak, Graphs and digraphs, 2nd edition, Wadsworth & Brooks/Cole, Monterey, 1986.

[2] Z. Chen, On super edge-magic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **38** (2001), 55-64.

[3] D. Craft and E.H. Tesar, On a question by Erdos about edge-magic graphs, *Discrete Math.* **207** (1999), 271-276.

[4] H. Enomoto, A. S. Llato, T. Nakamigawa, A. Ringel, Super edge-magic graphs, *SUT J. Math.* Vol. 34, No. 2 (1998), 105-109.

[5] H. Enomoto, K. Masuda and T. Nakamigawa, Induced graph theorem on magic valuations, *Ars Combinatorica* **56** (2000) 25-32.

[6] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* 5, 2001 ed., 95 pages.

[7] R.D. Godbold and P. Slater, All cycles are edge-magic, *Bull. Inst. Combin. Appl.* **22** (1998) 93-97.

[8] N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *Journal of Recreational Mathematics* **21** (1989), 116-124.

[9] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canada Math. Bull.* **13** (1970), 451-461.

[10] A. Kotzig and A. Rosa, Magic valuations of complete graphs, Publications du Centre de Recherches Mathematiques Universite de Montreal 175 (1972)

[11] Sin-Min Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congressus Numerantium* **105** (1994), 214-222.

[12] Sin-Min Lee, E. Seah and S.K. Tan , On edge-magic graphs, *Congressus Numerantium* **86** (1992), 179-191.

[13] Sin-Min Lee, E. Seah and Siu-Ming Tong, On the edge-graceful and edge-magic total graphs conjecture, *Congressus Numerantium*,, **141** (1999), 37-48.

[14] G. Ringel, A.S. Llado, Another tree conjecture, *Bull. ICA* **18** (1996), 83-85.

[15] K. Schaffer and Sin-Min Lee, Edge-graceful and edge-magic labelings of cartesian product of graphs, *Congressus Numerantium* **141** (1999), 119-134.

[16] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium* **132** (1998), 9-18.

[17] B.M. Stewart, Magic graphs, *Canada J. Math.* **18** (1966), 1031-1059.

[18] J. Sedlacek, Problem 27, "Theory of Graphs and its applications," Proc. Symposium, Smolenica, 1963, Prague, 163-164.

[19] J. Sedlacek, On magic graphs, *Math. Slov.* **26** (1976), 329-335.

[20] W.D. Wallis, E.T. Baskoro, M. Miller and Slamin, Edge-magic total labelings, *Australas. J. Combin.* **22** (2000) 177-190.