Average Cayley Genus for Cayley Maps of Dihedral Groups Generated by Their Reflections

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Abstract

Let Γ be a finite group and let Δ be a generating set for Γ. A Cayley map associated with Γ and Δ is an oriented 2-cell embedding of the Cayley graph $G_\Delta(\Gamma)$ such that the rotation of arcs emanating from each vertex is determined by a unique cyclic permutation of generators and their inverses. A formula for the average Cayley genus is known for the dihedral group with generating set consisting of all the reflections. However, the known formula involves sums of certain coefficients of a generating function and its format does not specifically indicate the Cayley genus distribution. We determine a simplified formula for this average Cayley genus as well as provide improved understanding of the Cayley genus distribution.

1 Introduction and Preliminaries

A *surface* is a closed orientable 2-manifold, which can be thought of as a sphere with handles. The number of handles is the genus of the surface. For an integer $k \geq 0$, let S_k denote the surface of genus *k*. For a connected graph *G* the *genus* $\gamma(G)$ of G is the minimum non-negative integer k such that G is 2-cell embedded on S_k and the *maximum genus* $\gamma_M(G)$ is the maximum such integer. A *rotation embedding scheme* $\wp$ is a collection of cyclic permutations $\rho_v : N(v) \rightarrow N(v)$, one for each $v \in V(G)$, where $V(G)$ is the set of vertices

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of G and $N(v)$ denotes the neighborhood of v. It is well known (see Edmonds [1]) that the labeled 2-cell embeddings of a connected graph G are in one-to-one correspondence with the rotation schemes of G. Thus, for a connected graph G with $V(G)=\{1, 2, \ldots, p\}$, there are $\prod_{i=1}^{p}(\deg(i)-1)!$ many labeled 2-cell embeddings of G.

If $\wp$ is a rotation scheme for G, then the ordered pair $(G,\wp)$ is called a *map* and we say that the *genus* $g(G,\wp)$ *of the map* $(G,\wp)$ is k if $\wp$ determines a 2-cell embedding of G on S_k. Thus, $\gamma(G)=\min_{\wp} g(G,\wp)$ and $\gamma_M(G)=\max_{\wp} g(G,\wp)$. One of the major areas of research has been the study of all 2-cell embeddings of a labeled connected graph G and, in particular, the enumeration of the embeddings of G on a given surface and the determination of the average genus of G.

This paper focuses on a special class of embeddings of Cayley graphs, namely, those called Cayley maps. Let Γ be a finite group and Δ be a generating set for Γ such that the identity $e \notin \Delta$. Also let $\Delta^{-1}=\{\delta^{-1} | \delta \in \Delta\}$ and $\Delta^{*}=\Delta \cup \Delta^{-1}$. Furthermore, let Δ be chosen so that if $\delta \in \Delta \cap \Delta^{-1}$, then $\delta^2 = e$. That is, if δ is chosen as a generator, then δ^{-1} is not chosen, unless $\delta^2=e$ (δ is its own inverse). The *Cayley graph* $G_\Delta(\Gamma)$ is that graph whose vertex set is Γ and edge set is $\{\{x, x\delta\} | x \in \Gamma, \delta \in \Delta^{*}\}$. For a cyclic permutation $\rho: \Delta^{*} \to \Delta^{*}$, the *Cayley map* (Γ, Δ, ρ) is the map $(G_\Delta(\Gamma), \wp)$ where $\wp=\{\rho_x | x \in \Gamma\}$ is the rotation scheme for $G_\Delta(\Gamma)$ such that $\rho_x(y)=x\rho(x^{-1}y)$ for each $x \in \Gamma$ and each $y \in N(x)$. In other words, a Cayley map is a 2-cell embedding of a Cayley graph in which each vertex rotation ρ_x is determined by the same cyclic ordering of the elements of Δ^{*}.

If a given Cayley map (Γ, Δ, ρ) determines a 2-cell embedding of $G_\Delta(\Gamma)$ in S_k, then k is called the *genus* $g(\Gamma, \Delta, \rho)$ *of the Cayley map* (Γ, Δ, ρ). The *Cayley genus* is $\gamma(\Gamma, \Delta)=\min_{\rho} g(\Gamma, \Delta, \rho)$, the *maximum Cayley genus* is $\gamma_M(\Gamma, \Delta)=\max_{\rho} g(\Gamma, \Delta, \rho)$, and the *average Cayley genus* is the average of the genera of all Cayley maps for some group Γ with fixed generating set Δ and is denoted by $\overline{\gamma}(\Gamma, \Delta)$.

Specifically, in this paper, we improve the existing formula for the average Cayley genus of the dihedral group with the generating set consisting of all the reflections. We use D_m to denote the dihedral group of order $2m$, where $m \geq 3$. The presentation we use is $D_m = \langle x, y \mid x^m = y^2 = (xy)^2 = e \rangle$, where $e, x, x^2, \ldots, x^{m-1}$ are the rotations and $y, xy, x^2y, \ldots, x^{m-1}y$ are the reflections. Let $\Delta = \{y, xy, x^2y, \ldots, x^{m-1}y\}$. Then the Cayley graph $G_\Delta(D_m)$ is $K_{m,m}$. Actually, when m is odd, the genus of every Cayley map (D_m, Δ, ρ) is $\lfloor m/2 \rfloor (m-2)$, as was shown in [5]. Hence we are interested in the formula for D_{2n}, where $n \geq 2$, with generating set Δ consisting of all the reflections.

2 Existing Formula

Some notation is necessary. For positive integers $k \leq j$, the generating function $u_j(t, k)$ for the number of partitions having k unequal parts with no part greater than j is given by

$$u_j(t, k) = \begin{cases} t^{\binom{k+1}{2}} \dfrac{(1-t^j)(1-t^{j-1})\ldots(1-t^{j-k+1})}{(1-t)(1-t^2)\ldots(1-t^k)} & \text{if } k < j \\ t^{\binom{j+1}{2}} & \text{if } k = j. \end{cases}$$

(See Riordan [4], for example.) The coefficient of t^i in $u_j(t, k)$ will be denoted by $[t^i]u_j(t, k)$, that is, $[t^i]u_j(t, k)$ is the number of partitions of the integer i into k unequal parts having no part greater than j. The following formula for the average Cayley genus is given in [5].

Theorem A *Let $n \geq 2$ be an integer and let $a = 0$ if n is even and $a = n$ if n is odd. Then the average Cayley genus $\bar{\gamma}(D_{2n}, \Delta)$, where Δ is the generating set for D_{2n} consisting of all the reflections, is given by*

$$\bar{\gamma}(D_{2n}, \Delta) = \frac{n!(n-1)!}{(2n-1)!} \left\{ \sum_{i=0}^{n-1} \left[(2n^2 - 2n + 1 - \gcd(2n, a+2i)) \right. \right.$$
$$\left. \left. \times \sum_{j=0}^{n-1} \left[t^{\binom{n}{2} + i + jn} \right] u_{2n-1}(t, n-1) \right] \right\}.$$

[Note: In Theorem A, we take $\gcd(2n, 0)$ to mean $\gcd(2n, 2n)$ so that $\gcd(2n, 0) = \gcd(2n, 2n) = 2n$.]

Table 1 Example of $n = 12$ Using Existing Formula

i	gcd	genus	Sum of Generating Function Coefficients
0	24	241	1+76+1109+6300+18320+30554+30554+18320+6300+1109+76+1 = 112720
1	2	263	1+98+1317+7040+19496+31132+29849+17125+5597+921+56 = 112632
2	4	261	2+129+1564+7850+20696+31641+29087+15968+4962+766+42 = 112707
3	6	259	3+165+1838+8698+21863+32017+28218+14812+4368+628+30 = 112640
4	8	257	5+212+2156+9613+23034+32312+27302+13703+3836+515+22 = 112710
5	2	263	7+266+2505+10560+24152+32467+26295+12608+3342+415+15 = 112632
6	12	253	11+336+2907+11573+25261+32540+25261+11573+2907+336+11 = 112716
7	2	263	15+415+3342+12608+26295+32467+24152+10560+2505+266+7 = 112632
8	8	257	22+515+3836+13703+27302+32312+23034+9613+2156+212+5 = 112710
9	6	259	30+628+4368+14812+28218+32017+21863+8698+1838+165+3 = 112640
10	4	261	42+766+4962+15968+29087+31641+20696+7850+1564+129+2 = 112707
11	2	263	56+921+5597+17125+29849+31132+19496+7040+1317+98+1 = 112632

To illustrate the formula, Table 1 summarizes the necessary calculations for $n = 12$. The details for the calculation of the genus for each i is provided in [5] and uses Euler's formula as well as standard voltage graph theory. In particular, the genus for each i $(0 \le i \le 11)$ is $2(12)^2 - 2(12) + 1 - \gcd(2(12),\ 2i)$, taken from Theorem A. Thus, by using the values in the third and fourth columns of Table 1, we obtain

$$\bar{\gamma}(D_{24},\ \Delta) = (12!11!)/23!\big[241(112720) + 263(112632) + 261(112707) +$$
$$259(112640) + 257(112710) + 263(112632) + 253(112716) + 263(112632) +$$
$$257(112710) + 259(112640) + 261(112707) + 263(112632)\big] = \frac{174631897}{869193}.$$

We make two observations. First, for different values of i, we see the same greatest common divisor, the same genus, and the same coefficient sum. What this suggests is that it may be possible to reduce the number of cases, and by doing so, we would consequently arrive at a more convenient representation of the Cayley genus distribution. Second, it is tedious work to determine the coefficients of the generating function and then find certain sums of these coefficients. We will see that, in fact, it is possible to find the necessary sums of coefficients directly without using the generating function at all.

3 Preparation for New Formula

We begin with a study of the generating function $u_{2n-1}(t,\ n-1)$. Define $f(t) = \frac{1}{t^{\binom{n}{2}}} u_{2n-1}(t,\ n-1) = \frac{(1-t^{2n-1})(1-t^{2n-2})\dots(1-t^{n+1})}{(1-t^{n-1})(1-t^{n-2})\dots(1-t)}$ so that we may write $f(t)$ as $g(t) = \sum_{k=0}^{n(n-1)} a_k t^k$, where $a_k = \left[t^{\binom{n}{2}+k}\right] u_{2n-1}(t,\ n-1)$. Let $c_0 = a_0 + a_n + a_{2n} + \dots + a_{n(n-2)} + a_{n(n-1)}$ and let $c_i = a_i + a_{n+i} + a_{2n+i} + \dots + a_{n(n-2)+i}$ for each i $(1 \le i \le n-1)$. In this way, c_i $(0 \le i \le n-1)$ is the sum of the coefficients in the ith case of the existing formula. We proceed to set up a system of $n-1$ equations in the variables $c_1, c_2, \dots, c_{n-1}$. Since $\sum_{i=0}^{n-1} c_i = \binom{2n-1}{n-1}$, we will then be able to solve for c_0.

The $n-1$ equations are obtained by considering each expression for $f(t)$ near the non-trivial nth roots of unity. The nth roots of unity are the n solutions to the equation $z^n = 1$, so they are of the form $z = e^{2\pi i(\ell/n)}$, where $\ell = 0, 1, 2, \dots, n-1$. For simplicity, we define $e(\ell/n) = e^{2\pi i(\ell/n)}$. For a positive integer n, the set of all of the nth roots of unity forms a multiplicative group that is cyclic. An nth root of unity that generates this multiplicative group is called a *primitive* nth root of unity. Before proceeding any further, a few remarks are in order.

Fact 1 If $c \in \mathbb{Z}$, then $e(c) = 1$.

Fact 2 For $e(\alpha)$ and $e(\beta)$ being solutions to $z^n = 1$, we have $e(\alpha+\beta) = e(\alpha)\cdot e(\beta)$.

Fact 3 Let ξ be a primitive nth root of unity, then $\xi^{n-1} + \xi^{n-2} + \ldots + \xi = -1$ and $\xi^{n-1} + \xi^{n-2} + \ldots + \xi + 1 = 0$.

For the functions $f(t) = \dfrac{(1-t^{2n-1})(1-t^{2n-2})\ldots(1-t^{n+1})}{(1-t^{n-1})(1-t^{n-2})\ldots(1-t)}$ and $g(t) = \sum_{k=0}^{n(n-1)} a_k t^k$, we see that $g(t)$ is defined for all complex numbers and $f(t)$ is defined on $\mathbb{C} - \mathscr{K}$, where $\mathscr{K} = \bigcup_{N=1}^{n-1} \left\{ e\left(\frac{k}{N}\right) : 0 \le k \le N-1 \right\}$. Certainly $f(t) = g(t)$ for all $t \in \mathbb{C} - \mathscr{K}$ and $g(t)$ is continuous on $\mathbb{C}$. Thus, $g(t_0) = \lim_{t \to t_0} g(t) = \lim_{t \to t_0} f(t)$ and we use $t_0 = e(\ell/n)$ for $\ell = 1, 2, \ldots, n-1$ to get $n-1$ equations and we write this system of equations as the matrix equation $(*)$.

$$\begin{bmatrix} e(1/n) & e(2/n) & \cdots & e((n-1)/n) \\ e(2/n) & e(4/n) & \cdots & e(2(n-1)/n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ e((n-1)/n) & e(2(n-1)/n) & \cdots & e((n-1)^2/n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \lim_{t \to e(1/n)} f(t) - c_0 \\ \lim_{t \to e(2/n)} f(t) - c_0 \\ \cdot \\ \cdot \\ \cdot \\ \lim_{t \to e((n-1)/n)} f(t) - c_0 \end{bmatrix}.$$

Observe that the coefficient matrix is a Vandermonde matrix whose determinant is non-zero, which implies that this system not only has a solution, but that the solution is unique. We proceed to determine the solution. First, we calculate $\lim_{t \to e(\ell/n)} f(t)$.

Theorem 1 *Let ℓ be an integer such that* $1 \le \ell \le n-1$ *and let* $d = \gcd(n, \ell)$. *Then*

$$\lim_{t \to e(\ell/n)} f(t) = \binom{2d-1}{d-1}.$$

Proof We consider two cases.

Case 1 Suppose that $d = 1$. In this case, $e(\ell/n)$ is a primitive nth root of unity so that if $\xi = e(\ell/n)$, then $\xi^n = 1$. Since, $f(t)$ is defined at ξ and is continuous there, we have $\lim_{t \to \xi} f(t) = f(\xi) = \dfrac{(1-\xi^{2n-1})(1-\xi^{2n-2})\dots(1-\xi^{n+1})}{(1-\xi^{n-1})(1-\xi^{n-2})\dots(1-\xi)} =$ $\dfrac{(1-\xi^{n-1}\xi^n)(1-\xi^{n-2}\xi^n)\dots(1-\xi\xi^n)}{(1-\xi^{n-1})(1-\xi^{n-2})\dots(1-\xi)} = 1$. Also when $d = 1$, we have $\binom{2d-1}{d-1} = 1$.

Case 2 Suppose that d satisfies $1 < d < n$. Since $d = \gcd(n, \ell)$, we may write $n = dN$ and $\ell = dL$ for some integers N and L with $\gcd(N, L) = 1$. So $\lim_{t \to e(\ell/n)} f(t) = \lim_{t \to e(L/N)} f(t)$. Since $\gcd(N, L) = 1$, it follows that $e(L/N)$ is a primitive Nth root of unity. Let $\xi = e(L/N)$ and so $\xi^N = 1$. Evaluating $\lim_{t \to \xi} f(t)$, we obtain $\lim_{t \to \xi} f(t) = \lim_{t \to \xi} \dfrac{(1-t^{2n-1})(1-t^{2n-2})\dots(1-t^{n+i})\dots(1-t^{n+1})}{(1-t^{n-1})(1-t^{n-2})\dots(1-t^i)\dots(1-t)}$.

Consider the general term $\dfrac{(1-t^{n+i})}{(1-t^i)}$, for some i with $1 \le i \le n-1$. Observe that if $N \nmid i$, then since $n = dN$, $\xi^N = 1$, and $\xi^i \ne 1$, we obtain $\dfrac{(1-\xi^{n+i})}{(1-\xi^i)} =$ $\dfrac{(1-\xi^i\xi^{dN})}{(1-\xi^i)} = 1$. Using this, $\lim_{t \to \xi} f(t) = \lim_{t \to \xi} \left(\prod_{\substack{1 \le i \le n-1 \\ N \nmid i}} \dfrac{1-t^{n+i}}{1-t^i} \right) \left(\prod_{\substack{1 \le i \le n-1 \\ N \mid i}} \dfrac{1-t^{n+i}}{1-t^i} \right) =$

$$\lim_{t \to \xi} \left(\prod_{\substack{1 \le i \le n-1 \\ N \mid i}} \frac{1-t^{n+i}}{1-t^i} \right) = \lim_{t \to \xi} \left(\prod_{j=1}^{d-1} \frac{1-t^{(d+j)N}}{1-t^{jN}} \right) =$$

$$\lim_{t \to \xi} \left(\prod_{j=1}^{d-1} \frac{(1-t^N)(1+t^N+t^{2N}+\dots+t^{(d+j-1)N})}{(1-t^N)(1+t^N+t^{2N}+\dots+t^{(j-1)N})} \right) =$$

$$\lim_{t\to\xi}\left(\frac{1+t^N+t^{2N}+\ldots+t^{(d+j-1)N}}{1+t^N+t^{2N}+\ldots+t^{(j-1)N}}\right)=\prod_{j=1}^{d-1}\frac{d+j}{j} = \binom{2d-1}{d-1}. \quad \square$$

An immediate corollary follows.

Corollary 2 *For integers* ℓ_1, ℓ_2, *and* n *with* $1\le \ell_1,\ \ell_2\le n-1$, $\lim_{t\to e(\ell_1/n)} f(t) = \lim_{t\to e(\ell_2/n)} f(t)$ *if and only if* $\gcd(n,\ \ell_1)=\gcd(n,\ \ell_2)$.

In order to solve the matrix equation $(*)$, it is useful to have the following notation for certain column vectors. First, if $\vec{x}$ is a row vector, then we write $\vec{x}^T$ for the transpose of $\vec{x}$. Let n be a positive integer and let d be a divisor of n with $n=dN$. We define several $(n-1)\times 1$ column vectors. Let $\vec{c}=(c_1,\ c_2,\ \ldots,\ c_{n-1})^T$, $\vec{u}=(1,\ 1,\ \ldots,\ 1)^T$, and for each k $(1\le k\le n-1)$, $\vec{v}_d=(v_1,\ v_2,\ \ldots,\ v_{n-1})^T$, where $v_k=\begin{cases}1 & \text{if} \quad d=\gcd(k,\ n)\\ 0 & \text{if} \quad d\ne\gcd(k,\ n)\end{cases}$ for each k and $\vec{u}_d=(u_1,\ u_2,\ \ldots,\ u_{n-1})^T$, where $u_k=\begin{cases}0 & \text{if} \quad N\mid k\\ 1 & \text{if} \quad N\nmid k\end{cases}$ for each k. Then observe that $\vec{u}-\vec{u}_d=(1-u_1,\ 1-u_2,\ \ldots,\ 1-u_{n-1})^T$, where $1-u_k=\begin{cases}1 & \text{if} \quad N\mid k\\ 0 & \text{if} \quad N\nmid k\end{cases}$. Also let A be the coefficient matrix of the equation $(*)$ and $\vec{b}$ be the right side of the matrix equation $(*)$. Define $D=\{d : d\mid n,\ 1<d<n\}$ so that from Theorem 1 and Corollary 2, we may write

$$\vec{b}=(1-c_0)\vec{u}+\sum_{d\in D}\left(\binom{2d-1}{d-1}-1\right)\vec{v}_d .$$

Several lemmas are useful in solving the matrix equation $(*)$, which is $A\vec{c}=\vec{b}$ with the new notation.

Lemma 3 *Using A and $\vec{u}$ as defined previously,* $A\vec{u}=(-1,\ -1,\ \ldots,\ -1)^T$.

Proof Let $A\vec{u}=(j_1, j_2,\ \ldots,\ j_{n-1})^T$ so that j_k is the sum of the kth row of the matrix A, that is, $j_k=\sum_{\ell=1}^{n-1} e(k\ell/n)$ for each k where $1\le k\le n-1$. If

$\gcd(k, n)=1$, then by Fact 3 we have $\sum_{\ell=1}^{n-1} e(k\ell/n)=-1$. So suppose that the $\gcd(k, n)=d$, where $d \neq 1$. Then we may write $k=Kd$ and $n=Nd$ for integers K and N with $\gcd(K, N)=1$. In this case, j_k consists of the sum of all N Nth roots of unity $d-1$ times plus the $N-1$ nontrivial Nth roots of unity once, that is, $j_k=\sum_{\ell=1}^{n-1} e(k\ell/n)=(d-1)\left[\sum_{\ell=1}^{N} e(K\ell/N)\right]+\sum_{\ell=1}^{N-1} e(K\ell/N)$. Using Fact 3, we obtain $j_k=(d-1)\cdot(0)+(-1)=-1$. □

So Lemma 3 gives us that $A\left[(c_0-1)\vec{u}\right]=(1-c_0)\vec{u}$, which is the first term in our solution vector $\vec{b}$. Later we define values U_N for each $N \in D$ such that $A\left(\sum_{N\in D} U_N\, \vec{u}_N\right)=\sum_{d\in D}\left(\binom{2d-1}{d-1}-1\right)\vec{v}_d$, which is the second part of our solution vector. However, it is useful to first provide a few more helpful observations.

Lemma 4 *Let $N \in D$ and $n=dN$. Then $A\vec{u}_N=-N(\vec{u}-\vec{u}_d)$.*

Proof Notice that $\vec{u}_N=(u_1, u_2, \ldots, u_{n-1})^T$, where $u_k=\begin{cases}0 & \text{if} \quad d \mid k \\ 1 & \text{if} \quad d \nmid k\end{cases}$. Then $A\vec{u}_N=(i_1, i_2, \ldots, i_{n-1})$, where $i_k=\sum_{\substack{1\le \ell\le n-1 \\ d\nmid \ell}} e(k\ell/n)$. Since the sum of the entries in an entire row of A is -1, we have that $\sum_{\substack{1\le \ell\le n-1 \\ d\nmid \ell}} e(k\ell/n)=$ $-1-\sum_{\substack{1\le \ell\le n-1 \\ d\mid \ell}} e(k\ell/n)=-1-\sum_{\substack{1\le \ell\le n-1 \\ \ell=dL}} e(k\ell/n)=-1-\sum_{L=1}^{N-1} e(kL/N)$. If $N \mid k$, then $k=NK$ for some integer K so that by using Fact 1, $A\vec{u}_N=$ $-1-\sum_{L=1}^{N-1} e(NKL/N)=-1-\sum_{L=1}^{N-1} e(KL)=-1-(N-1)=-N$. If $N \nmid k$, then $k=$ $NK+r$ for some integers K and r with $1 \le r \le N-1$ so that using Facts 1 and 3, we have that $A\vec{u}_N=-1-\sum_{L=1}^{N-1} e((NK+r)L/N)=-1-\sum_{L=1}^{N-1} e(NKL/N)\, e(rL/N)=$ $-1-\sum_{L=1}^{N-1} e(rL/N)=-1-(-1)=0$. So $A\vec{u}_N=\begin{cases}-N \text{ if} N \mid k \\ 0 \quad \text{if} N \nmid k\end{cases}$. Thus, since

$\vec{u}-\vec{u}_d=(1-u_1,\ 1-u_2,\ \dots,\ 1-u_{n-1})^T$, where $1-u_k=\begin{cases}1 & \text{if} \quad N\mid k\\ 0 & \text{if} \quad N\nmid k\end{cases}$, it follows that $A\vec{u}_N=-N(\vec{u}-\vec{u}_d)$. □

Lemma 5 *Let* $D=\{d : d\mid n,\ 1<d<n\}$ *and let* $N\in D$. *Then*

$$\vec{u}-\vec{u}_{n/N}=\sum_{\substack{d\in D\\ N\mid d}}\vec{v}_d .$$

Proof Note that $\vec{u}-\vec{u}_{n/N}=(i_1,\ i_2,\ \dots,\ i_{n-1})^T$, where $i_k=\begin{cases}1 & \text{if} \quad N\mid k\\ 0 & \text{if} \quad N\nmid k\end{cases}$. Next, consider the sum $\sum_{\substack{d\in D\\ N\mid d}}\vec{v}_d$, whose index set is the set of positive proper divisors of n that are also multiples of N. Thus, $\sum_{\substack{d\in D\\ N\mid d}}\vec{v}_d=(v_1,v_2,\dots,v_{n-1})^T$, where for each k $(1\le k\le n-1)$, we have $v_k=\begin{cases}1 & \text{if } \gcd(k,\ n)\in\{d\in D : N\mid d\}\\ 0 & \text{if } \gcd(k,\ n)\notin\{d\in D : N\mid d\}\end{cases}$. We verify that $v_k=i_k$ for each $k=1,\ 2,\ \dots,\ n-1$. If $\gcd(k,\ n)\in\{d\in D : N\mid d\}$, then $N\mid k$. On the other hand, if $\gcd(k,\ n)\notin\{d\in D : N\mid d\}$, then $N\nmid\gcd(k,\ n)$ and so $N\nmid k$. □

The previous two lemmas provide a relationship between the vectors $\vec{u}_N$ $(N\in D)$ and a sum of vectors $\vec{v}_d$ $(d\in D)$. We are ready to define the values U_N, $N\in D$, such that $A\left(\sum_{N\in D}U_N\ \vec{u}_N\right)=\sum_{d\in D}\left(\binom{2d-1}{d-1}-1\right)\vec{v}_d$. For each $N\in D$, define $U_N=\frac{1}{N}\left[\binom{2N-1}{N-1}-1-\sum_{\substack{1<k<N\\ k\mid N}}k\ U_k\right]$. A final lemma is useful.

Lemma 6 *For each* $d\in D$, *the sum* $\sum_{\substack{N\in D\\ N\mid d}}NU_N=\binom{2d-1}{d-1}-1$.

Proof $\sum_{\substack{N\in D\\ N\mid d}}NU_N=\sum_{\substack{1<N<d\\ N\mid d}}NU_N+dU_d=\sum_{\substack{1<N<d\\ N\mid d}}NU_N+\binom{2d-1}{d-1}-1-\sum_{\substack{1<N<d\\ N\mid d}}NU_N$. □

With the help of the previous four lemmas, we are now prepared to solve $A\vec{c} = \vec{b}$ for $\vec{c}$.

Theorem 7 *The solution to the matrix equation $A\vec{c} = \vec{b}$ shown as $(*)$ is*

$$\vec{c} = (c_0 - 1)\vec{u} - \sum_{N \in D} U_N \; \vec{u}_N .$$

Recall the Euler ϕ-function $\phi(m)$, which denotes the number of positive integers less than *m* that are relatively prime to *m* for $m > 1$ and $\phi(1)$ is defined to be 1. Then, for each proper divisor *d* of *n*, the number of elements in $\{c_i : 1 \le i \le n-1,\ \gcd(i, n) = d\}$ is $\phi(n/d)$. A basic result of number theory (see [3], for example) that will be useful in the following proof is that $\sum_{d|n} \phi(n/d) = n$. For our purposes, we will use that $\sum_{\substack{1 \le d < n \\ d|n}} \phi(n/d) = n - 1$. Also, define $D^* = D \cup \{1\}$. The next result allows us to find the coefficient sums without having to determine all the coefficients of the generating function.

Corollary 8 *Let $n \ge 2$ be a positive integer. Let $u_{2n-1}(t,\ n-1) = \sum_{k=0}^{n(n-1)} a_k t^{\binom{n}{2}+k}$ be the generating function for partitions of integers into $n-1$ unequal parts and no part greater than $2n-1$. Let $c_0 = \sum_{j=0}^{n-1} a_{nj}$ and for each $i = 1, 2, \ldots, n-1$, let $c_i = \sum_{j=0}^{n-2} a_{i+nj}$. Then $c_i = c_j$ $(i, j \ne 0)$ if and only if $\gcd(i, n) = \gcd(j, n)$.*

Furthermore, $c_0 = \frac{1}{n}\left[\binom{2n-1}{n-1} + n - 1 + \sum_{k \in D^*} \phi(n/k) \sum_{\substack{N \in D \\ n | kN}} U_N\right]$ *and for each $i \in D^*$, the value* $c_i = c_0 - \left[1 + \sum_{\substack{N \in D \\ n | iN}} U_N\right]$.

Proof From Corollary 2, it follows that $c_i = c_j$ if and only if $\gcd(i, n) = \gcd(j, n)$. For each $i \in D^*$, $c_i = c_0 - \left[1 + \sum_{\substack{N \in D \\ n | iN}} U_N\right]$ follows from Theorem 7.

Finally, to solve for c_0, we know $c_0 + \sum_{i=1}^{n-1} c_i = \binom{2n-1}{n-1}$ so that by the comments preceding this corollary, we have that $\binom{2n-1}{n-1} - c_0 = \sum_{i=1}^{n-1} c_i = \sum_{k \in D^*} \phi(n/k) c_k =$

$$\sum_{k \in D^*} \left[\phi(n/k) \left[(c_0 - 1) - \sum_{\substack{N \in D \\ n \nmid kN}} U_N \right] \right] = \sum_{k \in D^*} \left[\phi(n/k)(c_0 - 1) \right] - \sum_{k \in D^*} \left[\phi(n/k) \sum_{\substack{N \in D \\ n \nmid kN}} U_N \right] =$$

$$(c_0 - 1)(n-1) - \sum_{k \in D^*} \left[\phi(n/k) \sum_{\substack{N \in D \\ n \nmid kN}} U_N \right]. \quad \square$$

So now, we have reduced the number of cases from n (in the original formula) to the number of divisors of n. To show what an improvement this is, from Hardy [2], we have that the number of divisors of n is $d(n) = O(n^\delta)$ for all positive δ. Also, from Corollary 8, we are now able to determine the coefficient sums of the original formula without having to use the generating function. Thus, we arrive at a more compact formula for finding the average Cayley genus for dihedral groups with generating set consisting of all the reflections.

4 New Formula and Special Cases

Since the genus corresponding to c_0 is $\begin{cases} 2n^2 - 4n + 1 & \text{if } n \text{ is even} \\ 2n^2 - 3n + 1 & \text{if } n \text{ is odd} \end{cases}$, while for $d \in D$, the genus corresponding to c_d is $\begin{cases} 2n^2 - 2n - 2d + 1 & \text{if } n \text{ is even} \\ 2n^2 - 2n - d + 1 & \text{if } n \text{ is odd} \end{cases}$, we obtain the simplified version of the formula for calculating the average Cayley genus. As before, we use $D = \{d : d \mid n,\ 1 < d < n\}$ and $D^* = D \cup \{1\}$.

Theorem 9 *Let $n \geq 2$ be an integer. Then the average Cayley genus $\overline{\gamma}(D_{2n}, \Delta)$ for the dihedral group with generating set consisting of all the reflections is given by $\overline{\gamma}(D_{2n}, \Delta) =$*

$$\begin{cases} \binom{2n-1}{n-1}^{-1}\left[c_0\left(2n^2-4n+1\right)+\sum_{d\in D^*} c_d\phi(n/d)\left(2n^2-2n-2d+1\right)\right] & \text{if } n \text{ is even} \\ \binom{2n-1}{n-1}^{-1}\left[c_0\left(2n^2-3n+1\right)+\sum_{d\in D^*} c_d\phi(n/d)\left(2n^2-2n-d+1\right)\right] & \text{if } n \text{ is odd} \end{cases},$$

where for $N \in D$ *and* $i \in D^*$, $U_N = \frac{1}{N}\left[\binom{2N-1}{N-1}-1-\sum_{\substack{k|N \\ 1<k<N}} kU_k\right]$, $c_0 =$

$$\frac{1}{n}\left[\binom{2n-1}{n-1}+n-1+\sum_{k\in D^*}\phi(n/k)\sum_{\substack{N\in D \\ n\nmid kN}} U_N\right], \textit{ and } c_i = c_0 - \left[1+\sum_{\substack{N\in D \\ n\nmid iN}} U_N\right].$$

Let us revisit the example of $n=12$. Table 2 contains the necessary information for using the new formula. We must find c_0 and c_d for each $d \in D^* = \{1, 2, 3, 4, 6\}$. Since n is even, we use the first equation in the formula. Now, we use the values from the table in the formula for when n is even. We obtain

$$\overline{\gamma}(D_{24}, \Delta) = \frac{12!11!}{23!}[241(112720)+4\cdot 263(112632)+2\cdot 261(112707)+$$
$$2\cdot 259(112640)+2\cdot 257(112710)+253(112716)] = \frac{174631897}{869193}.$$

Table 2 Example of $n=12$ Using New Formula

d	$2n^2-4n+1$	c_0
0	241	112720

d	$\phi(\frac{n}{d})$	$2n^2-2n-2d+1$	c_d
1	4	263	112632
2	2	261	112707
3	2	259	112640
4	2	257	112710
6	1	253	112716

Next, we consider some special cases of this formula. First, let $n = p$, where p is an odd prime. The only divisor d of p that satisfies $1 \le d < p$ is 1. So for an odd prime integer, there are only two distinct values of the coefficient sums, namely, c_0 and c_1. Using Corollary 8, we find $c_0 = \frac{1}{p}\left[\binom{2p-1}{p-1} + p - 1\right]$ and $c_1 = c_0 - 1 = \frac{1}{p}\left[\binom{2p-1}{p-1} - 1\right]$, so that by Theorem 9, we obtain a formula that depends only on p.

Corollary 10 *If p is an odd prime, then*

$$\overline{\gamma}(D_{2p}, \Delta) = \frac{p-1}{p}\binom{2p-1}{p-1}^{-1}\left[\binom{2p-1}{p-1}(2p^2-1) - p + 1\right].$$

Similar formulas depending only on an odd prime p can be obtained for $n = p^2$ and $n = 2p$.

Corollary 11 *If p is an odd prime, then* $\overline{\gamma}(D_{2p^2}, \Delta) =$

$$\frac{p-1}{p}\binom{2p^2-1}{p^2-1}^{-1}\left[\binom{2p^2-1}{p^2-1}(2p^4+2p^3-2) - \binom{2p-1}{p-1}(p-2) - p^2 + p\right].$$

Corollary 12 If p is an odd prime, then $\overline{\gamma}(D_{4p}, \Delta) =$

$$\frac{1}{p}\binom{4p-1}{2p-1}^{-1}\left[\binom{4p-1}{2p-1}(8p^3-4p^2-5p+3) - \binom{2p-1}{p-1}(2p-1) - 10p^2 + 18p - 10\right].$$

In conclusion, we now have developed a new formula for finding the average Cayley genus for the dihedral group with generating set consisting of all the reflections. This formula is an improvement in that it uses fewer cases and enables us to find directly the coefficient sums without having to use the generating function.

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References

[1] J. Edmonds, A combinatorial representation for polyhedral surfaces. *Notices Amer. Math. Soc.* 7 (1960) 646.

[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*. Claredon Press, New York (1979).

[3] I. Niven, H. S. Zuckerman and H. L. Montgomery, *An Introduction to the Theory of Numbers*. John Wiley & Sons, New York (1991).

[4] J. Riordan, *An Introduction to Combinatorial Analysis*. John Wiley & Sons, New York (1958).

[5] M. Schultz, Random Cayley maps for groups generated by involutions. *Journal of Combinatorial Theory, Series B*. **76** (1999) 247-261.