TRI-RESTRICTED NUMBERS AND POWERS OF PERMUTATION REPRESENTATIONS

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ABSTRACT. Let G be a transitive permutation gro ıp on a set Q. The orbit decompositions of the actions of G on the sets of ordered n-tuples with elements repeated at most three times are studied. The decompositions involve Stirling numbers and a new class of related numbers, the so-called tri-restricted numbers. The paper presents exponential generating functions for the numbers of orbits, and examines relationships between various powers of the G-set involving Stirling numbers, the tri-restricted numbers, and the coefficients of Bessel polynomials.

1. INTRODUCTION

Let G be a finite group. A *G-set* (Q, G) or *permutation representation* of the group G consists of a set Q, together with a (right) action of G on Q via a homomorphism

$$G \to Q!\ ;\ g \mapsto (q \mapsto qg) \tag{1.1}$$

from G into the group $Q!$ of all permutations of the set Q. A G-set (Q, G) may be construed as an algebra of unary operations on the set Q. For a positive integer n, the direct power $(Q, G)^n$ of this algebra is the G-set Q^n with *diagonal action*

$$g : (q_1, \ldots, q_n) \mapsto (q_1 g, \ldots, q_n g) \tag{1.2}$$

of the elements g of G.

The subset $Q^{[n]}$ of Q^n consisting of all n-tuples of distinct elements of Q, equipped with the restriction of the diagonal action of G, is called the n-th *irredundant power* of the G-set (Q, G), and denoted by $(Q, G)^{[n]}$. The subset $Q^{[[n]]}$ of Q^n consisting of all n-tuples in which no element is repeated more than once, equipped with the restriction of the diagonal action of G, is called the n-th *bi-restricted power* of the G-set (Q,G), and denoted by $(Q,G)^{[[n]]}$.

The exponential generating functions for the numbers of orbits in the various direct powers, irredundant powers and bi-restricted powers are respectively

$$(1.3)\ \frac{1}{|G|}\sum_{g\in G} e^{t\pi(g)},\ \frac{1}{|G|}\sum_{g\in G}(1+t)^{\pi(g)} \text{ and } \frac{1}{|G|}\sum_{g\in G}\left(1+t+\frac{t^2}{2!}\right)^{\pi(g)},$$

where $\pi(g)$ is the number of points of Q fixed by an element g of G (5.1[7], Th.6.4 & Th.7.7[3]). The two latter generating functions may be considered as drastic truncations of the exponential generating function for the numbers of orbits in the direct power G-sets, since

$$(1.4)\qquad \frac{1}{|G|}\sum_{g\in G} e^{t\pi(g)} = \frac{1}{|G|}\sum_{g\in G}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\cdots\right)^{\pi(g)}.$$

In this paper, we consider a slightly less drastic truncation,

$$(1.5)\qquad \frac{1}{|G|}\sum_{g\in G}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\right)^{\pi(g)},$$

and define an appropriate G-subset of (Q^n,G) so that (1.5) become the exponential generating function for the number of orbits in it. The numbers $T_1(n,k)$ related to (1.5), the so-called *tri-restricted numbers of the first kind*, are defined in Definition 5.1 and investigated in Section 5.

The G-subset $Q^{[[[n]]]}$ of Q^n consisting of all n-tuples in which no element appears more than three times is called the *tri-restricted power* G-set $(Q,G)^{[[[n]]]}$. Then orbit decompositions of the tri-restricted powers $(Q,G)^{[[[n]]]}$ are related to the orbit decompositions of the irredundant powers $(Q,G)^{[n]}$ via the tri-restricted numbers of the first kind. The tri-restricted powers are also related to the other powers, the direct powers and the bi-restricted powers, via Stirling numbers and the coefficients in Bessel polynomials.

The *tri-restricted number of the second kind* $T_2(n,k)$ is defined to be the (n,k)-entry of the inverse of the matrix whose (m,j)-entry for each m,j is the tri-restricted number of the first kind $T_1(m,j)$. This provides an inverse relation between the tri-restricted powers and the irredundant powers. Theorem 5.12 presents (1.5) as the exponential generating function for the numbers of orbits in the tri-restricted power G-sets.

Introductory sections briefly cover Stirling numbers and Bessel polynomials (Section 2), the duality between direct powers and irredundant powers (Section 3), and the bi-restricted powers (Section 4).

2. Stirling numbers and Bessel numbers

For each positive integer n, the product $X(X-1)(X-2)\dots(X-n+1)$ in the integral polynomial ring $\mathbb{Z}[X]$ over an indeterminate X is denoted by $[X]_n$. Since $\{X^n \mid n \in \mathbb{N}\}$ and $\{[X]_n \mid n \in \mathbb{N}\}$ are free generating sets for $\mathbb{Z}[X]$ as a $\mathbb{Z}$-module, each can be uniquely expressed as a linear combination of the others.

Definition 2.1. The *Stirling numbers of the first kind* $S_1(n,k)$ and the *Stirling numbers of the second kind* $S_2(n,k)$ are given by

$$(2.1)\quad X^n = \sum_{k=0}^{n} S_2(n,k)[X]_k \qquad \text{and} \qquad [X]_n = \sum_{k=0}^{n} S_1(n,k)X^k. \quad \square$$

Proposition 2.2. (Cf. 3.14 [1].) *The Stirling number of the second kind $S_2(n,k)$ is the number of partitions of an n-set into exactly k nonempty subsets.* □

For each natural number n, the *Bessel polynomial* $y_n(x)$ is defined to be the (unique) polynomial of degree n with unit constant term

$$(2.2)\qquad y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k$$

which satisfies the differential equation $x^2y''+(2x+2)y' = n(n+1)y$ [2, 4, 6]. Then for each positive integer n, the n-th Bessel polynomial may be written in the form

$$P_n(x) = \sum_{k=0}^{\infty} B(n,k)x^{n-k}, \tag{2.3}$$

where the *Bessel number* $B(n,k)$ is given by

$$B(n,k) = \quad \textbf{if} \quad n < k \quad \textbf{then} \quad 0 \quad \textbf{else} \quad \frac{(2n-k)!}{2^{n-k}(k)!(n-k)!}. \tag{2.4}$$

Proposition 2.3. *The Bessel numbers satisfy the recursion*

$$B(n,k) = (2n-k+1)\cdot B(n-1,k) + B(n-1,k-1) \tag{2.5}$$

for $n \geq k$. □

The combinatorial significance of the Bessel numbers is given by the following.

Theorem 2.4. (Th.7.1 [3]) For an indeterminate X, let

$$f(t) = \left(1+t+\frac{t^2}{2!}\right)^X.$$

Then

$$f^{(n)}(0) = \sum_{k=1}^{n} B(k, 2k-n)[X]_k, \tag{2.6}$$

where $f^{(n)}(0)$ is the n-th derivative of f with respect to t at $t = 0$. □

Corollary 2.5. (Co.7.2 [3]) For any positive integers n and k, the Bessel number $B(n,k)$ is the number of partitions of a $(2n-k)$-set of type $1^k 2^{n-k} 3^0 4^0 \dots n^0$. □

3. Direct powers and irredundant powers

For a finite group G, let $\underline{\underline{G}}$ be the variety of G-sets, construed as a category with homomorphisms (G-equivariant maps) as morphisms. For an object Q of $\underline{\underline{G}}$, let $[Q]$ denote the isomorphism class of Q in $\underline{\underline{G}}$. Let

$A^+(G)$ be the set of isomorphism classes of finite G-sets. This set becomes a commutative, unital semiring $(A^+(G), +, \cdot, 0, 1)$ under $[P] + [Q] = [P + Q]$, $[P] \cdot [Q] = [P \times Q]$, $0 = [\emptyset]$, and $1 = [1]$. It embeds canonically into a commutative ring, the *integral Burnside algebra* of the group G (§1.2 [8]).

For each positive integer n, the *irredundant power* G-set $Q^{[n]}$ is defined to be the complement in the direct power Q^n of the subset consisting of all n-tuples comprising at most $n-1$ distinct elements of Q (cf. II.1.10 [5]).

The following proposition shows that the irredundant power G-sets $(Q, G)^{[n]}$ are dual to the direct power G-sets $(Q, G)^n$ via the Stirling numbers of the first and second kinds.

Proposition 3.1. (Prop.5.1 [3])

$$(3.1) \quad [Q^{[n]}] = \sum_{k=1}^{n} S_1(n,k)[Q^k] \quad \text{and} \quad [Q^n] = \sum_{k=1}^{n} S_2(n,k)[Q^{[k]}]. \quad \square$$

For a G-set (Q, G), let $\pi(g)$ be the number of points of Q fixed by an element g of G. By Burnside's Lemma (V.20.4 [5]), the average number of fixed points

$$(3.2) \qquad \frac{1}{|G|} \sum_{g \in G} \pi(g)^n$$

is the number of orbits of G on the n-th direct power Q^n. By Proposition 3.1,

$$(3.3) \qquad \frac{1}{|G|} \sum_{g \in G} [\pi(g)]_n$$

is the number of orbits of G on the n-th irredundant power $Q^{[n]}$ (Lem.6.3 [3]). Recall that the exponential generating function for a sequence $(a_n)_{n=0}^{\infty}$ is $\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$.

Theorem 3.2. (5.1, Th.6.4 [7]) The exponential generating functions for the numbers of orbits on the direct power G-sets $(Q,G)^n$ and the irredundant power G-sets $(Q,G)^{[n]}$ are respectively

$$\frac{1}{|G|}\sum_{g\in G}(e^t)^{\pi(g)} \quad \text{and} \quad \frac{1}{|G|}\sum_{g\in G}(1+t)^{\pi(g)}, \tag{3.4}$$

where $\pi(g)$ is the number of points of Q fixed by an element g of G. □

4. Bi-restricted powers

Consider the n-th direct power Q^n as the set of functions to Q from the n-set $\{1,2,3,\ldots,n\}$. Then the n-th irredundant power $Q^{[n]}$ is the subset consisting of injective functions from the n-set into Q. For each positive integer n, the n-th *bi-restricted power set* of the set Q is defined to be

$$Q^{[[n]]} = \{f : \{1,2,3,\ldots,n\} \to Q \mid \forall q \in Q,\ |f^{-1}\{q\}| \leq 2\}\ . \tag{4.1}$$

Thus $Q^{[[n]]}$ is an intermediate set, included in Q^n and including $Q^{[n]}$. For a G-set (Q,G), the restriction of the direct power action of G on Q^n to $Q^{[[n]]}$ is called the n-th *bi-restricted power* of (Q,G), and denoted by $(Q,G)^{[[n]]}$.

The following proposition shows how the Bessel numbers yield a dual relation between the bi-restricted powers and the irredundant powers.

Proposition 4.1. (Props.7.5, 6 [3])

1. $[Q^{[[n]]}] = \sum_{k=1}^{n} B(k,2k-n)[Q^{[k]}]$.
2. $[Q^{[n]}] = \sum_{k=1}^{n}(-1)^{n-k}B(n-1,k-1)[Q^{[[k]]}]$. □

Bringing in the Stirling numbers, one can obtain a dual relation between the bi-restricted powers and the direct powers.

Proposition 4.2. (Rmk.7.8 [3])

1. $[Q^{[[n]]}] = \sum_{k=1}^{n}\left(\sum_{m=k}^{n} B(m,2m-n)\cdot S_1(m,k)\right)[Q^k]$.
2. $[Q^n] = \sum_{k=1}^{n}\left(\sum_{m=k}^{n}(-1)^{m-k}\cdot S_2(n,m)\cdot B(m-1,k-1)\right)[Q^{[[k]]}]$. □

The following theorem shows that the exponential generating function for the number of orbits on the bi-restricted powers is an intermediate function between the exponential generating functions in (3.4).

Theorem 4.3. (Th.7.7 [3]) The exponential generating function for the number of orbits on the bi-restricted powers $(Q, G)^{[[n]]}$ is

$$f(t) = \frac{1}{|G|} \sum_{g \in G} \left(1 + t + \frac{t^2}{2!}\right)^{\pi(g)}, \tag{4.2}$$

where $\pi(g)$ is the number of points of Q fixed by an element g of G. □

5. Tri-restricted numbers and powers

By analogy with Theorem 4.3, we now want to build a new G-subset of Q^n such that the truncation

$$\frac{1}{|G|} \sum (1 + t + \frac{t^2}{2!} + \frac{t^3}{3!})^{\pi(g)} \tag{5.1}$$

of $e^{t\pi(g)}$ is the exponential generating function for the number of orbits in the G-subset. The first task is to introduce the coefficients that will play the role of the Bessel numbers in Proposition 4.1.

Definition 5.1. For an indeterminate X, let

$$f(t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right)^X.$$

The *tri-restricted numbers of the first kind* $T_1(n, k)$ are defined by

$$f^{(n)}(0) = \sum_{k=1}^{n} T_1(n, k)[X]_k, \tag{5.2}$$

where $f^{(n)}(0)$ is the n-th derivative of f with respect to t at $t = 0$. □

An explicit form for the tri-restricted numbers of the first kind is given by the following proposition.

Proposition 5.2. *For any positive integers n and k, $T_1(n,k) =$*

$$\begin{array}{ll} \textbf{if} & \lceil \frac{n}{3} \rceil \le k \le n \\ \textbf{then} & \displaystyle\sum_{t=t_1}^{\lfloor \frac{3k-n}{2} \rfloor} \frac{n!}{(2!)^{3k-n-2t}(3!)^{n-2k+t} t!(3k-n-2t)!(n-2k+t)!} \\ \textbf{else} & 0, \end{array}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x, $\lceil x \rceil$ is the least integer greater than or equal to x, and

$$t_1 = \quad \textbf{if} \quad \lceil \frac{n}{2} \rceil \le k \le n \quad \textbf{then} \quad 2k-n \quad \textbf{else} \quad 0.$$

Proof. Let $g(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$. Then $f(t) = (g(t))^X$, and the terms of the n-th derivative $f^{(n)}(t)$ of $f(t)$ have the form

$$[X]_k (g(t))^{X-k} (g'(t))^t (g''(t))^{3k-n-2t} (g^{(3)}(t))^{n-2k+t} \tag{5.3}$$

for all non-negative integers t, $3k-n-2t$ and $n-2k+t$. For all non-negative integers $\lambda_1, \lambda_2, \dots \lambda_n$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = k$ and $\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n$, the number of k-partitions of an n-set of type $1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}$ is

$$\frac{n!}{(1!)^{\lambda_1}(2!)^{\lambda_2} \dots (n!)^{\lambda_n} (\lambda_1)!(\lambda_2)! \dots (\lambda_n)!}. \tag{5.4}$$

Since $t + (3k-n-2t) + (n-2k+t) = k$ and $t + 2(3k-n-2t) + 3(n-2k+t) = n$, there are

$$\frac{n!}{(1!)^t (2!)^{3k-n-2t} (3!)^{n-2k+t} t!(3k-n-2t)!(n-2k+t)!} \tag{5.5}$$

many terms $[X]_k (g(t))^{X-k} (g'(t))^t (g''(t))^{3k-n-2t} (g^{(3)}(t))^{n-2k+t}$ in $f^{(n)}(t)$. Since $g(0) = g'(0) = g''(0) = g^{(3)}(0) = 1$, one obtains the expression

(5.6)

$$\sum_{k=1}^{n} \left(\sum_{\substack{t \ge 0 \\ 3k-n-2t \ge 0 \\ n-2k+t \ge 0}} \frac{n!}{(1!)^t (2!)^{3k-n-2t} (3!)^{n-2k+t} t!(3k-n-2t)!(n-2k+t)!} \right) [X]_k$$

for $f^{(n)}(0)$. It is clear that $T_1(n,k) = 0$ for all $k > n$, since $f^{(n)}(t)$ cannot have any term containing $[X]_k$ for any $k > n$. If $k < \lceil \frac{n}{3} \rceil$, one has $3k < n$

and $3k - n - 2t < n - n - 2t \leq -2t$, which violates the constraints $t \geq 0$ and $3k - n - 2t \geq 0$. Therefore $T_1(n,k) = 0$ for all $k < \lceil \frac{n}{3} \rceil$. Solving $n - 2k + t \geq 0$ and $3k - n - 2t \geq 0$ for t, one obtains

$$t \geq 2k - n \qquad \text{and} \qquad t \leq \frac{3k-n}{2}.$$

Together with $t \geq 0$ and the fact that t is an integer, this gives

$$\text{(5.7)} \qquad max\{0, 2k - n\} \leq t \leq \lfloor \frac{3k-n}{2} \rfloor.$$

If $\lceil \frac{n}{3} \rceil \leq k < \lceil \frac{n}{2} \rceil$, one has $2k < n$, and if $\lfloor \frac{n}{2} \rfloor \leq k \leq n$, one has $n \leq 2k$. Then

$$\text{(5.8)} \qquad max\{0, 2k - n\} = \begin{cases} 0 & \text{if } \lceil \frac{n}{3} \rceil \leq k < \lceil \frac{n}{2} \rceil \\ 2k - n & \text{if } \lfloor \frac{n}{2} \rfloor \leq k \leq n. \end{cases}$$

The required form for $T_1(n,k)$ is furnished by (5.6), (5.7) and (5.8). □

As an immediate corollary, one obtains the following combinatorial interpretation of the tri-restricted numbers of the first kind.

Corollary 5.3. *For each positive integer n, $T_1(n,k)$ is the total number of k-partitions of an n-set of type $1^t 2^{3k-n-2t} 3^{n-2k+t} 4^0 \ldots n^0$.* □

The following table shows the first few tri-restricted numbers of the first kind. The empty cells are to be filled with 0's.

8	7	6	5	4	3	2	$1 = k$	$T_1(n,k)$
							1	$n = 1$
						1	1	2
					1	3	1	3
				1	6	7		4
			1	10	25	10		5
		1	15	65	75	10		6
	1	21	140	315	175			7
1	28	266	980	1225	280			8

The tri-restricted numbers of the first kind

By analogy with (4.1), one may now define the desired subset of Q^n.

Definition 5.4. The n-th *tri-restricted power* of a set Q is

(5.9) $Q^{[[[n]]]} = \{f : \{1,2,3,\ldots,n\} \to Q \mid \forall q \in Q, |f^{-1}\{q\}| \leq 3\}$.

For a G-set (Q,G), the restriction of the diagonal action of G on Q^n to $Q^{[[[n]]]}$ is called the n-th *tri-restricted power* of (Q,G), and denoted by $(Q,G)^{[[[n]]]}$. □

Lemma 5.5. $Q^{[n]} \subseteq Q^{[[n]]} \subseteq Q^{[[[n]]]} \subseteq Q^n$. □

Tri-restricted powers are related to irredundant powers via the tri-restricted numbers of the first kind.

Theorem 5.6.

$$[Q^{[[[n]]]}] = \sum_{k=1}^{n} T_1(n,k)[Q^{[k]}]. \tag{5.10}$$

Proof. Let $A_k^n = \{f \in Q^n \mid k = |\mathrm{Im}(f)| \text{ and } \forall q \in Q,\ |f^{-1}(q)| \leq 3\}$ and $Q_k^n = \{f \in Q^n \mid k = |\mathrm{Im}(f)|\}$. Then $A_k^n = Q_k^n \cap Q^{[[[n]]]}$, and $Q^{[[[n]]]}$ is the disjoint union of the A_k^n. For any partition π of the n-set $\{1,2,3,\ldots,n\}$ of type $1^t 2^{3k-n-2t} 3^{n-2k+t} 4^0 \ldots n^0$, let $Q_\pi = \{f \in Q^n \mid \pi = \ker(f)\}$. Then Q_π is in A_k^n, and is G-isomorphic to $Q^{[k]}$. Since there are $T_1(n,k)$ partitions of an n-set of the type $1^t 2^{3k-n-2t} 3^{n-2k+t} 4^0 \ldots n^0$, the G-set A_k^n is G-isomorphic to $T_1(n,k)$ copies of $Q^{[k]}$. Therefore

$$Q^{[[[n]]]} = \bigcup A_k^n \cong \bigcup T_1(n,k) Q^{[k]}. \tag{5.11}$$

Considering the isomorphism classes from (5.11), one obtains (5.10). □

By Proposition 3.1 and Proposition 4.1(2) taken together with Theorem 5.6, the tri-restricted powers can be expressed in terms of the direct powers or the bi-restricted powers as follows;

Corollary 5.7.

1. $[Q^{[[[n]]]}] = \sum_{k=1}^{n} \left(\sum_{m=k}^{n} T_1(n,m) \cdot S_1(m,k)\right) [Q^k]$

2. $[Q^{[[[n]]]}] = \sum_{k=1}^{n} \left(\sum_{m=k}^{n} (-1)^{m-k} T_1(n,m) \cdot B(m-1,k-1)\right) [Q^{[[k]]}]$ □

Now consider the matrix T_1 whose (n,k)-th entry is $T_1(n,k)$ for each n,k. Since T_1 is a lower triangular matrix whose diagonal elements are all 1, we can consider the inverse matrix of T_1.

Definition 5.8. The *tri-restricted number of the second kind* $T_2(n,k)$ is defined to be the (n,k)-entry of the inverse matrix of T_1. □

The following table shows the first few tri-restricted numbers of the second kind. The empty cells are to be filled with 0's.

8	7	6	5	4	3	2	$1=k$	$T_2(n,k)$
							1	$n=1$
						1	−1	2
					1	−3	2	3
				1	−6	11	−5	4
			1	−10	35	−45	10	5
		1	−15	85	−210	175	35	6
	1	−21	175	−700	1225	−315	−910	7
1	−28	322	−1890	5565	−5670	−6265	11935	8

The tri-restricted numbers of the second kind

Remark 5.9. Unlike the Stirling numbers of the second kind, the tri-restricted numbers of the second kind $T_2(n,k)$ do not take alternating signs. The sum of the last three numbers in each row of the above table becomes 0, i.e. for all positive integers $n \leq 8$, one has

$$T_2(n,3) + T_2(n,2) + T_2(n,1) = 0. \tag{5.12}$$

In fact, the relationship (5.12) holds for all positive integers, since $T_1(1,1) = T_1(2,1) = T_1(3,1) = 1$. This might be the reason for the non-alternating signs of $T_2(n,k)$, but to be sure one would need to find a formula or a combinatorial interpretation for the $T_2(n,k)$. □

One may now provide an inverse to the formula of Theorem 5.6 as follows.

Proposition 5.10.

$$[Q^{[n]}] = \sum_{k=1}^{n} T_2(n,k)[Q^{[[[k]]]}]. \quad \square \tag{5.13}$$

Applying Proposition 3.1 and Proposition 5.10 with Proposition 4.1(1), the direct powers and the bi-restricted powers can be expressed in terms of the tri-restricted powers as follows.

Corollary 5.11.

1. $[Q^n] = \sum_{k=1}^{n} \left(\sum_{m=k}^{n} S_2(n,m) \cdot T_2(m,k)\right) [Q^{[[[k]]]}]$
2. $[Q^{[[n]]}] = \sum_{k=1}^{n} \left(\sum_{m=k}^{n} B(m, 2m-n) \cdot T_2(m,k)\right) [Q^{[[[k]]]}]$ $\square$

Finally, we conclude that (5.1) generates the numbers of orbits in the tri-restricted powers of a G-set (Q, G) with permutation character π.

Theorem 5.12. *The exponential generating function for the number of orbits on the n-th tri-restricted power G-set $(Q,G)^{[[[n]]]}$ is*

$$f(t) = \frac{1}{|G|} \sum_{g \in G} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right)^{\pi(g)}, \tag{5.14}$$

where $\pi(g)$ is the number of points of Q fixed by an element g of G.

Proof. By Definition 5.1, the n-th derivative of f with respect to t at $t = 0$ is

$$\begin{aligned} f^{(n)}(0) &= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{k=1}^{n} T_1(n,k)[\pi(g)]_k\right) \\ &= \sum_{k=1}^{n} T_1(n,k) \left(\frac{1}{|G|} \sum_{g \in G} [\pi(g)]_k\right). \end{aligned} \tag{5.15}$$

By (3.3) and Theorem 5.6, it is easy to see that $f^{(n)}(0)$ is the number of orbits of G on the n-th tri-restricted power $Q^{[[[n]]]}$. $\square$

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