Isoperimetric Numbers of Cayley Graphs Arising from Generalized Dihedral Groups

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Abstract

Let n, x be positive integers satisfying $1 < x < n$. Let $H_{n,x}$ be a group admitting a presentation of the form $\langle a, b \mid a^n = b^2 = (ba)^x = 1\rangle$. When $x = 2$ the group $H_{n,x}$ is the familiar dihedral group, D_{2n}. Groups of the form $H_{n,x}$ will be referred to as generalized dihedral groups. It is possible to associate a cubic Cayley graph to each such group, and we consider the problem of finding the isoperimetric number, $i(G)$, of these graphs. In section two we prove some propositions about isoperimetric numbers of regular graphs. In section three the special cases when $x = 2, 3$ are analyzed. The former case is solved completely. An upper bound, based on an analysis of the cycle structure of the graph, is given in the latter case. Generalizations of these results are provided in section four. The indices of these graphs are calculated in section five, and a lower bound on $i(G)$ is obtained as a result. We conclude with several conjectures suggested by the results from earlier sections.

1 Introduction and Notation

Let G be a connected, finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $S \subset V(G)$. We define the *boundary* of S, denoted ∂S, to be the subset of $E(G)$ consisting of those edges with exactly one endpoint in S. Then we define the *isoperimetric number* of G, denoted $i(G)$ by

$$i(G) = \inf_S \frac{|\partial S|}{|S|},$$

where the infimum is taken over all subsets S of $V(G)$ satisfying $|S| \leq \frac{1}{2}|V(G)|$.

As defined above the isoperimetric number is a combinatorial analog of the Cheeger constant used by geometers to investigate the spectral properties of Riemann surfaces. It was introduced by Peter Buser in 1978. His idea was to translate, via the isoperimetric number, spectral information about graphs into spectral information about certain associated surfaces. This aspect remains of considerable interest to geometers, see [3] for example.

The isoperimetric number is of interest to combinatorialists for several reasons. One is that bounds on the eigenvalue spectrum of a graph can be obtained from it. See [5] for a thorough description of such results. The isoperimetric number can also be viewed as a measure of the connectedness of the graph and is therefore relevant to the problem of constructing good expanders. An introduction to this aspect of the subject can be found in [6].

It is generally a difficult problem to determine explicitly the isoperimetric number of a given graph, as the number of cutsets that need to be considered is generally quite large. Thus, most of the work in this area has been devoted to bounding the isoperimetric numbers of certain families of graphs. See [2] for example. In this paper we will consider an important family of cubic Cayley graphs, and try to determine the structural properties satisfied by their isoperimetric sets.

2 Three Propositions

For most of this paper we will restrict our attention to certain cubic Cayley graphs. But first we establish three propositions that are true for any regular graph. We start with a definition.

Definition 1. *Let $S \subset V(G)$ satisfying $|S| \leq \frac{1}{2}|V(G)|$. Then the quotient $\frac{|\partial S|}{|S|}$ is denoted by $i_S(G)$ and is called the* **isoperimetric quotient** *of S.*

Of course, it follows from the definition of $i(G)$ that $i(G) \leq i_S(G)$ for any subset S of the appropriate size. We call S an isoperimetric set for the graph G if $i(G) = i_S(G)$.

Proposition 1. *Let G be a k-regular graph of girth g. Assume $g \leq \frac{1}{2}|V(G)|$. Then $i(G) \leq k-2$*

Proof. Let S be a cycle of length g in G. Then we compute

$$i_S(G) = \frac{|\partial S|}{|S|} = \frac{g(k-2)}{g} = k-2.$$

Since $i(G) \leq i_S(G)$ the proof is complete. □

Proposition 2. *Let G be as above and let S be an isoperimetric set for G. Assume that $i(G) \neq k-2$. Then S contains no vertices of order one.*

Proof. Suppose that S contains a vertex v of order one. Define the set $S' = S - \{v\}$.

Observe that $|S'| = |S| - 1$. Also observe that $|\partial S'| = |\partial S| - (k-2)$. So we compute

$$\frac{|\partial S'|}{|S'|} = \frac{|\partial S| - k + 2}{|S| - 1} < \frac{|\partial S|}{|S|}.$$

The last inequality follows from our assumption, and the fact that, by Proposition one, we know that $i(G) \leq k-2$. So we have $i_{S'}(G) < i_S(G)$. This contradicts the assumption that S is an isoperimetric set and the proof is complete. □

Proposition 3. *Let G be as above and let $S \subset V(G)$. Assume that $|S| \leq \frac{1}{2}|V(G)|$ and that S contains no vertices of order one. Let V_n denote the set of vertices in S that are joined by exactly n edges to other elements of S. Then*

$$i_S(G) = (k-2) - \frac{\sum_{n=3}^{k}(n-2)|V_n|}{|S|}.$$

Proof. We know by assumption that $|V_1| = 0$ in this case. Observe that if $v \in V_n$, then v is the endpoint of precisely $k-n$ edges in ∂S. It follows that

$$|\partial S| = \sum_{n=2}^{k}(k-n)|V_n|.$$

To prove the proposition it will suffice to show that

$$|S|(k-2) - \sum_{n=3}^{k}(n-2)|V_n| = \sum_{n=2}^{k}(k-n)|V_n|.$$

We perform the following calculation:

$$\begin{aligned}
\sum_{n=3}^{k}(n-2)|V_n| + \sum_{n=2}^{k}(k-n)|V_n| &= (k-2)|V_2| + \sum_{n=3}^{k}((n-2)|V_n| + (k-n)|V_n|) \\
&= (k-2)|V_2| + \sum_{n=3}^{k}(k-2)|V_n| \\
&= (k-2)\sum_{n=2}^{k}|V_n| \\
&= |S|(k-2)
\end{aligned}$$

and the proof is complete. □

We will be using this proposition in the special case of cubic graphs, so we will restate it in a form more useful for that purpose. First, we need a definition.

Definition 2. *Let G be a cubic graph and let $S \subset V(G)$. Let $v \in S$. We will say that v is a claw in S if it is connected to three other vertices in S. The number of claws in S will be denoted by C_S.*

Proposition 4. *Let G be a cubic graph and let $S \subset V(G)$ satisfy $|S| \leq \frac{1}{2}|V(G)|$. Assume that S contains no vertices connected to only one other element in S. Then*

$$i_S(G) = 1 - \frac{C_S}{|S|}.$$

Proof. This follows immediately from proposition three. □

3 Two Examples

The abstract ideas to be presented in later sections will be more easily understood if they are first explored in the context of specific examples.

Definition 3. *Let H be a finite group and let Ω be a generating set for H. Assume that $1 \notin \Omega$ and that $\Omega = \Omega^{-1}$. We define the Cayley graph for H with respect to Ω, denoted $G(H, \Omega)$, as follows: the vertices of G are the elements of H. Vertex v_1 is connected to vertex v_2 if and only if $v_1 = v_2\omega$ for some $\omega \in \Omega$.*

As defined above, Cayley graphs are simple, connected, $|\Omega|$-regular, and vertex-transitive.

3.1 The Dihedral Groups D_{2n}

The groups $H_{n,2}$, with $n > 2$ are the familiar dihedral groups. We consider the isoperimetric numbers of the Cayley graphs $G(D_{2n}, \{a, a^{-1}, b\})$, where a has order n and b has order two.

This Cayley graph can be pictured as follows: let $(u_1, u_2, ..., u_n)$ and $(v_1, v_2, ..., v_n)$ be two cycles of length n. Complete the graph by adding one edge for each pair of vertices of the form (u_i, v_i) with $1 \leq i \leq n$. The resulting graph, sometimes referred to as a ladder graph, is the desired Cayley graph.

Proposition 5. *With G as above, we have $i(G) = \frac{4}{n}$.*

Proof. Let S be the set of vertices $\{u_1, u_2, ..., u_{\frac{n}{2}}, v_1, v_2, ..., v_{\frac{n}{2}}\}$. Then it is easy to see that

$$|S| = n = \frac{1}{2}|V(G)| \text{ and } i_S(G) = \frac{4}{n}.$$

To see that S is actually an isoperimetric set, reason as follows: in the xy-plane we take $2n$ unit squares and arrange them in the form of a rectangle with dimensions $2 \times n$. Now edges in G can be viewed as adjacencies between squares. Call this rectangle R.

Since the graph G is vertex-transitive, we can find an isoperimetric set S that excludes any particular edge. It follows that finding $i(G)$ is equivalent to finding the subset of R, composed of unit squares, that minimizes the ratio of perimeter to area. But it is a consequence of the classical isoperimetric theorem of Euclidean geometry that a rectangle of dimensions $2 \times \frac{n}{2}$ is the desired region. From this it follows that the set S above is an isoperimetric set, and the proof is complete. □

The method of proof used here is frequently useful for calculating the isoperimetric number of planar graphs.

We now point out three interesting properties of our set S.

- It can be expressed as the union of cycles in S.
- All of those cycles have mnimial length.
- $|S| = \frac{1}{2}|V(G)|$.

It would be interesting to know if these properties hold for isoperimetric sets of the generalized dihedral groups.

3.2 The Matrix Groups $PSL(2, \mathbb{Z}_n)$

The groups $H_{n,3}$ are isomorphic to a certain family of matrix groups. Let $\mathbb{Z}_n$ denote the set of integer residue classes mod n. Let $SL(2, \mathbb{Z}_n)$ denote the group of 2×2 matrices with entries in $\mathbb{Z}_n$ and determinant one. Let $PSL(2, \mathbb{Z}_n)$ denote the group $SL(2, \mathbb{Z}_n)/\{\pm 1\}$. In discussing the elements of this group, we will make no distinction between a matrix and the two-element coset it represents.

It is well known that the set

$$\Omega = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

generates the group $PSL(2, \mathbb{Z}_n)$. To simplify the notation we will denote the elements of Ω as T, T^{-1} and R respectively. Since $R = R^{-1}$, we observe that Ω is a symmetric generating set. We can therefore talk about the Cayley graphs $G(PSL(2,\mathbb{Z}_n),\Omega)$; for simplicity we will refer to these graphs as G_n. These graphs arise in an interesting way from the theory of automorphic forms. The details can be found in [7].

These graphs are far more complicated than the ones in the previous subsection. It seems reasonable, however, that a good candidate isoperimetric set could be formed by taking a union of cycles of shortest length. In everything that follows we will assume that $n > 6$.

Proposition 6. *The girth of G_n is six.*

Proof. It is easy to see that $(RT)^3 = (TR)^3 = I$, and therefore the girth of G_n is no greater than six. Showing that there is no shorter relation in Ω is a simple exercise in matrix multiplication. □

In light of this proposition we make the following definition:

Definition 4. *Given $v \in V(G_n)$, we define the six-cycle generated by v, denoted by O_v, to be the set $\{v, Rv, TRv, RTRv, (TR)^2v, R(TR)^2v\}$.*

Proposition 7. *Every cycle of length six in G_n is of the form O_v for some vertex v.*

Proof. A cycle of length six in G_n corresponds to a relation of six letters in Ω. It is easy to show that the only relations of length six in Ω are $(RT)^3$ and $(TR)^3$. Now let v be an arbitrary vertex. In addition to O_v, the only cycle of length six containing v is seen to be

$$O'_v = \{v, Tv, RTv, TRTv, (RT)^2v, T(RT)^2v\}.$$

But now we have $O'_v = O_{Tv}$. It follows that every cycle of length six has the desired form and the proof is complete. □

It is important to note that the statement $O_v = O_{v'}$ does not imply that $v = v'$. Indeed, it is easy to see that $O_v = O_{TRv} = O_{(TR)^2v}$. However, one can verify that $O_v = O_{v'}$ does imply one of the following three things: $v = v'$, $v = TRv'$ or $v = (TR)^2v'$.

Proposition 8. *Let $v_1, v_2 \in V(G_n)$. Assume that $O_{v_1} \neq O_{v_2}$ and $O_{v_1} \cap O_{v_2} \neq \emptyset$. Then $O_{v_1} \cap O_{v_2} = \{v, Rv\}$ for some vertex v.*

Proof. This is a tedious exercise in matrix multiplication. The details of the proof can be found in [8]. □

This proposition suggests a uesful way of picturing the six-cycles of G_n.

Definition 5. *Given two vertices v_1 and v_2, we will say that O_{v_1} is adjacent to O_{v_2} provided $O_{v_1} \neq O_{v_2}$ and $O_{v_1} \cap O_{v_2} \neq \emptyset$.*

Now define a new graph, which we shall denote G'_n, as follows: there wil be one vertex for each six-cycle of G_n. Two vertices will be connected if they represent adjacent six-cycles. The graph G'_n is plainly three-regular. Since every vertex in G_n is an element of exactly two six-cycles, we find that $|G'_n| = \frac{1}{3}|G_n|$.

We will now construct a candidate isoperimetric set for G_n. First, observe that $T^n = I$. It follows that the set of vertices

$$\{v, Tv, T^2v, ..., T^{n-2}v, T^{n-1}v\}$$

is a cycle of length n in G_n.

Now we define the following set:

$$S = \{\mathcal{O}_v, \mathcal{O}_{Tv}, \mathcal{O}_{T^2v}, ..., \mathcal{O}_{T^{n-2}v}, \mathcal{O}_{T^{n-1}v}\}.$$

Note that $\mathcal{O}_{T^iv} \cap \mathcal{O}_{T^{i+1}v} = \{T^iv, RT^iv\}$, where $0 \leq i \leq n-1$. It follows that $\mathcal{O}_{T^iv}$ and $\mathcal{O}_{T^{i+1}v}$ represent adjacent vertices in G'_n. In particular, the set S is a cycle of length n in G'_n. (This shows, incidentally, that the girth of G'_n is no greater than n).

We now evaluate $i_S(G_n)$. To evaluate $|S|$, observe that $\mathcal{O}_v$ contributes six vertices to S, each of the next $n-2$ six-cycles in S contributes four vertices each, and $\mathcal{O}_{T^{n-1}v}$ then contributes two vertices. It follows that

$$|S| = 6 + 4(n-2) + 2 = 4n.$$

How many of these vertices are claws? Each of the n six-cycles contains four claws and two non-claws. This makes a total of $4n$ claws. However, each claw is an element of two adjacent six-cycles. It follows that the actual number of claws is $2n$.

We see, therefore, that

$$i(G_n) \leq i_S(G_n) = 1 - \frac{\mathcal{C}_S}{|S|} = 1 - \frac{2n}{4n} = \frac{1}{2}.$$

4 Generalized Dihedral Groups

We now let $H_{n,x}$ be a group with presentation $\langle a, b \mid a^n = b^2 = (ba)^x = 1\rangle$. Here x represents an arbitrary positive integer. For each value of x we get an infinite family of graphs, one for each integer n satisfying $n \geq x$. For reasons that will become clear later, we will restrict our attention to values of n satisfying $n > 2x$. We let $G_{n,x}$ denote the Cayley graphs of $H_{n,x}$ with respect to the symmetric generating set $\Omega = \{a, a^{-1}, b\}$. In what follows we will assume that n and x are given, and simply refer to a particular graph as G.

Our goal is to derive an upper bound on $i(G)$ in terms of n and x. Since isoperimetric sets can not exceed half the number of vertices, we must first establish a crude lower bound on the size of G.

Definition 6. *Let $v \in V(G)$. Then the **n-orbit generated by v** is the set*

$$\mathcal{N}_v = \{v, av, a^2v, ..., a^{n-2}v, a^{n-1}v\}.$$

We can define an equivalence relation on $V(G)$ by declaring v, v' to be equivalent if and only if $\mathcal{N}_v = \mathcal{N}_{v'}$. The equivalence classes of this relation are clearly given by the n-orbits. It follows that the n-orbits partition the graph G. Further, multiplying a vertex by a or a^{-1} moves you to a different vertex in the same n-orbit, while multiplying by b moves you to a different orbit.

Proposition 9. *With G defined as above, we have $|G| \geq n^{x-1}$.*

Proof. Note that the Cayley graph $G(H, \Omega)$ is the unique graph satisfying the following two properties:

1. Every vertex is of degree three, with one edge for each of the three generators.
2. Every relation is satisfied at every vertex of G.

We now construct our Cayley graph as follows: We will continue to add vertices to our graph until we are compelled, by condition two, to identify two vertices. We begin with n vertices corresponding to $\mathcal{N}_1$. We then multiply each of these vertices by b, and complete each of the n-orbits arising as a result of this multiplication. Let us refer to these cycles as second generation n-orbits. We can assume that there are precisely n of them, as otherwise our graph would exhibit a relation not mandated by our presentation.

We can now repeat the procedure on each of our n second generation n-orbits. Each such n-orbit has $n-1$ vertices of degree two. We multiply each of these by b, creating $n(n-1)$ third generation n-orbits. We can continue this procedure through $x-1$ generations before being forced, by the relation $(ba)^x = 1$, to identify two of the n-orbits so produced.

The result will be that after $x-1$ generations we will have produced

$$\prod_{i=0}^{x-1} (n-i) \geq n^{x-2}$$

n-orbits. Since each of these n-orbits contains n vertices, we conclude that $|G| \geq n^{x-1}$ as claimed. □

Just as in the last section, we now see what bound we can achieve by taking clever unions of cycles whose length is the girth of G. We need one more definition, generalizing the idea of a six-cycle.

Definition 7. *Given a vertex $v \in V(G)$, we define the $2x$-cycle generated by v to be the set of vertices*

$$\mathcal{O}_v = \{v, bv, abv, babv, (ba)^2v, \cdots, a(ba)^{x-1}v\}.$$

Proposition 10. *Given G as above, we have $i(G) \leq \frac{x-2}{x-1}$.*

Proof. Our assumption that $2x < n$ implies that the girth of G is $2x$. Construct the following set $S \subset V(G)$:

$$S = \{\mathcal{O}_v, \mathcal{O}_{av}, \mathcal{O}_{a^2v}, \cdots, \mathcal{O}_{a^{n-1}v}\}.$$

Then we have

$$\begin{aligned} |S| &= 2x + (2x-2)(n-2) + 2x + 4 \\ &= 2x + (2nx - 2n - 4x + 4) + 2x - 4 \\ &= 2nx - 2n \\ &= 2n(x-1). \end{aligned}$$

This is easily seen to be less than $\frac{1}{2}|G|$. To evaluate C_S we observe that

- $\mathcal{O}_v$ contributes four claws to C_S.
- $\mathcal{O}_{a^iv}$, with $1 \leq i \leq n-2$ contributes two additional claws each.
- $\mathcal{O}_{a^{n-1}v}$ contributes no claws that have not already been counted.

From this we conclude that

$$C_S = 4 + 2(n-2) = 2n.$$

Therefore, we have that

$$\frac{C_S}{|S|} = \frac{2n}{2n(x-1)} = \frac{1}{x-1} \text{ and } i(G) \leq 1 - \frac{1}{x-1} = \frac{x-2}{x-1}$$

and the proof is complete. □

It would be possible to make small improvements in this bound by cleverly choosing additional $2x$-cycles for our set S. The improvements gained by doing this are not worth the trouble of refiguring the arithmetic.

5 Lower Bounds on $i(G)$

Bounding the isoperimetric number of a graph from above is a relatively simple matter. One simply chooses an arbitrary set of vertices S of the appropriate size and evaluates $I_S(G)$. Lower bounds are a different matter entirely. We will briefly consider one interesting example of such a bound in this section, but we need some definitions first. In the definitions below we assume that G is a general, vertex-transitive graph. Such graphs are regular, say of degree k. Of course, the class of vertex-transitive graphs includes as a subset the set of all Cayley graphs.

Definition 8. *Let G be as above. Given two edges $e_1, e_2 \in E(G)$ we say e_1 is related to e_2 if there is an element $\phi \in Aut(G)$ such that $\phi(e_1) = e_2$. This defines an equivalence relation on $E(G)$. We will suppose that there are r equivalence classes, which we will denote by $E_1, E_2, ...E_r$.*

Definition 9. *Let G be as above. Then the **index** of G, denoted by $ind(G)$, is given by the formula*

$$ind(G) = \inf_i \frac{|E(G)|}{|E_i|}.$$

If G is edge-transitive as well as vertex-transitive then there will be only one equivalence class of edges and it will have size $|E(G)|$. On the other hand, it is easy to see that $|E_i| \geq \frac{k}{2}$. It follows that $1 \leq ind(G) \leq k$.

Proposition 11. *Let G be as above. Then*

$$i(G) \geq \frac{1}{2D(ind(G))} \geq \frac{1}{2Dk},$$

where D represents the diameter of G.

A proof of this result can be found in [5].

Determining the index of the Cayley graphs for the generalized dihedral groups is actually quite simple.

Proposition 12. *Let $G = \langle a, b \mid a^n = b^2 = (ba)^x = 1\rangle$ be a presentation of the generalized dihedral group. Then G has two equivalence classes of edges given by*

$$E_1 = \{(v, av) : v \in V(G)\} \text{ and } E_2 = \{(v, bv) : v \in V(G)\}.$$

Further, we have $|E_1| = \frac{2}{3}|E(G)|$ and $|E_2| = \frac{1}{3}|E(G)|$.

Proof. It is easy to show that given $g \in G$, the function $\phi_g : V(G) \to V(G)$ given by $v \mapsto gv$ is a graph automorphism. Let $v_1, v_2 \in V(G)$. Lety e_1 denote the edge (v_1, av_1) and let e_2 denote (v_1, bv_1). Let e_3, e_4 denote the edges (v_2, av_2) and (v_2, bv_2) respectively. Then we find that $\phi_{v_2 v_1^{-1}}(e_1) = e_3$ and $\phi_{v_2 v_1^{-1}}(e_2) = e_4$. This shows that all of the edges in E_1 are equivalent, as are all of the elements in E_2. It remains to show that no edge of the form (v, av) is similar to any edge of the form (v, bv).

To do this we note that any edge of the form (v, bv) is contained in precisely two different $2x$-cycles; specifically $\mathcal{O}_v$ and $\mathcal{O}_{av}$. By contrast, edges of the form (v, av) are contained in only one cycle of length $2x$; specifically $\mathcal{O}_{av}$. We conclude that there is no graph automorphism mapping an element of E_1 to an element of E_2. Since E_1 and E_2 partition $E(G)$, we see that we have found all of the equivalence classes.

Finally, observe that every vertex is the endpoint of two edges of the form (v, av) but is the endpoint of only one edge of the form (v, bv). This shows that $|E_1| = \frac{2}{3}|E(G)|$ and $|E_2| = \frac{1}{3}|E(G)|$ and the proof is complete. □

It follows that $ind(G) = \frac{3}{2}$. So by the inequality of proposition 11, we have

$$i(G) \geq \frac{1}{3D}.$$

6 Conjectures

The usefulness of the results given above depend partly on the truth or falsity of the following conjectures. If they are true, then there is hope of finding formulas for the isoperimetric numbers of an important family of graphs.

Conjecture 1. *The graphs $G_{n,x}$ have an isoperimetric set such that every $v \in S$ is part of a cycle in S.*

Conjecture 2. *The graphs $G_{n,x}$ have an isoperimetric set such that every edge with both endpoints in S is contained in a cycle in S.*

Conjecture 3. *The graphs $G_{n,x}$ have an isoperimetric set S that can be constructed as a union of cycles of shortest length.*

Conjecture 4. *Every isoperimetric set of the graphs $G_{n,x}$ is expressible as the union of cycles of shortest length.*

These conjectures clearly increase in strength as you go down the list. Proving conjecture four would imply the other three. Also, all of these conjectures are clearly true for the Cayley graphs associated to D_{2n}. There seems to be more results to be discovered here, and I hope that others will be encouraged to take up the challenge.

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