Kirkman Triple Systems of Orders 27, 33, and 39

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Abstract

In the search for doubly resolvable Kirkman triple systems of order v, systems admitting an automorphism of order $(v-3)/3$ fixing three elements, and acting on the remaining elements in three orbits of length $(v-3)/3$, have been of particular interest. We have established by computer that 100 such Kirkman triple systems exist for $v = 21$, 90,598 for $v = 27$, at least 4,494,390 for $v = 33$, and at least 1,626,684 for $v = 39$. This improves substantially on known lower bounds for numbers of Kirkman triple systems. We also establish that the KTS(27)s so produced yield 47 nonisomorphic doubly resolved KTS(27)s admitting the same automorphism.

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1 Introduction

A *Steiner triple system* of *order* v, denoted STS(v), is a pair $(V, \mathcal{B})$, where V is a set of v *elements*, and $\mathcal{B}$ is a set of 3-element subsets of V called *triples* or *blocks*, so that every 2-element subset of V occurs in precisely one triple of $\mathcal{B}$. Steiner triple systems have been extensively investigated; see [4].

A *parallel class* in an STS(v) $(V, \mathcal{B})$ is a set of disjoint triples whose union is the set V; a parallel class therefore contains $v/3$ triples, and hence an STS(v) having a parallel class can exist only when $v \equiv 3 \pmod 6$. When the entire block set $\mathcal{B}$ can be partitioned into parallel classes, such a partition $\mathcal{R}$ is called a *resolution* of the STS, and the STS is *resolvable*. If $(V, \mathcal{B})$ is an STS(v) and $\mathcal{R}$ is a resolution of it, then $(V, \mathcal{B}, \mathcal{R})$ is a *Kirkman triple system*, and $(V, \mathcal{B})$ is its *underlying* STS. The distinction between resolvable STSs and KTSs is that a resolvable STS may underlie many nonisomorphic KTSs, since in a KTS the specific resolution is given.

If $(V, \mathcal{B})$ and $(X, \mathcal{D})$ are STSs, an *isomorphism* from $(V, \mathcal{B})$ to $(X, \mathcal{D})$ is a one-to-one mapping π from V to X for which $\{x, y, z\} \in \mathcal{B}$ if and only if $\{\pi(x), \pi(y), \pi(z)\} \in \mathcal{D}$. The systems are *isomorphic* if there is at least one isomorphism from one to the other, and *nonisomorphic* otherwise. Extending this to Kirkman triple systems, we require an isomorphism to preserve parallel classes, i.e. to map all triples of a parallel class of the first system to triples of a parallel class of the second. An *automorphism* is an isomorphism from a system to itself. The set of automorphisms forms a group under composition, the *automorphism group* of the system. The *order* of the automorphism group is the number of automorphisms which it contains, while the *order of an automorphism* is the smallest positive number of times that it can be applied in order to obtain the identity map.

A parallel class contains $v/3$ triples, and hence a resolution $\mathcal{R}$ consists of $r = (v-1)/2$ parallel classes, $\mathcal{R} = \{R_1, \ldots, R_r\}$. A parallel class T is *orthogonal* to the resolution $\mathcal{R}$ if $T \cap R_i$ contains zero or one triples for each $1 \leq i \leq r$. Let $\mathcal{R} = \{R_1, \ldots, R_r\}$ and $\mathcal{T} = \{T_1, \ldots, T_r\}$ be resolutions of the same STS. These two resolutions are *orthogonal* if the number of triples in $R_i \cap T_j$ is either zero or one for all $1 \leq i, j \leq r$. When a system has two orthogonal resolutions, it is *doubly resolvable*.

Kirkman [8] first asked about the existence of Kirkman triple systems in 1850, and solved the case when $v = 15$ (the *Kirkman 15-schoolgirl problem*). Ray-Chaudhuri and Wilson [13] published the first solution to the existence question for KTSs for all $v \equiv 3 \pmod 6$.

There is a unique STS(9) up to isomorphism, and it is resolvable. Indeed, it underlies a unique KTS(9). Of the eighty nonisomorphic STS(15)s, four are resolvable; together they underlie seven nonisomorphic KTS(15)s. The catalogue of seven KTS(15)s was presented by Woolhouse [14, 15] in 1862–63, although the systems themselves were known prior to that time. The KTS(9) and seven KTS(15)s do not admit an orthogonal resolution, and so no STS(v) is doubly resolvable for $v < 21$.

The determination of Kirkman triple systems of the next order, $v = 21$, has remained far from complete, although all KTS(21)s with nontrivial automorphism group have now been enumerated [2]. There are at least 63,745 nonisomorphic KTS(21)s, a substantial increase from the 192 previously known [5].

Doubly resolvable STS(v)s do not exist when $v \in \{9, 15\}$ but do exist for all $v \geq 21$ with $v \equiv 3 \pmod 6$ with 23 possible exceptions [3]. The smallest possible exception occurs when $v = 21$, so that the smallest known (nontrivial) doubly resolvable STS(v) has $v = 27$.

2 A special automorphism

Fuju-Hara and Vanstone [6] and Mathon and Vanstone [11] observed that the doubly resolved KTS(27) from the affine geometry admits an automorphism of order 8 fixing three points, and mapping the rest in three cycles of length 8; hence they suggested the study of KTS(v)s admitting an automorphism with a similar structure, of order $(v-3)/3$, fixing three elements. Centore and Vanstone [1] established that no doubly resolvable KTS(21) exists admitting such an automorphism.

For $v = 6t+3$, let $V = (\mathbb{Z}_{2t} \times \{0,1,2\}) \cup \{\infty_0, \infty_1, \infty_2\}$. (We often write

x_i for $(x,i) \in \mathbb{Z}_{2t} \times \{0,1,2\}$.) We suppose that there is an automorphism of order $2t$ fixing the three elements in $\{\infty_0, \infty_1, \infty_2\}$, and developing the remaining points modulo $(2t, -)$. A KTS($6t+3$) has $3t+1$ parallel classes, and we suppose that the automorphism of order $2t$ induces orbits of length $2t$, t, and 1 on the parallel classes. We further suppose that the only triple fixed by the automorphism is the infinite block $\{\infty_0, \infty_1, \infty_2\}$.

The parallel class fixed by the automorphism can be represented by a single triple, which is necessarily of the form $\{x_0, y_1, z_2\}$; the parallel class contains all translates of this triple together with the infinite block. Since the automorphism has order $2t$, the orbit of length t on the parallel classes must be fixed by the tth power of the automorphism. Thus each parallel class in the orbit must be fixed by the addition of $(t, -)$ to each non-infinite element, and hence whenever $\{i,j,k\} \subseteq \{0,1,2\}$ and $\{x_i, y_j, z_k\}$ is in a parallel class of this orbit, we find also $\{(x+t)_i, (y+t)_j, (z+t)_k\}$ in the same parallel class. In order to place pairs of the form $\{x_i, (x+t)_i\}$ for $0 \le x < t$ and $i \in \{0,1,2\}$ into triples, observe that such pairs lie in an orbit of t pairs and hence must appear in the triples of parallel classes in the orbit of length t. Indeed, we may suppose without loss of generality that the xth parallel class of this orbit contains the triples $\{\infty_i, x_i, (x+t)_i\}$ for $i \in \{0,1,2\}$. Hence to determine the orbit of t parallel classes, we can specify $t-1$ triples so that adjoining the $t-1$ triples obtained by addition of $(t, -)$, and adjoining the three triples containing infinite points, we obtain a parallel class.

The orbit of $2t$ parallel classes has ∞_0 appearing with pairs of the form $\{x_1, y_2\}$; a similar constraint holds for the two other infinite elements.

In order to make the determination of KTSs admitting such an automorphism feasible, we first enumerate all possible patterns for the second coordinate (from $\{0,1,2\}$) for the non-infinite elements. By taking into account the number of pairs of each type, a relatively small set of such patterns exists. For $v = 21$, for example, there is only one pattern. For $v = 27$, there are seven patterns, named as in Table 1.

Case	Orbit of $2t$ PCs						Orbit of t PCs		
1a	001	001	022	022	112	112	011	122	002
1b	001	001	022	022	112	112	001	022	112
1c	001	002	011	022	112	122	001	112	022
3a	111	001	002	022	122	012	001	122	012
3b	001	001	122	122	012	012	111	002	022
7a	012	012	012	012	012	012	000	111	222
7b	000	111	222	012	012	012	012	012	012

Table 1: Patterns for $v = 27$

Adjoining triples with infinite elements, and adjoining the fixed parallel class, we can then proceed by backtracking to determine all assignments to the first coordinate

Group Order	Number in Case							Total Number
	1a	1b	1c	3a	3b	7a	7b	
8	2744	1900	24110	49680	8448	1394	1143	89419
16	42	42		223	560	111	75	1053
24	2	1						3
32					31	23		54
48	2	1					3	6
72		4				24	1	29
96						4		4
144				1		7	1	9
288					1	3		4
648						6		6
1296						6		6
1944		1						1
2592						1		1
3888		1					1	2
303264						1		1
total	2790	1950	24110	49904	9040	1580	1224	90598

Table 2: Certain KTS(27)s

(elements of $\mathbb{Z}_{2t}$) which lead to a KTS($6t + 3$). We completed this procedure for $v \leq 27$. When $v = 21$, the single pattern leads to the 100 nonisomorphic KTS(21)s having an automorphism of order six. When $v = 27$, each of the seven patterns leads to solutions, yielding 90,598 nonisomorphic KTS(27)s. The numbers and automorphism group orders in each case are detailed in Table 2.

Of most interest is that each of the designs constructed not only has an automorphism group whose order is a multiple of eight, but the stronger property that its automorphism group has a *cyclic* subgroup of order eight. Replacing $\mathbb{Z}_8$ by $\mathbb{Z}_4 \times \mathbb{Z}_2$ or by $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ may lead to more nonisomorphic KTS(27)s. It is also of interest to determine whether a KTS(27) exists that has three orthogonal resolutions. We examined systems which admit a pair of orthogonal resolutions, both having the structure preserved under an automorphism consisting of three 8-cycles and three fixed elements. An example is shown in Table 3, using ι in place of ∞. We found 47 nonisomorphic examples of this type, arising from 15 nonisomorphic STS(27)s. For each STS(27), we list its automorphism group order and the number of doubly resolved STS(27)s of this type which it underlies: (303264,11), (96,5), (32,7), (32,2), (32,1), (16,5), (16,2), (16,1) twice, (8,4) twice, and (8,1) four times. All of the doubly resolved STS(27)s found have an automorphism group of order eight. We determined for each whether it admits a third resolution of any type which is orthogonal to the two specified resolutions, in an attempt to find a triply resolvable STS(27). However, none of the 47 nonisomorphic doubly resolved STS(27)s admits a third orthogonal resolution.

$0_2 5_2 6_2$	$0_1 1_1 3_1$		$0_0 1_0 3_0$	$2_0 4_1 3_2$	$6_0 2_1 4_2$		$5_0 6_1 7_2$		$7_1 2_2 \infty_0$		$7_0 5_1 \infty_2$	$4_0 1_2 \infty_1$
$1_0 2_0 4_0$	$1_2 6_2 7_2$	$1_1 2_1 4_1$		$3_0 5_1 4_2$	$5_0 2_2 \infty_1$	$7_0 3_1 5_2$		$6_0 7_1 0_2$		$0_1 3_2 \infty_0$		$0_0 6_1 \infty_2$
	$2_0 3_0 5_0$	$2_2 7_2 0_2$	$2_1 3_1 5_1$	$4_0 6_1 5_2$	$1_0 7_1 \infty_2$	$6_0 3_2 \infty_1$	$0_0 4_1 6_2$		$7_0 0_1 1_2$		$1_1 4_2 \infty_0$	
$3_1 4_1 6_1$		$3_0 4_0 6_0$	$3_2 0_2 1_2$	$5_0 7_1 6_2$		$2_0 0_1 \infty_2$	$7_0 4_2 \infty_1$	$1_0 5_1 7_2$		$0_0 1_1 2_2$		$2_1 5_2 \infty_0$
$4_2 1_2 2_2$	$4_1 5_1 7_1$		$4_0 5_0 7_0$	$6_0 0_1 7_2$	$3_1 6_2 \infty_0$		$3_0 1_1 \infty_2$	$0_0 5_2 \infty_1$	$2_0 6_1 0_2$		$1_0 2_1 3_2$	
$5_0 6_0 0_0$	$5_2 2_2 3_2$	$5_1 6_1 0_1$		$7_0 1_1 0_2$		$4_1 7_2 \infty_0$		$4_0 2_1 \infty_2$	$1_0 6_2 \infty_1$	$3_0 7_1 1_2$		$2_0 3_1 4_2$
	$6_0 7_0 1_0$	$6_2 3_2 4_2$	$6_1 7_1 1_1$	$0_0 2_1 1_2$	$3_0 4_1 5_2$		$5_1 0_2 \infty_0$		$5_0 3_1 \infty_2$	$2_0 7_2 \infty_1$	$4_0 0_1 2_2$	
$7_1 0_1 2_1$		$7_0 0_0 2_0$	$7_2 4_2 5_2$	$1_0 3_1 2_2$		$4_0 5_1 6_2$		$6_1 1_2 \infty_0$		$6_0 4_1 \infty_2$	$3_0 0_2 \infty_1$	$5_0 1_1 3_2$
				$\infty_0 \infty_1 \infty_2$	$0_0 0_1 0_2$	$1_0 1_1 1_2$	$2_0 2_1 2_2$	$3_0 3_1 3_2$	$4_0 4_1 4_2$	$5_0 5_1 5_2$	$6_0 6_1 6_2$	$7_0 7_1 7_2$
$3_0 7_0 \infty_0$		$1_2 5_2 \infty_2$	$0_1 4_1 \infty_1$			$5_0 2_1 0_2$	$4_0 7_1 3_2$	$2_0 1_1 6_2$		$1_0 6_1 4_2$	$0_0 3_1 7_2$	$6_0 5_1 2_2$
$1_1 5_1 \infty_1$	$4_0 0_0 \infty_0$		$2_2 6_2 \infty_2$		$7_0 6_1 3_2$		$6_0 3_1 1_2$	$5_0 0_1 4_2$	$3_0 2_1 7_2$		$2_0 7_1 5_2$	$1_0 4_1 0_2$
$3_2 7_2 \infty_2$	$2_1 6_1 \infty_1$	$5_0 1_0 \infty_0$			$2_0 5_1 1_2$	$0_0 7_1 4_2$		$7_0 4_1 2_2$	$6_0 1_1 5_2$	$4_0 3_1 0_2$		$3_0 0_1 6_2$
	$4_2 0_2 \infty_2$	$3_1 7_1 \infty_1$	$6_0 2_0 \infty_0$		$4_0 1_1 7_2$	$3_0 6_1 2_2$	$1_0 0_1 5_2$		$0_0 5_1 3_2$	$7_0 2_1 6_2$	$5_0 4_1 1_2$	

Table 3: A doubly resolvable KTS(27)

A complete enumeration for larger values of v appears to be infeasible. According to [10], the number of known Kirkman triple systems is 192 for $v = 21$, 909 for $v = 27$, 28 for $v = 33$, and 88 for $v = 39$. Our results using this simple automorphism seemed to suggest that these numbers could be easily improved. Having completed the generation when $v = 27$, we therefore generated a large number of nonisomorphic KTSs for $v = 33$ and $v = 39$, using only one pattern. Even the restriction to a single pattern was not sufficient, and so we abandoned the search having produced 4,449,390 KTS(33)s and 1,626,684 KTS(39)s. On this basis, we expect that the actual numbers will be very much larger.

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