Perfect Distance-d Placements in 3-Dimensional Tori

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Abstract
A necessary and sufficient condition for the existence of a perfect distance-d placement in 3−dimensional tori is given for both the regular and the irregular cases.

1 Introduction

Let $k_1, ..., k_n \in N$, the set of natural numbers, and C_i be a cycle of length i. Then an n-dimensional torus $\mathcal{T} = C_{k_1} \times .. \times C_{k_n}$ is the direct product of n cycles of the respective lengths k_i. If $k_1 = ... = k_n = k$, then the torus is denoted by $\mathcal{T}(n, k)$. A set $\mathcal{L}$ of vertices of $\mathcal{T}$ is a perfect distance-d placement in $\mathcal{T}$ if each vertex of $\mathcal{T}$ is at Manhattan distance at most d from exactly one vertex in $\mathcal{L}$. If $2d + 1 > k_i$ for some $i \in \{1, 2, ..., n\}$ then the perfect distance-d placement is called irregular.

The question of the existence of perfect distance-d placements in tori $\mathcal{T}$ has been extensively studied. Most papers on the topic use the language of perfect Lee codes. It is not difficult to see that the existence of a perfect Lee d-error correcting code over Z_k^n, the n-fold cartesian product of the Z_k, is equivalent to the existence of a perfect distance-d placement in $\mathcal{T}(n, k)$. We have chosen the language of placements as our research was motivated by an application in computer science. Processing elements in a supercomputer communicate through an interconnection network. Tori have been used as topologies for these networks in several existing supercomputers among which Cray T3D and Cray T3E. Frequently there are a limited number of shared resources, such as I/O devices, that every processing element needs to access. A perfect distance-d placement of these resources minimizes maximum access time by the processing elements. This results in a better overall performance of the system.

The first paper in this area goes back 35 years, see [6]. The authors showed that a perfect Lee d-error correcting code over Z_k^n exists with an appropriate value of k, for

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(i) $n = 1$ and all d;
(ii) $n = 2$ and all d;
(iii) $d = 1$ and all n.

Moreover, they conjectured that:

Conjecture 1 *A perfect Lee d-error correcting code over Z_k^n exists only in the cases (i),(ii), and (iii).*

In [9] it was proved that there is no perfect Lee d-error correcting code over Z_k^n for $3 \leq n \leq 5, d \geq n-1, 2d+1 \leq k$, and for $n \geq 6, d \geq \frac{\sqrt{2}}{2}(n-\frac{3}{2}-\frac{\sqrt{2}}{4}), 2d+1 \leq k$. In [1] the author surveys other triples (n, d, k) for which a perfect Lee d-error correcting code over Z_k^n does not exist. If $2d + 1 > k$, then the code is called a perfect Lee code over small alphabet. In this case the non-existence of the code has been shown for some triples in [2] and [7].

In connection with the computer science application mentioned above, the problem has been studied in a more general setting when the sizes of the individual dimensions of a torus are not equal. In [8] the question of the existence of perfect distance-d placement is answered for 2-dimensional tori. In this paper we deal with the 3-dimensional case, that is, we consider perfect distance-d placements in a torus $\mathcal{T} = C_n \times C_m \times C_k$, $2 \leq n \leq m \leq k$. If $2d+1 > n$, then the placement is irregular.

It is easy to see that there is no perfect regular distance-$d, d \geq 2$, placement in a 3-dimensional torus, as otherwise the periodic repetition of such a placement would provide a perfect distance-d placement in $\mathcal{T}(3, t)$, t being any multiple of n, m, and k, contradicting a result mentioned above. It has been noted by several authors that there exists a perfect distance-1 placement in $\mathcal{T} = C_n \times C_m \times C_k$ if each of the numbers n, m, k is a multiple of 7. In such a case a perfect placement is obtained by periodically repeating the perfect distance-1 placement in $C_7 \times C_7 \times C_7$. In [4] it is conjectured:

Conjecture 2 *Let $n, m, k \geq 3$. There exists a perfect distance-1 placement in $\mathcal{T} = C_n \times C_m \times C_k$ if and only if each of n, m, and k is a multiple of 7.*

As to perfect irregular distance-1 placements in $\mathcal{T} = C_n \times C_m \times C_k$, a torus where at least one of n, m, k equals 2, it was shown in [8] that such a placement exists for $\{n, m, k\} = \{2, 3i, 6j\}$, where $i, j \in N$. In [3] it is proved that the condition is also a necessary one.

In Section 3 we prove Conjecture 2. In Section 4 we state a necessary and sufficient condition for the existence of a perfect irregular distance-$d, d \geq 2$, placement.

We finish this section with a conjecture. We strongly believe, although at this moment we are not able to prove it, that Theorem 5, which answers in the affirmative Conjecture 2, can be generalized for higher dimensions as well. Therefore we raise:

Conjecture 3 *Let $k_1, ..., k_n \geq 3$. There exists a perfect distance-1 placement in $\mathcal{T} = C_{k_1} \times .. \times C_{k_n}$ if and only if k_i is a multiple of k for each $i, 1 \leq i \leq n$, where k is a number such that there is a perfect Lee single error correcting code over Z_k^n.*

2 Preliminaries

In this section we introduce the necessary definitions and notation to facilitate our discussion.

For the reader's convenience we start with a definition of an n-dimensional torus $\mathcal{T} = C_{k_1} \times .. \times C_{k_n}$, the direct (cartesian) product of cycles of respective lengths $k_1, k_2, ..., k_n$. Then $\mathcal{T}$ has as its vertex set the set $\{(a_1, ..., a_n);\ a_i \in N,\ 0 \leq a_i \leq k_i - 1\}$, the i-the coordinate of any vertex is always taken *mod* k_i. Two vertices $P = (a_1, ..., a_n)$ and $R = (b_1, ..., b_n)$ are joined by an edge if there is a $j \in \{1, ..., n\}$ so that $a_i = b_i$ for $i \neq j$, and $|a_j - b_j| = 1$ or $k_j - 1$. The distance ρ of two vertices of $\mathcal{T}$ is their distance along edges of $\mathcal{T}$. As the distance of two vertices of a cycle $C = v_0 v_1 ... v_k v_0$ is $\rho(v_l, v_j) = \min(|l - j|, k - |l - j|)$ the distance of P and R is given by $\rho(R, P) = \sum_{i=1}^{n} \min(|a_i - b_i|, k_i - |a_i - b_i|)$.

Let P be a vertex of $\mathcal{T}$, and let r be a non-negative integer. Then the radius-r sphere of $\mathcal{T}$ centered at P, $R_r(P)$, is the set of all vertices of $\mathcal{T}$ at the distance $\leq r$ from P. Formally, $R_r(P) = \{V \in \mathcal{T}, \rho(P, V) \leq r\}$. If $V \in G = R_r(P)$ we will also say that the sphere G covers the vertex V.

Let $\mathcal{L}$ be a perfect distance-d placement. It follows immediately from the definition of $\mathcal{L}$ that:

(a) for every two vertices $V_1, V_2 \in \mathcal{L}$, $\rho(V_1, V_2) \geq 2d + 1$, which is equivalent to $R_d(V_1) \cap R_d(V_2) = \emptyset$.
(b) each vertex of $\mathcal{T}$ is covered by exactly one radius-d sphere centered at a vertex of $\mathcal{L}$.

In what follows we confine ourselves to the 3-dimensional case.

Let $\mathcal{T} = C_n \times C_m \times C_k$ and $0 \leq s \leq k-1$. Then by $\mathcal{S} = C_n \times C_m \times \{s\}$ we denote a subtorus of $\mathcal{T}$ induced by the vertices $\{(a_1, a_2, s), 0 \leq a_1 \leq n-1, 0 \leq a_2 \leq m-1\}$.

Consider a vertex $P = (a, b, s)$ of a 3-dimensional torus $\mathcal{T} = C_n \times C_m \times C_k$ and its subtorus $\mathcal{S} = C_n \times C_m \times \{c\}$. Then the intersection of the sphere $G = R_r(P)$ with $\mathcal{S}$ is either a radius-r' sphere $R_{r'}(P'), r' \leq r$, centered at $P' = (a, b, c)$ or it is an empty set. If $k \geq 2r$, we get, setting c gradually to $s - r, s - r + 1, ..., s - 1, s, s + 1, ..., s + r - 1, s + r$ spheres $R_0, R_1, ..., R_{r-1}, R_r, R_{r-1}, ..., R_1, R_0$ of radii $0, 1, ..., r - 1, r, r - 1, ...1, 0$. We denote the first and the last by $L(G)$, and $H(G)$, or simply L and H, respectively (low and high spheres). Note that for $k = 2r$ the L and the H spheres coincide.

Let $\mathcal{L}$ be a perfect distance-d placement in $\mathcal{T} = C_n \times C_m \times C_k$. Then the spheres centered at vertices in $\mathcal{L}$ cover all vertices in $\mathcal{T}$. Obviously, the covering provides an induced covering of $\mathcal{S} = C_n \times C_m \times \{c\}$, where each vertex in $\mathcal{S}$ is covered by exactly one sphere in $F = \{R_d(V) \cap \mathcal{S},\ V \in \mathcal{L}\}$.

The following straightforward lemma is a special case of Lemma 1 in [5].

Lemma 4 *Let G_1 and G_2 be two radius-0 spheres in F centered at V_1 and V_2, respectively, so that they are both L spheres or both H spheres. Then $\rho(V_1, V_2) \geq 2d + 1$.*

We note that there is a typo in Lemma 1 in [5]. It should be $d(X_u, Y_u) \geq 1+r_1+r_2$ instead of $d(X_u, Y_u) \geq 3 + r_1 + r_2$.

3 Perfect distance-1 placements in 3-dimensional tori

In this section we prove Conjecture 2.

Theorem 5 *Let $\mathcal{T} = C_n \times C_m \times C_k$ be a 3-dimensional torus, where $n, m, k \geq 3$. If there exists a perfect distance-1 placement in $\mathcal{T}$ then each of the numbers n, m, and k is a multiple of 7.*

As the converse statement to Theorem 5 is true as well, see Introduction, we have a necessary and sufficient condition for the existence of a perfect distance-1 placement in a 3-dimensional torus.

Let $\mathcal{L}$ be a perfect distance-1 placement in $\mathcal{T}$. Consider the covering of $\mathcal{S} = C_n \times C_m \times \{s\}$, induced by $\mathcal{L}$. Set $F = \{R_d(V) \cap \mathcal{S},\ V \in \mathcal{L}\}$. As $\mathcal{L}$ is a distance-1 placement, spheres in F are of radius 0 or of radius 1. To prove Theorem 5 we derive first some properties of F. For the reader's convenience, proofs of several lemmas and properties contain pictures depicting a part of the induced placement in $\mathcal{S}$. In these pictures we depict the placement in a way which is standard for depicting Lee codes. That is, each vertex is depicted by a unit square, two unit squares sharing a side depict neighboring vertices. A perfect distance-d placement is then a tiling by radius-d tiles. Two dimensional tiles of radii 0, 1 and 2as well as the sphere $R_4 \cap C_6 \times C_k \times \{t\}$, $t = 0, 1, 2, 3$, where $C_6 \times C_k \times \{t\}$ is a subtorus of $C_6 \times C_k \times C_4, k \geq 9$, are depicted in Figure 1. Further, in some figures the sign "up" or "down" indicates that to the left of the arrow we depict (a part) of the induced tiling of $\mathcal{S} = C_n \times C_m \times \{s\}$, while to the right of the arrow the tiling of $\mathcal{S} = C_n \times C_m \times \{s + 1\}$ or $\{s - 1\}$. The squares framed by dashed lines are vertices of $\mathcal{S}$ we refer to in proofs which might but does not have to be covered by radius-0 spheres.

Let $G = R_1(P) \in F$, $P = (a, b, s)$. Then the four vertices $(a + \varepsilon, b + \eta, s)$, $\epsilon, \eta \in \{-1, 1\}$ are called the corners of G.

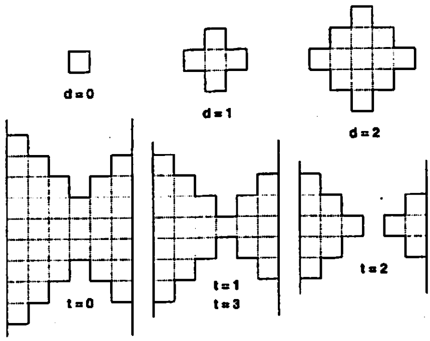

Figure 1: Tiles

Lemma 6 *If a corner* $(a+\varepsilon, b+\eta, s)$ *of* $G = R_1((a,b,s))$ *is covered by a radius-*0 *sphere in* F *then the corners* $(a-\varepsilon, b+\eta, s)$ *and* $(a+\varepsilon, b-\eta, s)$ *of* G *are covered by radius-*1 *spheres in* F.

Proof: Suppose that two neighboring corners of G are covered by radius-0 spheres in F. By Lemma 4, one is an L and the other is an H sphere. The rest of the proof is given in Figure 2. We note that in part (a) at least one vertex with the label $\times$ has to be covered by a radius-0 sphere, in (b) both those vertices have to be covered by radius-0 spheres. In both cases this contradicts Lemma 4. ■

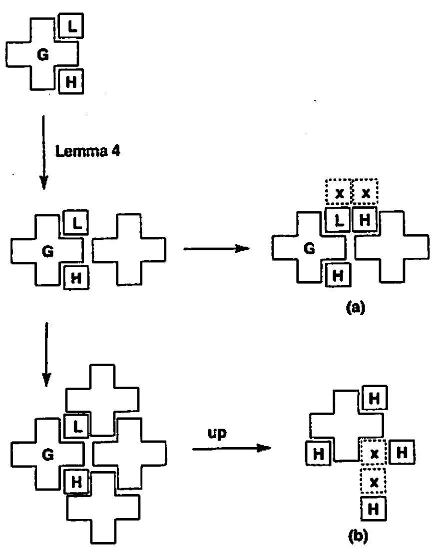

Figure 2: Proof of Lemma 6

Lemma 7 *If a corner* $(a+\varepsilon, b+\eta, s)$ *of* $G = R_1((a,b,s))$ *is covered by a radius-*1 *sphere in* F *then the corners* $(a-\varepsilon, b+\eta, s)$ *and* $(a+\varepsilon, b-\eta, s)$ *of* G *are covered by radius-*0 *spheres in* F.

Proof: Suppose that two neighboring corners of G are covered by two radius-1 spheres, see Figure 3.(a) and (b). Note that in (b), by Lemma 6, at least one of the two vertices with the label $\times$ is covered by radius-0 sphere leading to the same situation as in (a). In Figure 3.(c) and (d) we deal with the case when the vertex with the label z is covered by a radius-0 and a radius-1 sphere, respectively. Lemma 6 contradicts the case (c). In part (d), the four vertices with the label $\times$ would have to be covered by radius-0 spheres, contradicting Lemma 4. ■

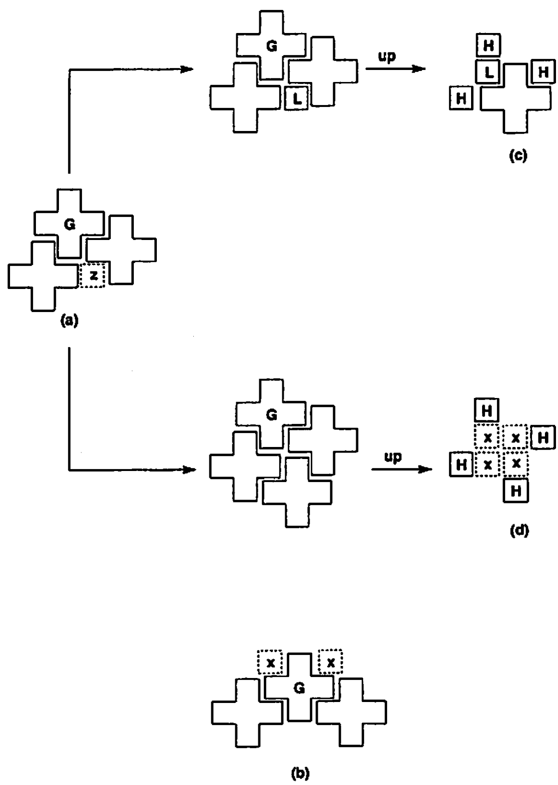

Figure 3: Proof of Lemma 7

Combining Lemma 6 and Lemma 7 we get:

Corollary 8 *Let $G = R_1((a,b,s)) \in F$. Then two opposite corners of G are covered by radius-0 spheres in F, and the other two opposite corners are covered by radius-1 spheres in F.*

Lemma 9 *If $G = R_0(A) \in F, A = (a,b,s)$, then (a,b,s) is a corner of exactly one radius-1 sphere of F.*

Proof: First suppose that A is a corner of two radius-1 spheres, G and G', see Figure 4. No vertex with the label $\circ$ in Figure 4 (a) can be the center of a radius-1 sphere which implies that $A' = (a,b,s-1)$ is an H sphere. Note that, by Lemma 4, no vertex with the label $\circ$ in Figure 4 (a), (b), and (c) can be covered by a radius-0 sphere. Further, by Lemma 7, the vertex with the label $\times$ in Figure 4 (d) cannot be covered by radius-1 sphere in F, hence, by Lemma 4, it is covered by an L sphere. However, then at least one of the two vertices with the label $\circ$ in Figure 4 (e) is covered by a radius-0 sphere in F, contradicting Lemma 4. Thus each vertex in S is a corner of at most one radius-1 sphere in F.

Suppose now that A is not a corner of any radius-1 sphere in F, see Figure 5. This means that no vertex with the label $\times$ is the center of a radius-1 sphere in F. Further, by Lemma 4, at least three of the four vertices with the label $\circ$ are covered by radius-1 spheres in F. However, then the vertex with the label z would be a corner of two radius-1 spheres in F. The statement follows. ∎

Corollary 10 *The number of radius-0 spheres in F equals twice the number of radius-1 spheres in F. In particular, the total number of vertices in S equals the total number of radius-1 spheres in F multiplied by 7.*

Proof: Let G be a radius-1 sphere in F. By Corollary 8, exactly two corners of G are covered by radius-0 spheres, G_1 and G_2. Further, by Lemma 9, $G_i, i = 1,2$, does not cover a corner of another radius-1 sphere of F. As every radius-0 sphere of F covers a corner of a radius-1 sphere of F, the first part of the statement follows. Since a radius-1 sphere and two radius-0 spheres cover in total 7 vertices, the proof is complete. ∎

Before finishing the proof of Theorem 5 we state one more lemma providing information on the location of centers of radius-1 spheres in F.

Lemma 11 *Let $G = R_1((a,b,s)) \in F$. If $G' = R_1((a+1,b+2,s)) \in F$, then $R_1((a+i,b+2i,s)) \in F$ for each $i \in N$.*

Proof: Let $G = R_1((a,b,s)) \in F$, $G' = R_1((a+1,b+2,s)) \in F$. By Lemma 7, the corner $(a+2,b+3,s)$ of G' is covered by T, a radius-1 sphere in F. Suppose, by the way of contradiction, that T is centered at $(a+3,b+3,s)$, see Figure 6. Then, by Lemma 6, the vertices with the label $\circ$ are covered by radius-0 spheres in F, which in turn implies, by Lemma 9, that the vertex with the label $\times$ is not the center of a radius-1 sphere in F. Hence, the vertex with the label $\cdot$ is covered by a radius-0 sphere. However, this contradicts Lemma 4. Therefore, the vertex

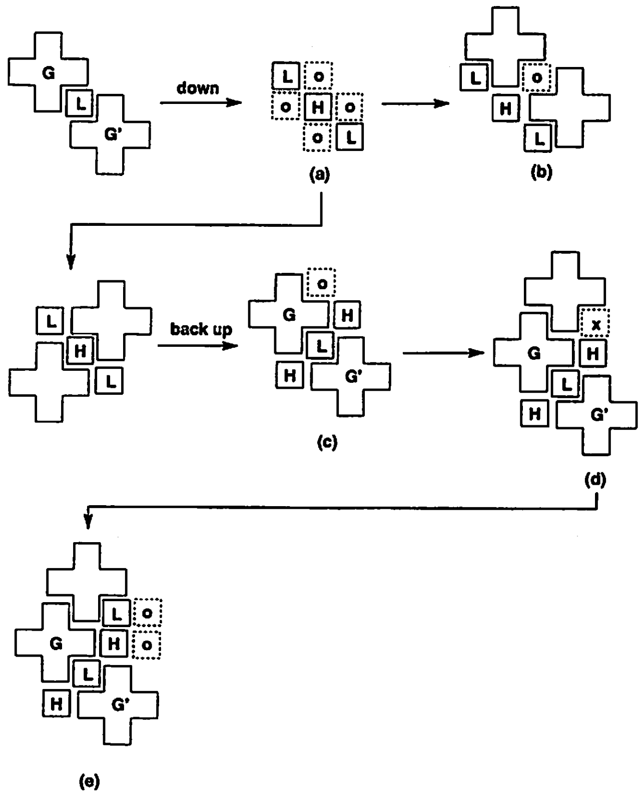

Figure 4: Proof of Lemma 9

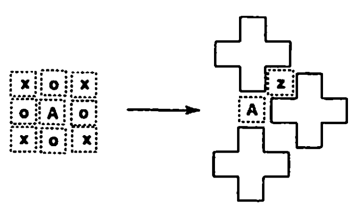

Figure 5: Proof of Lemma 9

$(a+2, b+3, s)$ is covered by a sphere centered at $(a+2, b+4, s)$. Repeating the above argument finishes the proof of the lemma. ■

Now we are ready to prove the theorem.

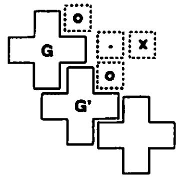

Figure 6: Proof of Lemma 11

Proof of Theorem 5: Let $G = R_1((a,b,s)) \in F$. Two corners of G are covered by radius-1 spheres in F. Without loss of generality, we may assume that $R_1((a+1,b+2,s)) \in F$ as well. Then, by Lemma 11, $R_1((a+i,b+2i,s)) \in F$ for each natural number i. As S is finite, $Z = \{R_1((a+i,b+2i,s)),\ i \in N\}$, the orbit of $R_1((a,b,s))$, is finite as well. The indices $a+i$ and $b+2i$ are taken *mod* n and *mod* m, respectively. Therefore, $|Z| = \frac{nm}{t\,\gcd(n,m)}$, where $t \in \{1,2\}$. (Let $n = 2^\alpha x, m = 2^\beta y$, where x, y are not divisible by 2. Then $t = 1$ for $\alpha \geq \beta$, otherwise $t = 2$). If Z does not contain all radius-1 spheres in F, then the set of all radius-1 spheres in F can be partitioned into orbits, each of the same size as Z. Thus, the total number of radius-1 spheres in F is $c|Z| = c\frac{nm}{t\,\gcd(n,m)}$.

The total number of vertices in S equals nm. At the same time, by Corollary 10, the total number of vertices of S equals 7 times the total number of radius-1 spheres in F, hence $nm = 7c\frac{nm}{t\,\gcd(n,m)}$. Hence, $\gcd(n,m) = 7\frac{c}{t}$, which in turn implies that both n and m are divisible by 7. By the same argument applied to $S = C_n \times \{s\} \times C_k$ we get that k is a multiple of 7 as well. The statement follows. ∎

Remark. The proof of Theorem 5 is "algebraic" is nature. As pointed out by a referee, there is a "geommetric" proof of the theorem as well. This proof would be based on the observation that any 1-dimensional torus S obtained from T by fixing two coordinates consists of repetitions of the patterrn $AAABHLC$, where different letters represent vertices of S covered by distinct spheres. We present here the details of the "algebraic" proof as it seems to us that the ideas of the proof might be generalized to prove Conjecture 3.

4 Perfect distance-d placements for $d \geq 2$

We recall, cf. [6] and [9], that there is no perfect regular distance-d placement, $d \geq 2$, for a 3-dimensional torus. In this section we state a necessary and sufficient condition for the existence of a perfect distance-d placement, $d \geq 2$, in an irregular case. For $d = 1$, the condition is stated in the introduction.

Theorem 12 *Let $1 < n \leq m \leq k$. Then there exists a perfect irregular distance-d placement, $d \geq 2$, in $T = C_n \times C_m \times C_k$ if and only if either (i) $1 < n \leq m$ are even, $n + m < 2d + 1$, and $k = 2t(2d + 1 - \frac{n}{2} - \frac{m}{2})$, $t \geq 1$, or (ii) $1 < n \leq$*

$m \le k < 2d+1$, and either $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + \lfloor \frac{k}{2} \rfloor \le d$, or $n+m \ge 2d+2$, n, m are even, and $k = 2(2d+1-\frac{n}{2}-\frac{m}{2})$.

Before proving the theorem we state a series of lemmas and propositions. As mentioned above, if all three numbers n, m, k are at least $2d+1$ then there is no perfect distance-$d, d \ge 2$, placement in $\mathcal{T}$. Now we consider the case when exactly one of the numbers n, m, k is smaller than $2d+1$.

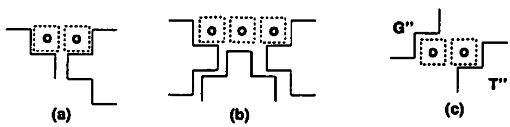

Figure 7: Proof of Proposition 13

Remark 13 *For $k = 2d$ each radius-0 sphere is both an L and an H sphere. Thus, for $k = 2d$, we can apply Lemma 4 to any two radius-0 spheres in F.*

Proposition 14 *Let $1 < n < 2d+1 \le m \le k$. Then there is no perfect distance-$d, d \ge 2$, placement in $\mathcal{T} = C_n \times C_m \times C_k$.*

Proof: Suppose for the sake of contradiction, that $\mathcal{L}$ is a perfect distance-d placement in $\mathcal{T}$, $O = (0,0,0) \in \mathcal{L}$. First we prove that then $n = 2d$. In this regard it suffices to show that $F = \{R_d(V) \cap \mathcal{S}, V \in \mathcal{L}\}$, where $\mathcal{S} = \{0\} \times C_m \times C_k$, contains a radius-0 sphere. To see this note that for the radius r of any sphere in F it is $d - \lfloor \frac{n}{2} \rfloor \le r \le d$. In order to show that F contains a radius-0 sphere, it suffices to show that there are in F two (three) spheres positioned as in Figure 7 (a) or in Figure 7 (b). In such a case at least one vertex with the label $\circ$ is covered by a radius-0 sphere.

Set $G = R_d(O)$, and $A = \{(0,i,j); i+j = d+1, 1 \le i \le d-1, 1 \le j \le d-1\} \subset \mathcal{S}$. The vertices of A are at distance $d+1$ from O, thus they are not covered by the sphere $G' = G \cap \mathcal{S}$. Let $T \in F$ be a sphere covering vertices, not necessarily all, of A. If T does not cover all vertices of A, then G' and T are positioned as in Figure 7 (a). Otherwise, all vertices of A are covered either by (i) a radius-d sphere $T' = R_d((0,y,z)) = T \cap \mathcal{S} \in F$. Clearly, $T = R_d((0,y,z))$. Then either $(y,z) = (d, d+1)$, or $(y,z) = (d+1, d)$. Consider now $\mathcal{S}' = \{1\} \times C_m \times C_k$. Let $G'' = G \cap \mathcal{S}'$, and $T'' = T \cap \mathcal{S}'$, see Figure 7 (c). Then both vertices with the label $\circ$ would have to be covered by L spheres, which contradicts Lemma 4; or by (ii) a radius-$(d-1)$ sphere in F centered at $(0,d,d)$. In this case consider the set of vertices $A' = \{(0,-i,j); i+j = d+1, 1 \le i \le d-1, 1 \le j \le d-1\}$. Let $R \in F$ be a sphere covering vertices, not necessarily all, of A'. As above, if R does not cover all vertices of A' or R is as in (i) we are done. So we are left with the case when R is a radius-$(d-1)$ sphere centered at $(0,-d,d)$. However, then G', R, and T' are positioned as in Figure 7 (b).

We have proved that $n = 2d$. Set $\mathcal{S} = C_n \times C_m \times \{2\}$. Let $G' = G \cap \mathcal{S}$, see Figure 8. The vertices with the label $\times$ are either covered by one or three spheres in $F = \{R_d(V) \cap \mathcal{S}, V \in \mathcal{L}\}$. In the former case, at least two vertices with the label $\circ$ in Figure 9 are covered by two L spheres in F, which contradicts Lemma 4 as the distance of these vertices is less than $2d+1$. To see that any radius-0 sphere covering a vertex with the label $\times$ or $\circ$ is an L sphere it suffices to note that $R_d(O)$ covers vertices which are in $C_n \times C_m \times \{1\}$ or in $C_n \times C_m \times \{0\}$ "below" $\times$ and $\circ$ vertices. In the latter case, at least one vertex, say C, with the label $\times$ in Figure 8 is covered by an L sphere in F and at least one vertex with the label $\circ$ is covered by an L sphere as well. Lemma 4 finishes the proof. ■

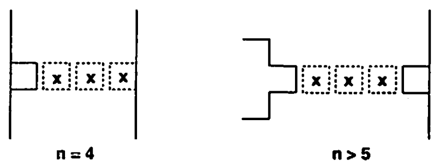

Figure 8: Proof of Proposition 13

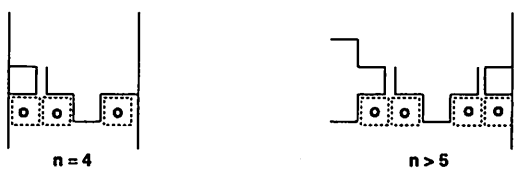

Figure 9: Proof of Proposition 13

Now we deal with the case when exactly two of the numbers of n, m, k are smaller than $2d+1$.

Lemma 15 *Let $1 < n < 2d+1, 1 < m < 2d+1$, $n \le k, m \le k$. If there exists a perfect distance-d placement $\mathcal{L}, |\mathcal{L}| \ge 2$, in $\mathcal{T} = C_n \times C_m \times C_k$, then both n and m are even numbers.*

Proof: Suppose that n is an odd number. Let $\mathcal{L}, |\mathcal{L}| \ge 2$, be a perfect distance-$d$ placement in $\mathcal{T}$, $O = (0,0,0) \in \mathcal{L}$. First we show that if $|\mathcal{L}| \ge 2$, then (*) $d - \frac{n-i}{2} \le k - (d - \frac{n-i}{2}), i \in \{-1,+1\}$. To prove (*) it suffices to show that $2d+1 \le k+n$. Assume first that $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor \le d$. Then $\rho(O,V) \le d$ for all vertices of $\mathcal{S}' = C_n \times C_m \times \{0\}$. Therefore, if $A = (x,y,z) \in \mathcal{L}, A \ne O$, then $z \ge d+1$ and $k - z \ge d+1$, that is, $k \ge 2d+1$, and (*) follows. Otherwise, $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor > d$, which implies $n + m \ge 2d+1$. As $k \ge n$, (*) follows in this case as well. Set $F = \{R_d(V) \cap \mathcal{S}, V \in \mathcal{L}\}$, $\mathcal{S} = C_n \times \{1\} \times C_k$. Then $G = R_d(O) \cap \mathcal{S}$ covers the vertices $V_1 = (\frac{n-3}{2}, 1, d - \frac{n-1}{2})$, $V_2 = (\frac{n+3}{2}, 1, d - \frac{n-1}{2})$, $V_3 = (\frac{n-1}{2}, 1, d - \frac{n+1}{2})$, and $V_4 = (\frac{n+1}{2}, 1, d - \frac{n+1}{2})$ as (*) implies $\rho(O, V_i) = d$ for $i = 1, ..., 4$. Further, G does not cover the vertices $W_1 = (\frac{n-1}{2}, 1, d - \frac{n-1}{2})$ and $W_2 = (\frac{n+1}{2}, 1, d - \frac{n-1}{2})$ as (*) implies $\rho(O, W_i) = d+1$ for $i = 1, 2$. See Figure 10, where W_i are the vertices with the label $\circ$. Clearly, $W_i's$ are covered by two different spheres in F.

At least one of $W_i's$, has to be covered by a radius-0 sphere in F. However, the existence of a radius-0 sphere in F is possible only when $m = 2d$. By symmetry, $V_1' = (\frac{n-3}{2}, 1, -(d-\frac{n-1}{2}))$, $V_2' = (\frac{n+3}{2}, 1, -(d-\frac{n-1}{2}))$, $V_3' = (\frac{n-1}{2}, 1, -(d-\frac{n+1}{2}))$, and $V_4' = (\frac{n+1}{2}, 1, -(d-\frac{n+1}{2}))$ are covered by G, while $Z_1 = (\frac{n-1}{2}, 1, -(d-\frac{n-1}{2}))$ and $Z_2 = (\frac{n+1}{2}, 1, -(d-\frac{n-1}{2}))$ are not. At least one of Z_1 and Z_2, the vertices with the label $\times$ in Figure 10, is covered by radius-0 sphere. However, for $i, j \in \{1, 2\}$, $\rho(W_i, Z_j) \leq 2d - n + 2 < 2d + 1$ since $n \geq 3$. With respect to Remark 13 this contradicts Lemma 4. ■

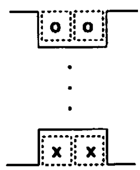

Figure 10: Proof of Lemma 14

Lemma 16 *Let $n \geq m > 1$ be even numbers, $n < 2d+1, m < 2d+1, k \geq 2d+1$. If there exists a perfect distance-d placement in $\mathcal{T} = C_n \times C_m \times C_k$, then $n+m < 2d+1$.*

Proof: Assume that $O = (0,0,0) \in \mathcal{L}$, and $n + m \geq 2d + 1$. As n, m are even, it is $n+m \geq 2d+2$. Let $\mathcal{S} = C_n \times \{d - \frac{n}{2} + 1\} \times C_k$. As $\frac{m}{2} \geq d - \frac{n}{2} + 1$, the sphere $G = R_d(O) \cap \mathcal{S}$ is of radius $d - (d - \frac{n}{2} + 1) = \frac{n}{2} - 1 > 0$ since $n+m \geq 2d+2, n \geq m$, implies $n \geq 4$. Further, G does not cover the vertex $W = (\frac{n}{2}, d - \frac{n}{2} + 1, 0)$ and vertices with the label $\times, \circ$, and $\cdot$ in Figure 11. However, G covers all vertices $(x, d - \frac{n}{2} + 1, 0)$ for $x \neq \frac{n}{2}$. Clearly, the sphere T of $F = \{R_d(V) \cap \mathcal{S}, V \in \mathcal{L}\}$ covering W cannot cover both vertices with the label $\times$ and vertices with the label $\circ$. Suppose T covers $\times$ vertices. Then T can possibly cover the vertex with the label $\cdot$, this would imply $k = 2d + 1$. In any case, two of the three vertices with the label $\circ$ or $\cdot$ are covered by radius-0 spheres in F. The existence of a radius-0 sphere in F implies $m = 2d$. This contradicts Lemma 4, cf. Remark 13. ■

Lemma 17 *Let n, m be even numbers so that $n + m < 2r + 1$, let $k \geq 2r + 1$. Then each radius-r sphere in $\mathcal{T} = C_n \times C_m \times C_k$ covers $nm(2r + 1 - \frac{n}{2} - \frac{m}{2})$ vertices of $\mathcal{T}$.*

Proof: Let $G = R_r((0,0,0)) \in \mathcal{T}$. First we count the number α of vertices covered by G in $\mathcal{S} = C_n \times \{0\} \times C_k$. According to $n < 2r + 1$, G covers in $\mathcal{S}$ $2r + 1 - 2i$ vertices with the first coordinate equal to i for $0 \leq i \leq \frac{n}{2}$, and $2r+1-2(n-i)$ vertices with the first coordinate equal to i for $\frac{n}{2}+1 \leq i \leq \frac{n}{2} - 1$. Set $x = 2r + 1$. Then, $\alpha = x + 2(x-2) + 2(x-4) + ... + 2(x - 2(\frac{n}{2} - 1)) + x - 2\frac{n}{2}$ $= nx - n - 4(1 + 2 + ... + (\frac{n}{2} - 1)) = n(x - \frac{n}{2}) = n(2r + 1 - \frac{n}{2})$.

Further, let $\mathcal{S}_i = C_n \times \{i\} \times C_k$. The sphere $G \cap \mathcal{S}_i$ is of radius $r - i$ for $1 \leq i \leq \frac{m}{2}$, and of radius $r - (m - i)$ for $\frac{m}{2} + 1 \leq i \leq m - 1$. As $n + m < 2r + 1$, $G \cap \mathcal{S}_i$ is

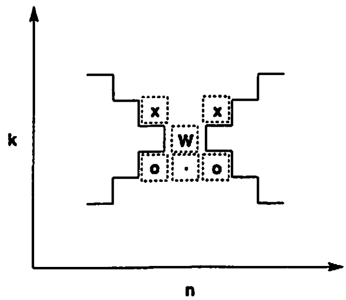

Figure 11: Proof of Lemma 15

of radius $\geq \frac{n}{2}$ for all $1 \leq i \leq m-1$. Therefore, G covers in S_i exactly $\alpha - i2n$ vertices for $1 \leq i \leq \frac{m}{2}$, and $\alpha-(m-i)2n$ vertices for $\frac{m}{2}+1 \leq i \leq m-1$. Hence, G covers in total $\alpha + 2(\alpha-2n)+2(\alpha-4n)+...+2(\alpha-2n(\frac{m}{2}-1))+\alpha-2n\frac{m}{2} = m\alpha - nm - 4n(1+2+...+(\frac{m}{2}-1)) = m\alpha - nm\frac{m}{2} = mn(2r+1-\frac{n}{2}) - nm\frac{m}{2} = mn(2r+1-\frac{n}{2}-\frac{m}{2})$. The proof is complete. ■

Now we are ready to prove the theorem in the case when exactly two numbers in $\{n, m, k\}$ are smaller than $2d+1$.

Proposition 18 *Let $1 < n \leq m < 2d+1 \leq k$. Then there exists a perfect distance-d placement in $\mathcal{T} = C_n \times C_m \times C_k$ if and only if n, m are even, $n+m < 2d+1$, and $k = 2s(2d+1-\frac{n}{2}-\frac{m}{2})$, $s \geq 1$.*

Proof: Let there be a perfect distance-d placement $\mathcal{L}$ in $\mathcal{T} = C_n \times C_m \times C_k$, $1 < n \leq m < 2d+1 \leq k$. By Lemma 15 and Lemma 16, n, m are even ($k \geq 2d+1$ implies $|\mathcal{L}| \geq 2$), and $n+m < 2d+1$. Each radius-d sphere covers $nm(2d+1-\frac{n}{2}-\frac{m}{2})$ vertices of $\mathcal{T}$. As the total number of vertices in $\mathcal{T}$ is nmk, k is a multiple of $2d+1-\frac{n}{2}-\frac{m}{2}$. To finish the necessary part of the proof we need to show that k is an even multiple of $2d+1-\frac{n}{2}-\frac{m}{2}$.

Assume that $O = (0,0,0) \in \mathcal{L}$. Let $G = R_d(O)$. The vertex $W = (\frac{n}{2}, \frac{m}{2}, t), t = d - \frac{n}{2} - \frac{m}{2} + 1$, is not covered by G ($\rho(O, W) = d+1$ follows from $k - t \geq t$) but the four vertices $(\frac{n}{2} \pm 1, \frac{m}{2} \pm 1, t)$ are. Therefore W has to be covered by a sphere centered at a vertex $Z = (\frac{n}{2}, \frac{m}{2}, t+d)$, that is, $Z \in \mathcal{L}$. Note that, for $0 \leq x \leq n$, $0 \leq y \leq m$, $0 \leq z \leq t+d-1$, the vertex $V = (x, y, z) \notin \mathcal{L}$ as $\rho(O, V) \leq \frac{n}{2}+\frac{m}{2}+2d-\frac{n}{2}-\frac{m}{2} = 2d$. Applying the same argument to Z we get that the vertex $U = (0, 0, 2(t+d)) \in \mathcal{L}$. Repeating the argument a sufficient number of times we get that all vertices in $\mathcal{L}$ are either of the type $(0, 0, 2i(t+d))$ or of the type $(\frac{n}{2}, \frac{m}{2}, (2i-1)(t+d))$. Clearly, there have to be in $\mathcal{L}$ the same number of vertices of both types, thus k is an even multiple of $t + d = 2d+1-\frac{n}{2}-\frac{m}{2}$.

Let $k = 2s(2d+1-\frac{n}{2}-\frac{m}{2})$. To prove the sufficiency of the condition, set $\mathcal{L} = \{(0, 0, 2i(2d+1-\frac{n}{2}-\frac{m}{2})), (\frac{n}{2}, \frac{m}{2}, (2i-1)(2d+1-\frac{n}{2}-\frac{m}{2})), i = 1, ..., s\}$, that is, $\mathcal{L}$ comprises $2s$ vertices. It is easy to check that the distance of any two vertices V, V' in $\mathcal{L}$ is at least $2d+1$, hence $R_d(V) \cap R_d(V') = \emptyset$. Each sphere $R_d(V)$ covers $nm(2d+1-\frac{n}{2}-\frac{m}{2})$ vertices implying that each vertex of $\mathcal{T}$ is at a distance at most d from exactly one vertex in $\mathcal{L}$. The proof is complete. ■

Now we prove the theorem for the case when all three numbers n, m, k are smaller than $2d+1$

Proposition 19 *Let $1 < n \le m \le k < 2d+1$. Then there exists a perfect distance-d covering of $\mathcal{T} = C_n \times C_m \times C_k$ if and only if either $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + \lfloor \frac{k}{2} \rfloor \le d$, or $n+m \ge 2d+2$, n, m are even, and $k = 2(2d+1-\frac{n}{2}-\frac{m}{2})$.*

Proof: Let $\mathcal{L}$ be a perfect distance-d placement in $\mathcal{T} = C_n \times C_m \times C_k$, and let $O \in \mathcal{L}$. Suppose first that $|\mathcal{L}| = 1$. It is $|\mathcal{L}| = 1$ if and only if $\rho(O, V) \le d$ for all $V \in \mathcal{T}$ if and only if $\max_{V \in \mathcal{L}} \rho(O, V) \le d$. Since $\max_{V \in \mathcal{L}} \rho(O, V) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + \lfloor \frac{k}{2} \rfloor$, we are done with this case. Before treating the case $|\mathcal{L}| \ge 2$ we state a simple technical result.

Claim 20 *Let $\mathcal{S} = C_n \times C_m \times \{t\}, 0 \le t \le k-1$, and let $O' = (0,0,t), A = (\frac{n}{2}, \frac{m}{2}, t)$. Then $\rho(O', V) + \rho(V, A) = \frac{n}{2} + \frac{m}{2}$ for each $V \in \mathcal{S}$. In particular, if $G = R_r(O') \in \mathcal{S}$, and $T = R_{r'}(A) \in \mathcal{S}$, $r' = \frac{n}{2} + \frac{m}{2} - r - 1$, then $G \cap T = \emptyset$, and $G \cup T = \mathcal{S}$.*

Proof: First suppose $1 \le x \le \frac{n}{2}$, $1 \le y \le \frac{m}{2}$. Then $\rho(O', V) + \rho(V, A) = (x+y) + (\frac{n}{2} - x + \frac{m}{2} - y) = \frac{n}{2} + \frac{m}{2}$. Further, suppose $\frac{n}{2} + 1 \le x \le n-1$, $1 \le y \le \frac{m}{2}$. Then $\rho(O', V) + \rho(V, A) = (n - x + y) + (x - \frac{n}{2} + \frac{m}{2} - y) = \frac{n}{2} + \frac{m}{2}$. The proof of the other two cases $1 \le x \le \frac{n}{2}$, $\frac{m}{2} + 1 \le y \le m-1$, and $\frac{n}{2} + 1 \le x \le n-1$, $\frac{m}{2} + 1 \le y \le m-1$ is analogous.

Let $V \in \mathcal{S}$. Then $\rho(O', V) > r$ if and only if $\rho(A, V) = \frac{n}{2} + \frac{m}{2} - \rho(O', V) \le \frac{n}{2} + \frac{m}{2} - r - 1$, that is, $G \cap T = \emptyset$, and $G \cup T = \mathcal{S}$. ■

So now suppose that $|\mathcal{L}| \ge 2$. Set $\mathcal{S} = C_n \times C_m \times \{0\}$. By Lemma 15, n, m are even numbers. If we had $n+m \le 2d$ then $G = R_d(O) \cap \mathcal{S} = \mathcal{S}$ since $\rho(O, V) \le \frac{n}{2} + \frac{m}{2} \le d$ for all $W \in \mathcal{S}$. This in turn implies that for all $V = (x, y, z) \in \mathcal{L}, V \ne O$, it is $z \ge 2d+1$ which contradicts the assumption $k < 2d+1$. Therefore, $n+m \ge 2d+2$. By Claim 20, the vertices of $\mathcal{S}$ not covered by G form a sphere $T = R_r(A)$, where $A = (\frac{n}{2}, \frac{m}{2}, 0)$, and $r = \frac{n}{2} + \frac{m}{2} - d - 1$ since $\rho(O, A) = \frac{n}{2} + \frac{m}{2}$. Note that $r < d$ as $n + m \le 4d$.

We show now that all vertices in T are covered by the same sphere, that is, $T = R_d(Z) \cap \mathcal{S}$, where $Z \in \mathcal{L}$. This is trivially satisfied if $\mathcal{S}$ is a radius-0 sphere. Otherwise, set $B = (\frac{n}{2} - r, \frac{m}{2}, 0)$, and $C = (\frac{n}{2} + r, \frac{m}{2}, 0)$. Then $B \ne C$ since $\frac{n}{2} > r$. Suppose by the way of contradiction that B and C are covered by spheres $T_B = R_{r'}(X_B) = R_d(Z_B) \cap \mathcal{S}$, $Z_B \in \mathcal{L}$, and $T_C = R_{r''}(X_C) = R_d(Z_C) \cap \mathcal{S}$, $Z_C \in \mathcal{L}$, respectively, $T_A \ne T_B$. Then $X_B = (b, \frac{m}{2}, 0)$, and $X_C = (c, \frac{m}{2}, 0)$, otherwise T_B (T_C) would cover at least one of two vertices $(\frac{n}{2} - r, \frac{m}{2} \pm 1, 0)$ (one of $(\frac{n}{2} + r, \frac{m}{2} \pm 1, 0)$) which are covered by G, contradicting that $\mathcal{L}$ is a perfect placement. However, then $\rho(Z_B, Z_C) \leq \frac{n}{2} + \lfloor\frac{k}{2}\rfloor \leq 2d$ contradicting again that $\mathcal{L}$ is a perfect cover. Thus, B and C are covered by the same sphere, which in turn implies that all vertices of T are covered by the same sphere. Hence $T = R_r(\frac{n}{2}, \frac{m}{2}, 0) = R_d(Z) \cap \mathcal{S}$, $Z = (\frac{n}{2}, \frac{m}{2}, z)$. Moreover, $T = R_d(Z) \cap \mathcal{S}$ and $G = R_d(O) \cap \mathcal{S}$ cover all the vertices of $\mathcal{S}$. Using the same argument as when proving that $n + m \geq 2d + 2$,we get that O and Z are the only vertices in $\mathcal{L}$, that is $|\mathcal{L}| = 2$. In other words, the spheres $R_d(O)$ and $R_d(Z)$ cover all vertices of $\mathcal{T}$. As $O \in \mathcal{S}$, $G = R_d(O) \cap \mathcal{S}$ is of radius d implying that both spheres $R_d(O) \cap \mathcal{S}_1, \mathcal{S}_1 = C_n \times C_m \times \{1\}$, and $R_d(O) \cap \mathcal{S}_{-1}, \mathcal{S}_{-1} = C_n \times C_m \times \{-1\}$ are of radius $d - 1$, which in turn implies that the two spheres $R_d(Z) \cap \mathcal{S}_1$, and $R_d(Z) \cap \mathcal{S}_{-1}$ are of the same radius $r + 1 = \frac{n}{2} + \frac{m}{2} - d$ as well. This is possible if and only if $z = k - z = d - (\frac{n}{2} + \frac{m}{2} - d - 1) = 2d + 1 - \frac{n}{2} - \frac{m}{2}$ (we recall that z is the third coordinate of Z). Hence $k = 2(2d + 1 - \frac{n}{2} - \frac{m}{2})$ and we are done with the necessary part of the proof.

To show the sufficiency, let $\mathcal{T} = C_n \times C_m \times C_k$, $n + m \geq 2d + 2$, n, m are even, and $k = 2(2d + 1 - \frac{n}{2} - \frac{m}{2})$. Set $\mathcal{L} = \{O, Z\}$, where $Z = (\frac{n}{2}, \frac{m}{2}, 2d + 1 - \frac{n}{2} - \frac{m}{2})$. Then $\rho(O, Z) = 2d + 1$. To finish the proof we need to show that $R_d(O) \cup R_d(Z)$ covers all vertices of $\mathcal{T}$. To give a better insight we provide two proofs of the fact. First, let $\mathcal{S} = C_n \times C_m \times \{t\}$, $t \leq 2d + 1 - \frac{n}{2} - \frac{m}{2}$. Then $G = R_d(O) \cap \mathcal{S}$ is a radius-$(d - t)$ sphere centered at $O' = (0, 0, t)$ while $T = R_d(Z) \cap \mathcal{S}$ is a radius-r, $r = d - (2d + 1 - \frac{n}{2} - \frac{m}{2} - t) = \frac{n}{2} + \frac{m}{2} + t - d - 1$, sphere, centered at $Z' = (\frac{n}{2}, \frac{m}{2}, t)$. For $2d + 1 - \frac{n}{2} - \frac{m}{2} \leq t \leq k - 1$, T is a radius-$(d - t)$ sphere and G is radius-$(\frac{n}{2} + \frac{m}{2} + t - d - 1)$ sphere. In both cases Claim 20 finishes the proof.

The second proof is based on a different idea. First we calculate the volume (= the number of vertices covered by) of the radius-d sphere T in $\mathcal{T} = C_n \times C_m \times C_k$, where $n + m \geq 2d + 2$, n, m are even, and $k = 2(2d + 1 - \frac{n}{2} - \frac{m}{2})$. It is well known that the volume of the radius-d sphere R in $\mathcal{R} = C_n \times C_m \times C_k$ with $2d + 1 \leq n \leq m \leq k$, that is, the radius-$d$ sphere in a regular torus, is $V(d) = (2d + 1)(\frac{2}{3}d(d + 1) + 1)$. Suppose that R is centered at $O = (0, 0, 0)$. Clearly, for $t = -d, ..., 0, .., d$, the sphere $R_r = \mathcal{S} \cap \{t\} \times C_m \times C_k$ is of radius-r, $r = 0, ..., d, ..., 0$. Consider a radius-d sphere T' in $C_n \times C_m \times C_k$, where $2d + 1 \leq m \leq k$, and $n < 2d + 1$, n is even. T' might be seen as obtained from R by removing vertices of the set $R(n) = \{V = (x, y, z) \in R;\ -d \leq x \leq -\frac{n}{2}$, or $\frac{n}{2} + 1 \leq x \leq d\}$, that is, $T' = R - R(n)$. Now we evaluate $R(n)$. For $-d \leq t \leq -\frac{n}{2}$, and $\frac{n}{2} + 1 \leq t \leq d$, $R_r = T' \cap \{t\} \times C_m \times C_k$ is a sphere of radius-r, $r = 0, ..., 2d - 2k, ...0$, Thus, $|R(n)| = V(2d - 2n)$. To calculate the volume of the sphere T it is sufficient to realize that T results from R by chopping off $R(n), R(m)$, and $R(k)$, where $R(m) = \{V = (x, y, z) \in R;\ -d \leq y \leq -\frac{m}{2}$, or $\frac{m}{2} + 1 \leq x \leq d\}$, $|R(m)| = V(2d - 2m)$, and $R(k) = \{V = (x, y, z) \in R;\ -d \leq z \leq -\frac{k}{2}$, or $\frac{k}{2} + 1 \leq x \leq d\}$, $|R(k)| = V(2d - 2k)$. The condition $n + m \geq 2d + 2$ guarantees that $R(n) \cap R(m) = \emptyset$. To see this, let $W = (x, y, z) \in R(n) \cap R(m)$. Then $|x| + |y| \geq \frac{n}{2} + \frac{m}{2} \geq d + 1$ implies $\rho(O, W) \geq d + 1$, hence $W \notin R$, a contradiction. As $n + m \geq 2d + 2$ implies $n + k \geq 2d + 2$, and $m + k \geq 2d + 2$, $R(n), R(m)$, and $R(k)$ are mutually disjoint. So we get $V(T) = V(2d - 2n) - V(2d - 2m) - V(2d - 2k)$. Substituting $k = 2d + 1 - \frac{n}{2} - \frac{m}{2}$, after tedious calculation, we get $V(T) = nm(2d + 1 - \frac{n}{2} - \frac{m}{2})$.

The torus $\mathcal{T} = C_n \times C_m \times C_k$ comprises $nmk = nm2(2d+1-\frac{n}{2}-\frac{m}{2})$ vertices. Since, for $Z = (\frac{n}{2}, \frac{m}{2}, 2d+1-\frac{n}{2}-\frac{m}{2})$, $\rho(O,Z) = 2d+1$, the spheres $R_d(O)$ and $R_d(Z)$ are disjoint. They cover in total $2V(\mathrm{T}) = nm2(2d+1-\frac{n}{2}-\frac{m}{2})$ vertices, hence $\mathcal{L} = \{O, Z\}$ is a perfect distance-d placement in $\mathcal{T}$.

Proof of Theorem 12: Combining Propositions 14, 18, and 19 gives the desired result. ∎

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