Investigating the Antimagic Square

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Abstract

The magic square is probably the most popular and well studied topic in recreational mathematics. We investigate a variation on this classic puzzle — the antimagic square. We review the history of the problem, and the structure of the design. We then present computational results on the enumeration and construction. Finally, we describe a construction for all orders.

1 Introduction

A magic square of order n is a $n \times n$ square array filled with the numbers $1, 2, \ldots, n^2$ in such a way that the sum of every row, every column and both diagonals is a constant. Figure 1 shows a third order magic square. Notice that the sum of any three entries in a straight line is 15.

While when $n = 3$ there is only one magic square (disregarding reflections and rotations) for larger n, the number of magic squares grows very rapidly. Since with volume comes flexibility, recreational mathematicians have asked what additional properties can be added to the magic squares they create. One finds in the literature terms like "pan-magic square", "doubly-super-magic squares" and so on used to describe various extra requirements the author has

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8	1	6
3	5	7
4	9	2

Figure 1: A magic square of order $n = 3$

				34
1	14	7	13	35
15	2	12	4	33
11	6	10	5	32
3	9	8	16	36
30	31	37	38	29

Figure 2: An antimagic square of order $n = 4$

imposed on the square. For example, a "pan-magic square" has the property that the entries on broken diagonals also sum to the magic constant. A doubly prime magic square requires that only primes be used as entries, and that the array (a_{ij}^2) is also magic in the usual sense. Another variation is that the product, not the sum, must be constant over rows, columns and diagonals.

The term antimagic square has also been used to mean some kind of twist on the magic square problem. If "magic" means all sums are the same, then perhaps antimagic should mean all sums must be different? In fact, this type of design is usually called a *heterosquare* and Madachy [2] shows that they are easy to construct [1].

While even more meanings have been attributed to the term "antimagic" besides the heterosquare confusion, we define this term as follows:

Definition 1 *Let A be an $n \times n$ square array filled with n^2 integers. Let $S(A)$ denote the set of row, column and diagonal sums of A. If $S(A)$ is a set of $2n+2$ consecutive integers, then A is a* general antimagic square *of order n.*

If we use consecutive numbers as entries (usually the natural numbers 1 to n^2), then A is an antimagic square *of* order n *(an* AMS(n)*).*

In the square shown in figure 2 the row, column, and diagonal sums form the set $S = \{30, 31, 37, 38, 29, 36, 32, 33, 35, 34\} = [29, 38]$

[1] **At least in two dimensions. For n odd, spiral increasing entries out from the center, as in**

7	6	5
8	1	4
9	2	3

Constructing the n-dimensional hetero-hypercube may be a more interesting problem.

2 Notation and Conventions

Since we deal exclusively with integers we let $[a, b] = \{i \in \mathbf{Z} |\ a \le i \le b\}$.

An integer array or matrix (a_{ij}) is a function $a : [1, n] \times [1, n] \to \mathbf{Z}$, where $a_{ij} = a(i, j)$ is the value of a at (i, j) and also the entry in the ij^{th} cell. If $B \subset [1, n] \times [1, n]$ then $f : B \to \mathbf{Z}$ is a square that is partially filled in, with the domain of f consisting of those cells that have entries. The i^{th} row sum of the partially filled in square is then

$$r_i(f) = \sum_{\substack{1 \le j \le n, \\ (i,j) \in Dom(f)}} f(i, j).$$

The j^{th} column sum $c_j(f)$, the main diagonal sum $d_1(f)$ and the back diagonal sum $d_2(f)$ of f are defined similarly.

Given a partially filled in square, f, that we are trying to complete to an antimagic square it is extremely useful to compare the entries of f with the average value of an entry in the finished square, $\alpha = \frac{1}{2}(n^2 + 1)$. The i^{th} *relative row sum* of f is

$$\overline{r}_i(f) = \sum_{\substack{1 \le j \le n, \\ (i,j) \in Dom(f)}} [f(i, j) - \alpha],$$

with similar definitions for $\overline{c}_j(f)$, $\overline{d}_1(f)$ and $\overline{d}_2(f)$. These relative sums tell us how much more or how much less f contributes to a particular row, column or diagonal compared to what an equal number of average entries would contribute. The relative row, column and diagonal sums for the AMS(4) in figure 2 are $0, \pm 1, \pm 2, \pm 3, \pm 4$ and -5, so naturally we also call these relative sums *differences*, as they are the differences between the actual sums that occur and the corresponding sums of average entries.

3 Structure

We first consider magic squares since their structure is related to that of the antimagic square. The following is well known.

Lemma 1 *If M is a magic square of order n, and c is the magic constant for M, then $c = \frac{1}{2}n(n^2 - 1)$.*

The constant c is halfway between $1+2+\ldots+n$ and $(n^2-n+1)+(n^2-n+1)+\ldots+(n^2-1)+n^2$. It also has the largest number of n-part compositions with distinct parts of all the numbers in $[1, n^2]$. The value c is also important for antimagic squares, as we will see.

The definition of a magic square stipulates a requirement: all sums must be equal to each other; but we can prove that this implies something more restrictive: all sums must equal $\frac{1}{2}n(n^2 - 1)$. In the same way, the definition of an antimagic square implies more restrictions on the design than it explicitly states. Specifically, while the definition merely requires the set of sums to be consecutive, we can show that there are actually only two possibilities for this set.

Theorem 2 *If A is an antimagic square of order n, S is the set of sums of A, and $c = \frac{1}{2}n(n^2+1)$ then either:*

1. $S = [c-n-1, c+n]$ *and* $(d_1+d_2) = 2c-n-1$ *(and A is called* negative*), or*

2. $S = [c-n, c+n+1]$ *and* $(d_1+d_2) = 2c+n+1$ *(and A is called* positive*).*

Proof: We will sum the elements of S in two different ways. Let s_{min} be the smallest element of S. Let r_i, c_i, and d_i be the i^{th} row, column and diagonal sum respectively. Firstly, S must be the set of $2n+2$ consecutive integers starting with s_{min}. Therefore $\sum_{s\in S} = \frac{1}{2}(2n+2)(2s_{min}+2n+1)$. Secondly, we know the elements of S to be row, column and diagonal sums so $\sum_{s\in S} = \sum_{i=1}^{n} r_i + \sum_{j=1}^{n} c_j + (d_1+d_2) = 2nc + (d_1+d_2)$. Since we're summing the same thing both ways, we've established that

$$(d_1+d_2) = \frac{1}{2}(2n+2)(2s_{min}+2n+1) - 2nc.$$

But both d_1 and d_2 are in the range $[s_{min}, s_{min}+2n+1]$ so

$$2s_{min}+1 \le (d_1+d_2) \le 2s_{min}+4n+1.$$

Substituting the expression for (d_1+d_2) into the inequality and simplifying yields

$$(c-n) - \frac{3}{2} \le s_{min} \le (c-n) + \frac{1}{2}.$$

Since all quantities are integers, we must have either: $s_{min} = c-n-1$ which implies $S = [c-n-1, c+n]$, or $s_{min} = c-n$ which implies $S = [c-n, c+n+1]$.

Moreover, if we substitute each s_{min} back into our expression for (d_1+d_2) and simplify, we get exactly what was claimed about the diagonals in each case. □

So, we find that in the antimagic square, the set S of sums must be roughly balanced around the magic constant c. The design is classified based on which way S is "lopsided." There is a natural operation that interchanges positive and negative antimagic squares. Given an $n \times n$ square $M = (m_{ij})$ filled with the integers $[1, n^2]$ the *complement* of M is the square $M' = (n^2+1-m_{ij})$. It is easy to verify that the complement of a negative square is a positive square, and vice versa. This result is useful in showing that there are no small antimagic squares. We write AMS$(n)^+$ and AMS$(n)^-$ to indicate a positive and a negative antimagic square (respectively).

Lemma 3 *There is no antimagic square of order 1, 2 or 3.*

Proof: Clearly no 1×1 square can be antimagic. As noted in [2] a 2×2 antimagic square has six distinct sums, whereas only the five distinct sums 3,4,5,6 and 7 can be formed by adding a pair of numbers from $\{1,2,3,4\}$, hence there is no 2×2 antimagic square.

We rule out an antimagic square of order 3 by case analysis.

For the sake of contradiction, assume such a square exists. Then there exists a AMS$(3)^-$ with sums $[11,18]$, which we label as

a	b	c
d	e	f
g	h	i

For definiteness let $d_1 = a+e+i$ and $d_2 = g+e+c$. By Lemma 2 we have $d_1+d_2 = 26$, so we need only rule out the two cases $(d_1,d_2) = (15,11)$ and $(d_1,d_2) = (14,12)$. Consider the case $(d_1,d_2) = (15,11)$. There is only one way to partition the remaining sums 12, 13, 14, 16, 17 and 18 into two groups of three that each add up to 45, so by symmetry we may take $\{r_1,r_2,r_3\} = \{12,16,17\}$ and $\{c_1,c_2,c_3\} = \{13,14,18\}$. Then $45 = r_2+c_2+d_1+d_2-3e = r_2+c_2+15+11-3e$, hence $r_2+c_2 = 19+3e \in \{22, 25, 28, 31, 34, 37, 40, 43, 46\}$. On the other hand $r_2 + c_2 \in \{12, 16, 17\} + \{13, 14, 18\} = \{25, 26, 30, 29, 30, 34, 30, 31, 35\}$ (a componentwise sum of two ordered sets that yields an ordered multiset). Thus $(r_2,c_2,e) = (12,13,2), (17,14,4), (16,18,5)$ constitute three subcases of the case $(d_1,d_2) = (15,11)$. Consider the subcase $(r_2,c_2,e) = (12,13,2)$. Since $e=2$ we have $g+c=9$, $d+f=10$, $b+h=11$ and $a+i=13$. Furthermore, $\{\{g,c\},\{d,f\},\{b,h\},\{a,i\}\}$ is a partition of $[1,9]-\{2\}$ into pairs. Below we tabulate the possibilities for each of the four pairs, and beneath these possibilities we tabulate the three partitions of $[1,9]-\{2\}$ into pairs that arise.

$\{g,c\}$	$\{d,f\}$	$\{b,h\}$	$\{a,i\}$	
$\{1,8\}$	$\{1,9\}$	$\{3,8\}$	$\{4,9\}$	
$\{3,6\}$	$\{3,7\}$	$\{4,7\}$	$\{5,8\}$	
$\{4,5\}$	$\{4,6\}$	$\{5,6\}$	$\{6,7\}$	
$\{1,8\}$	$\{3,7\}$	$\{5,6\}$	$\{4,9\}$	partition 1
$\{3,6\}$	$\{1,9\}$	$\{4,7\}$	$\{5,8\}$	partition 2
$\{4,5\}$	$\{1,9\}$	$\{3,8\}$	$\{6,7\}$	partition 3

Considering the last partition, there are 8 possible ways to have $\{g,c\} = \{4,5\}$, $\{b,h\} = \{3,8\}$ and $\{a,i\} = \{6,7\}$, namely $(g,c) = (4,5)$ or $(g,c) = (5,4)$, $(b,h) = (3,8)$ or $(8,3)$ etc., and none of these 8 possible orientations give $\{r_1,r_3\} = \{a+b+c, g+h+i\} = \{16,17\}$, as must be the case here. We summarize this failure by saying that partition 3 is not orientable for rows 1 and 3. An easy way to detect the non-orientability for rows 1 and 3 is to inspect the array

$$\begin{array}{ccc|c} 4 & 3 & 6 & 16 \\ 5 & 8 & 7 & 17 \end{array}$$

Numbers 16, 17 are r_1 and r_3 in some order. We form the sums 4+3+7, 4+3+6, 4+8+6, 4+8+7 to see if we can get 16 or 17. We cannot, therefore the last partition is not orientable for rows 1 and 3. Similarly, inspecting the arrays

$$\begin{array}{ccc|c} 3 & 4 & 5 & 16 \\ 6 & 7 & 8 & 17 \end{array} \qquad \begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 8 & 7 & 9 & 18 \end{array}$$

shows that partition 2 is not orientable for rows 1 and 3, and partition 1 is not orientable for columns 1 and 3 (respectively). This rules out the subcase $(r_2, c_2, e) = (12, 13, 2)$ of the case $(d_1, d_2) = (15, 11)$. The subcase $(r_2, c_2, e) = (17, 14, 4)$ yields two partitions, neither of which is orientable for rows 1 and 3, while subcase $(r_2, c_2, e) = (16, 18, 5)$ yields one partition, which *is* orientable for rows 1 and 3 and also for columns 1 and 3; but these essentially unique orientations are not compatible with each other.

In the other case, $(d_1, d_2) = (14, 12)$, there is again an essentially unique way to partition the remaining sums into row and column sums, so without loss of generality we take $\{r_1, r_2, r_3\} = \{11, 16, 18\}$ and $\{c_1, c_2, c_3\} = \{13, 15, 17\}$. This case also splits into three subcases. Each subcase yields just one partition, which is always found to be non-orientable for rows 1 and 3. We leave the details to the reader. □

The following lemma is useful in restricting the search for negative antimagic squares of order four.

Lemma 4 *If* $A = (a_{ij})$ *is an* AMS(4)$^-$ *then* $27 \leq a_{22} + a_{23} + a_{32} + a_{33} \leq 36$.

Proof: Let $N = a_{22} + a_{23} + a_{32} + a_{33}$ and let W be the sum of the entries in the AMS(4)$^-$, then

$$r_2 + r_3 + c_2 + c_3 + d_1 + d_2 - W = 2N.$$

By Lemma 2 we have $d_1 + d_2 - W = 63 - 136 = -73$. Since $\{r_i, c_i | 1 \leq i \leq 3\} = [29, 38] - \{29, 34\}, [29, 38] - \{30, 33\}$ or $[29, 38] - \{31, 32\}$ we find that $126 \leq r_2 + r_3 + c_2 + c_3 \leq 146$. It follows that $53 \leq 2N \leq 73$, hence $27 \leq N \leq 36$. □

Figure 3 shows a square that achieves one of the extremes of Lemma 4 and a square that can never be completed to a negative antimagic square (although it might be possible to complete it to a positive square).

We now consider the group of symmetries of 4×4 squares. Let a be a 4×4 matrix with entries a_{ij}. Rotation clockwise by 90°, transposition and complementation are described by the respective formulae $(Ra)_{ij} = a_{5-j,i}$, $(Ta)_{ij} = a_{ji}$, and $(Ca)_{ij} = 17 - a_{ij}$. There is also the $1-4$ *exchange* E_{14} and the $2-3$ *exchange* E_{23} given by

8	2	16	3
4	15	6	13
12	14	1	10
7	5	11	9

	15	6	
	9	7	

Figure 3: An AMS(4)$^-$ with maximum N, and an incompletable square.

$$E_{14}(a) = \begin{array}{|c|c|c|c|}\hline a_{44} & a_{42} & a_{43} & a_{41} \\ \hline a_{24} & a_{22} & a_{23} & a_{21} \\ \hline a_{34} & a_{32} & a_{33} & a_{31} \\ \hline a_{14} & a_{12} & a_{13} & a_{11} \\ \hline \end{array} \qquad E_{23}(a) = \begin{array}{|c|c|c|c|}\hline a_{11} & a_{13} & a_{12} & a_{14} \\ \hline a_{31} & a_{33} & a_{32} & a_{34} \\ \hline a_{21} & a_{23} & a_{22} & a_{24} \\ \hline a_{41} & a_{43} & a_{42} & a_{44} \\ \hline \end{array}$$

We let $\mathcal{G}_4 = \langle R, T, C, E_{14}, E_{23} \rangle$ be the group generated by these five operations. The group $\mathcal{G}_4$ acts on the set of all antimagic squares of order 4, taking a given antimagic square and creating many new ones. We leave it to the reader to compute the size of this group, the sizes of the orbits and how to unambiguously pick out an orbit representative:

Lemma 5 *The group $\mathcal{G}_4$ of symmetries of 4×4 antimagic squares is of order 32, all orbits are full, and under the action of $\mathcal{G}_4$ any* AMS(4) *can be reduced to a square $A = (a_{ij})$ such that*

1. *A is a negative antimagic square,*
2. $d_2 < d_1$,
3. $a_{32} < a_{23}$,
4. $a_{41} < a_{14}$,
5. $a_{11} < a_{44}$.

4 Computational Results

Computers proved indispensable in the research for this paper. Numerous programs were written in the C language to investigate this design. However, the limits of computation were quickly reached, as even enumerating all order AMS(5)'s was infeasible.

4.1 Enumeration

A backtracking algorithm was written to enumerate all antimagic squares of small orders, disregarding rotations and reflections. The results are summarized in this table and are compared to what is known about magic squares.

n	MS(n)	AMS(n)
1	1	0
2	0	0
3	1	0
4	880	1,198,840
5	≈320M[2]	very large

Table 1: Magic and Antimagic enumeration for small n

Although Madachy presents something very close to an order 3 antimagic square, (see [2], p.104) the program's exhaustive checking showed that no such design exists, in agreement with Lemma 3.

In the order 4 enumeration the symmetries in Lemma 5 and the inequality of Lemma 4 were exploited to speed up the program. Up to the full group of symmetries, the number of fully reduced AMS(4)'s is 299,710. As a double-check a simple (and much slower), unoptimized backtrack obtained 32*299,710 = 9,590,720 unreduced antimagic squares, in agreement with the faster program.

The order five antimagic squares proved too numerous to count. If the same fraction of order 5 squares are antimagic as order 4 squares are, there would be on the order of 10^{17} order five antimagic squares.

4.2 Probabilistic Methods

Given the large numbers of antimagic squares for $n > 3$ it is perhaps not so surprising to learn that the first examples of such squares where constructed by hand implementation of a probabilistic algorithm (see [2]).

A hill climbing algorithm of a slightly different flavour was implemented to create antimagic squares of large order. Starting with a large magic square, we swap entries within a row or column to produce sums offset from the magic constant c. For example if we swap two elements in the same row of a magic square, which differ by an amount d, we end up with $c + d$ as one new column sum and $c - d$ as another new column sum without disturbing the row sum. By finding mutually independent pairs of entries differing by d for each $0 < d \leq n$ and for each row and column, we can fix the row and column sums. Of course this is not always possible, but for large ($n > 15$) magic squares the computer could usually accomplish the task.

Next the program adjusts the diagonals. If the swaps in the previous step are chosen to avoid diagonal elements, then each diagonal still sums to the magic constant. If rows a and b are chosen at random and exchanged, and subsequently columns $n - a$ and $n - b$ are exchanged, then one diagonal can be adjusted without disturbing the other.

In this way, the computer produced antimagic squares with $[c - n, c - 1] \cup [c + 1, c + n]$ as row, and column sums, and with c and $c \pm n \pm 1$ as diagonal sums. The interested reader may visit [4] to obtain this and the other programs mentioned here.

5 Even Construction: Bordering Antimagic Squares

Let

$$B_n = \{1, n+2\} \times [1, n+2] \cup [1, n+2] \times \{1, n+2\},$$

so that B_n consists of the coordinates of all the border cells of a $(n+2) \times (n+2)$ square. The cells in B_n are naturally partitioned into pairs of *opposite cells.* The cells $(1,1)$, $(n+2, n+2)$ and $(1, n+2)$, $(n+2, 1)$ are opposite cells, and so are $(1, j)$, $(n+2, j)$ and $(i, 1), (i, n+2)$ for $2 \le i, j \le n+1$. A *bordering* of a square of order n is any bijection

$$f : B_n \to [1, 2n+2] \cup [n^2 + 2n + 3, (n+2)^2].$$

We will refer to the numbers in the interval $[1, 2n+2]$ as *small* and the numbers in $[n^2 + 2n + 3, (n+2)^2]$ as *large.*

The idea of course is that given a square of order n filled with the numbers $[1, n^2]$ we may add the constant $2n + 2$ to all the entries and then obtain the remaining entries from the bordering to get a square of order $n + 2$ filled with the numbers $[1, (n+2)^2]$.

We work with relative sums in this section, so the reader must pay close attention to where the domains of partially filled in squares come from, since two squares of different sizes have different average entries (the α's of section 2).

Definition 2 *A* border design *of* order n *is a bordering*

$$f : B_n \to [1, 2n+2] \cup [n^2 + 2n + 3, (n+2)^2]$$

such that the following conditions hold:

(A) *Rows* 1 *and* $n+2$ *and columns* 1 *and* $n+2$ *each contain* $(n+2)/2$ *large numbers and* $(n+2)/2$ *small numbers.*

(B) *The sums of the design are:*

$$\begin{array}{rclrclrcl} \overline{r}_1(f) & = & -(n+2), & \overline{r}_2(f) & = & 2, & \overline{r}_{n+2}(f) & = & n, \\ \overline{c}_1(f) & = & -(n+1), & \overline{c}_{n+2}(f) & = & n+1, & \overline{d}_2(f) & = & -2, \end{array}$$

and $\overline{d}_1(f) = \overline{r}_i(f) = \overline{c}_j(f) = 0$ *for all* $i \ne 1, 2, n+2$ *and* $j \ne 1, n+2$.

The next Lemma shows that a border design is just the right thing to make larger antimagic squares from smaller ones.

Lemma 6 *If there exists an* $\mathrm{AMS}(n)^-$ *with difference 0 on a diagonal and there exists a bordering design of order* n*, then there exists an* $\mathrm{AMS}(n+2)^-$ *with difference 0 on a diagonal.*

Proof: Let A be an AMS$(n)^-$ with difference 0 on one of the diagonals. By Lemma 2 we know that the other diagonal has difference $-(n+1)$. Without loss of generality we may assume that $\overline{d}_1(A) = 0$, and $\overline{d}_2(A) = -(n+1)$ and $\overline{r}_k(A) = n$. By permuting the rows of the border design we obtain a new bordering function f' such that

$$\begin{array}{rclrclrcl} \overline{r}_1(f') & = & -(n+2), & \overline{r}_k(f') & = & 2, & \overline{r}_{n+2}(f') & = & n, \\ \overline{c}_1(f') & = & -(n+1), & \overline{c}_{n+2}(f') & = & n+1, & \overline{d}_2(f') & = & -2, \end{array}$$

and $\overline{d}_1(f') = \overline{r}_i(f') = \overline{c}_j(f') = 0$ for all $i \neq 1, k, n+2$ and $j \neq 1, n+2$.

Define a $(n+2) \times (n+2)$ matrix A' by

$$A'_{ij} = \begin{cases} A_{ij} + (2n+2) & \text{if } (i,j) \in [2, n+1] \times [2, n+1] \\ f'(i,j) & \text{if } (i,j) \in B_n \end{cases}$$

Since $\overline{r}_i(A') = \overline{r}_i(A) + \overline{r}_i(f')$, with similar relationships for the column and diagonal sums, it is simple to check that A' is an AMS$(n+2)^-$ with difference 0 on the main diagonal. □

We now show how to construct some border designs.

Lemma 7 *There exists a border design of order n for each even $n \geq 4$.*

Proof: The proof is by induction on n. For $n = 4$ and $n = 6$ we exhibit the designs below:

27	9	30	1	34	4
33					6
8					29
2					35
5					32
31	28	7	36	3	10

60	55	4	52	11	51	12	7
58							9
6							59
2							63
62							3
1							64
8							57
56	10	61	13	54	14	53	5

We will show that the existence of a border design of order n implies the existence of a border design of order $n+4$

Let f be a border design of order n. Let h be the function

$$h : [1, 2n+2] \cup [n^2 + 2n + 3, (n+2)^2] \rightarrow [1, 2n+10] \cup [n^2 + 10n + 27, (n+6)^2]$$

$$h(k) = \begin{cases} k + 4 & \text{if } k \in [1, 2n+2] \\ k + 8n + 8 & \text{if } k \in [n^2 + 2n + 3, (n+2)^2] \end{cases}$$

Note that h maps small numbers to small numbers, and large numbers to large numbers. Also note that the first four and the last four numbers in the intervals $[1, 2n+10]$ and $[n^2+10n+27, (n+6)^2]$ are not in the image of h .

Define the border function

$$g : B_{n+4} \to [1, 2n+10] \cup [n^2+10n+27, (n+6)^2]$$

by the formulas

$$\begin{aligned}
g(1,j) &= h(f(i,j)), \quad 1 \le j \le n+1, && (1)\\
g(n+6,j) &= h(f(n+2,j)), \quad 1 \le j \le n+1, && (2)\\
g(1,n+2) &= 1 && (3)\\
g(1,n+3) &= (n+6)^2 - 2, && (4)\\
g(1,n+4) &= 2, && (5)\\
g(1,n+5) &= (n+6)^2 - 3. && (6)\\
g(n+6,j) &= (n+6)^2 + 1 - g(1,j), \quad n+2 \le j \le n+5 && (7)
\end{aligned}$$

$$\begin{aligned}
g(i,1) &= h(f(i,1)), \quad 1 \le i \le n+1, && (8)\\
g(i,n+6) &= h(f(i,n+2)), \quad 1 \le i \le n+1, && (9)\\
g(n+2,1) &= 2n+7, && (10)\\
g(n+3,1) &= n^2+10n+28 && (11)\\
g(n+4,1) &= 2n+8, && (12)\\
g(n+5,1) &= n^2+10n+27 && (13)\\
g(i,n+6) &= (n+6)^2 + 1 - g(i,1), \quad n+2 \le i \le n+5. && (14)\\
g(n+6,n+6) &= k(f(n+2,n+2)) && (15)
\end{aligned}$$

We claim that g is a border design of order $n+4$. Since h preserves the "largeness" or the "smallness" of a number the equations (1), (2), (9) and (15) contribute an equal number of large and small numbers to the rows. Equations (3) to (7) contribute two large and two small numbers to each row, hence there are an equal number of large and small numbers in the first and last rows. Similarly we can check that the columns also contain the same number of large and small numbers.

Note that the definition of a border design implies that in a pair of opposite border cells one cell contains a large number and the other cell contains a small number. This gives the relation $\overline{r}_i(g) = \overline{r}_i(f)$ for $2 \le i \le n+1$, so $\overline{r}_2(g) = 2$ and $\overline{r}_i(g) = 0$ for $3 \le i \le n+1$. In the first row we compute the relative sum $g(1,n+2) + g(1,n+3) + g(1,n+4) + g(1,n+5) = -4$, while the sum of entries in position $(1, n+6)$ and positions $(1,1)$ to $(1, n+1)$ is $\overline{r}_1(f) = -(n+2)$ (since these other entries come from the first row of f and f distributes them equally among large and small numbers). Therefore $\overline{r}_1(g) = -(n+2) - 4 = -(n+6)$, as desired. Having worked out the first row sum for g, the last row sum and the column and diagonal sums are just as easy to work out. We leave this for the reader.

The proof is now complete, since we may start with the border designs above and apply recursion to get border designs of all even orders $n \geq 4$. □

Theorem 8 *For each even $n \geq 4$ there exists an* AMS$(n)^-$ *with 0 as one of its diagonal sums.*

Proof: It suffices to exhibit one AMS$(4)^-$ with 0 as a diagonal sum. Then by Lemma 6 and Lemma 7 we may repeatedly border this square to obtain an AMS(n) for all even $n \geq 4$. Such a square is exhibited in figure 2. □

6 Odd Constructions

6.1 High and low order squares

Most types of magic squares conventionally use the numbers $[1, n^2]$ as entries, but equivalently we will use $[0, n^2 - 1]$. This allows us to think of a square as filled with all two digit numbers to the base n. We will construct two square designs, one to provide the most significant digit and the other to provide the least significant digit of each entry. Call the high and low order squares H and L respectively.

A latin square is *left semi-diagonal* if $a_{11}, a_{22}, \ldots, a_{nn}$ are all distinct, it is *right semi-diagonal* if $a_{1n}, a_{2.n-1}, \ldots, a_{n1}$ are all distinct. It is known that if we use semi-diagonal latin squares with the extra property that the non-latin diagonal sums to $\frac{1}{2}n(n-1)$ for each of the high and low order squares, then juxtaposing them in this way yields a magic square [1]. Can we make an antimagic square in a similar way, perhaps by manipulating the low order digits only?

Lemma 9 *If H is a semi-back-diagonal Latin square of order n with the property that the main diagonal sum is $\frac{1}{2}n(n-1)$ and if L is a general antimagic square orthogonal to H and if we form M by*

$$M = nH + L,$$

then M is an AMS(n).

Proof: Since H and L are orthogonal M has entries 0 to $n^2 - 1$. The i^{th} row sum of M is

$$r_i(M) = nr_i(H) + r_i(L) = nc + r_i(L).$$

Similarly the column and diagonal sums of M are nc plus the corresponding sum for L, so $S(M) = \{nc + \sigma \mid \sigma \in S(L)\}$. Since $S(L)$ consists of consecutive

									36
4	5	6	7	8	0	1	2	3	36
3	4	5	6	7	8	0	1	2	36
2	3	4	5	6	7	8	0	1	36
1	2	3	4	5	6	7	8	0	36
0	1	2	3	4	5	6	7	8	36
8	0	1	2	3	4	5	6	7	36
7	8	0	1	2	3	4	5	6	36
6	7	8	0	1	2	3	4	5	36
5	6	7	8	0	1	2	3	4	36
36	36	36	36	36	36	36	36	36	36

Figure 4: H_9 and its sums.

integers, so does $S(M)$. This shows M is an antimagic square . □

Let us place subscripts on H, L to indicate their size. We may obtain a suitable H_n by permuting rows and columns of the addition table of Z_n (n odd), as in Figure 4. We see in the figure that all diagonals parallel to the main diagonal (sloping left to right) are constant with $c = (n-1)/2$ on the main diagonal. Thus there are many ways to construct an H_n, n odd, satisfying the requirements of Lemma 9 by filling in the left-right diagonals with different integers from $[0, n-1]$ and putting c on the main diagonal.

Half of the construction is complete at this point and we proceed to the more difficult task of building an L_n.

6.2 Staircases of consecutive integers on the torus

We think of the low order square as a mapping $Z_n \times Z_n \rightarrow [0, n-1]$, that is, as a mapping of the torus to the integers $[0, n-1]$, so in the upcoming discussion all indices are to be taken modulo n. As we move across a row the first coordinate of (i, j) increases from 0 to $n-1$, and as we move down a column the second coordinate increases similarly. Let $|a|_n$ denote the smallest non-negative integer such that $|a|_n \equiv a \pmod n$

Definition 3 *A* staircase *on an order* n *matrix is a bijection from a set of cells* $\{(i,j) \mid n-i+j-1 = K\}$ *to the entries* $[0, n-1]$.

For $e, f \in [0, n-1]$, $0 < |d| \leq c$ *and* $p = \pm 1$ *we define one particular pair of staircases* $S^p_{d,e,f}$ *as:*

$$\begin{array}{llcll} \text{I.} & S^p_{d,e,f}(c-d+t, c+d+t) & = & |e+t|_n & \textit{for } 0 \leq t < n \\ \text{II.} & S^p_{d,e,f}(c-d-e-t, c+d-e-t+p) & = & |f+t|_n & \textit{for } 0 \leq t < n \end{array}$$

where $c = \frac{1}{2}(n-1)$, *the centre row and column coordinate.*

Visually, a pair of staircases $S^p_{d,e,f}$ is two diagonals of consecutive numbers zero to n, (I) ascending and (II) descending as we move down and right parallel to the main diagonal. Band (I) is distance d from the main diagonal, aligned so that element e intersects the back diagonal and so that zero is in the same row as element f in (II), either 1 unit to the right (p positive) or 1 unit to the left (p negative). Note that a positive d places e *below* the main diagonal, and a negative d places e *above* the main diagonal.

We are interested in what contributions a particular staircase will make to the row, column and diagonal sums.

Lemma 10 *$S^p_{d,e,f}$ contributes to the row, column, and diagonal sums of a square in the following way:*

$$
\begin{aligned}
r_i(S^p_{d,e,f}) &= \begin{cases} f & \text{if } i \in [c+d-e, c+d-e+f] \\ n+f & \text{otherwise} \end{cases} \\
c_j(S^p_{d,e,f}) &= \begin{cases} |f+p|_n & \text{if } j \in [c-d-e, c-d-e+|f+p|_n] \\ n+|f+p|_n & \text{otherwise} \end{cases} \\
d_2(S^p_{d,e,f}) &= e+|f+|-e+p(\frac{n+1}{2})|_n|_n \\
d_1(S^p_{d,e,f}) &= 0
\end{aligned}
$$

Proof: Consider the following diagram:

$f+2$								$\dots$
$n-1$	$f+1$							$n+f$
	0	f						f
		1	$f-1$					f
			$\dots$	$\dots$				$\dots$
				$f-1$	1			f
					f	0		f
						$f+1$	$n-1$	$n+f$
							$f+2$	$\dots$
$n+f+1$	$f+1$	$f+1$	$\dots$	$\dots$	$f+1$	$f+1$	$n+f+1$	

The uppermost $\mathbf{0}$ is in cell $(c-d-e, c+d-e)$ and comes from part I of Definition 3. Regardless of what f is the rows sum to f as we move downward from row $c+d-e$ to row $c+d-e+f$ (row indices taken modulo n). The other rows then sum to $n+f$. Since the row sums do not change if p changes sign we obtain the formula for row sums.

If $f \neq n-1$ then the entry $f+1$ in cell $(c-d-e-1, c+d-e)$ (immediately above the uppermost $\mathbf{0}$) is not reduced modulo n, hence columns $c-d-e$ to $c-d-e+f+1$ sum to $f+1$. On the other hand, if $f = n-1$ then $f+1$ reduces to 0 modulo n and column $c-d-e$ sums to 0 while all other columns sum to n. This is in agreement with the column sum formula for $p = +1$. In a similar manner one checks that the column sum formula is correct for $p = -1$.

Clearly the staircases we defined do not intersect the main diagonal, since $d \neq 0$, hence they contribute 0 to the main diagonal. On the back diagonal we will have e from part I of the definition plus whatever intersects the back diagonal in II. Now a_{ij} is on the back diagonal iff $i + j = n - 1$, so we solve for the parameter t in part II of the definition yielding $t \equiv -e - 2^{-1}p \pmod{n}$. But $2^{-1} = \frac{n+1}{2}$ since n is odd, so the entry from II on the back diagonal is $|f + t|_n = |f + |-e + p(\frac{n+1}{2})|_n|_n$.

□

Lemma 11 *Any matrix L_n entirely defined by staircases is orthogonal to H_n.*

Proof: A given diagonal of H_n that slopes from left to right has a constant entry. The same diagonal in L_n has the numbers 0 to $n - 1$ for entries, each exactly once, because L_n is a union of staircases. Therefore in the juxtaposition of H_n and L_n this diagonal gives n different pairs (x, y), since all the y's are different. Different left-right diagonals will have no pairs in common, since they will have different x's due to the way H_n is constructed. Altogether there are n^2 different pairs in the juxtaposition, as desired. □

6.3 The construction of L_n for $n = 4k + 1$

So, if we can construct a general antimagic square using staircases, the orthogonality to H will follow. We will construct a low order square using a centre band, a transparent band, and a projection band. The number $c = \frac{n-1}{2}$ plays a special role in what follows since it is the average value of the numbers $[0, n - 1]$ and also marks the central row and column.

The *centre band* is defined by

$$C(c - t, c - t) = t \quad \text{for } 0 \leq t < n$$

C gives the consecutive numbers we wish to project onto the row and column sums. Clearly, C is a staircase which contributes the sums:

$$\begin{aligned} r_i(C) &= |c - i|_n, \\ c_j(C) &= |c - j|_n, \\ d_1(C) &= nc, \\ d_2(C) &= 0. \end{aligned}$$

Next, we define the *transparent band.* This band is almost magic in the sense that it contributes the same amount to row, column and diagonal sums, except that it makes no contribution to the main diagonal. Collect some special staircases together and form

$$T = \bigcup_{d=1}^{\frac{n-5}{4}} \left(S^{+}_{d,c,n-1} \cup S^{-}_{-d,c+2d-1,n-1}\right)$$

Compute the row sums using lemma 10:

$$\begin{aligned}
r_i(T) &= r_i(\bigcup_{d=1}^{\frac{n-5}{4}} \left(S^{+}_{d,c,n-1} \cup S^{-}_{-d,c+2d-1,n-1})\right) \\
&= \sum_{d=1}^{\frac{n-5}{4}} \left(r_i(S^{+}_{d,c,n-1}) + r_i(S^{-}_{-d,c+2d-1,n-1})\right) \\
&= \sum_{d=1}^{\frac{n-5}{4}} \left((n-1) + (n-1)\right) \\
&= (n-5)c
\end{aligned}$$

Now compute the column contributions:

$$\begin{aligned}
c_j(T) &= c_j(\bigcup_{d=1}^{\frac{n-5}{4}} \left(S^{+}_{d,c,n-1} \cup S^{-}_{-d,c+2d-1,n-1}\right)) \\
&= \sum_{d=1}^{\frac{n-5}{4}} \left(c_j(S^{+}_{d,c,n-1}) + c_j(S^{-}_{-d,c+2d-1,n-1})\right) \\
&= \sum_{d=1}^{\frac{n-5}{4}} \left\{ \begin{array}{lll} 0 & \text{if} & j \equiv -d \\ n & \text{if} & j \not\equiv -d \end{array} \right\} + \left\{ \begin{array}{lll} n-2 & \text{if} & j \not\equiv -d \\ 2n-2 & \text{if} & j \equiv -d \end{array} \right\} \\
&= \sum_{d=1}^{\frac{n-5}{4}} (2n-2) \\
&= (n-5)c
\end{aligned}$$

Finally, the back diagonal:

$$d_2(T) = d_2(\bigcup_{d=1}^{\frac{n-5}{4}} \left(S^{+}_{d,c,n-1} \cup S^{-}_{-d,c+2d-1,n-1}\right))$$

$$
\begin{aligned}
&= \sum_{d=1}^{\frac{n-5}{4}} \left(d_2(S^{+}_{d,c,n-1}) + d_2(S^{-}_{-d,c+2d-1,n-1}) \right) \\
&= \sum_{d=1}^{\frac{n-5}{4}} \Big(c + |n-1+|1|_n|_n \\
&\qquad + c + 2d - 1 + |n-1+|-n-2d+1|_n|_n \Big) \\
&= \sum_{d=1}^{\frac{n-5}{4}} (2n-2) \\
&= (n-5)c
\end{aligned}
$$

T contributes $(n-5)c$ to the row, column and back diagonal sums. T allows the differences created by the centre band to pass through unchanged, as it acts as a group of average elements in each case.

As it stands, we have $[-\frac{(n-1)}{2}, \frac{(n-1)}{2}]$ in the set of sums twice – once from the rows, and once from the column sums. The *projection band*, P, maps $2[-\frac{(n-1)}{2}, \frac{(n-1)}{2}]$ made by the centre band onto $[-n, 1] \cup [1, n]$ evenly distributed among row and column sums:

$$P = S^{+}_{-\frac{1}{2}c,\frac{3}{2}c,c-1} \cup S^{-}_{-\frac{1}{2}c,\frac{1}{2}c-1,n-2}$$

By applying Lemma 10, we can calculate P's contribution to row, column and diagonal sums:

$$
\begin{aligned}
d_2(P) &= d_2(S^{+}_{-\frac{c}{2},\frac{3}{2}c,c-1}) + d_2(S^{-}_{\frac{c}{2},\frac{1}{2}c-1,n-2}) \\
&= \frac{3c}{2} + |(c-1) + n - \frac{c}{2} + 1|_n \\
&\quad + \frac{c}{2} - 1 + |(2c-1) + n - \frac{3c}{2}|_n \\
&= (\frac{3c}{2} + \frac{c}{2}) + (\frac{c}{2} - 1 + \frac{c}{2} - 1) \\
&= 3c - 2
\end{aligned}
$$

The details of the remaining calculations are omitted.

$$
r_i(P) = \begin{cases} 5c-1 & \text{if } i = 0 \\ 3c-2 & \text{if } i \in [1, c-1] \\ 5c-1 & \text{if } i \in [c, 2c] \end{cases}
$$

$$
c_j(P) = \begin{cases} 3c-2 & \text{if } i = 0 \\ 5c-1 & \text{if } j \in [1, c-2] \\ 7c & \text{if } j \in [c-1, c] \\ 3c-2 & \text{if } j \in [c+1, 2c] \end{cases}
$$

									26
4	6	2	**8**	**8**	0	**3**	8	0	39
8	**3**	5	3	**7**	0	**1**	**2**	0	29
1	7	**2**	4	4	**6**	**1**	**2**	**1**	28
0	2	6	**1**	3	5	**5**	**2**	**3**	27
4	**8**	3	5	**0**	2	6	**4**	**3**	35
4	**5**	**7**	4	4	**8**	1	7	**3**	43
2	**5**	**6**	**6**	5	3	**7**	0	8	42
0	**1**	**6**	**7**	**5**	6	2	**6**	8	41
7	1	**0**	**7**	**8**	**4**	7	1	**5**	40
30	38	37	45	44	34	33	32	31	36

Figure 5: L_9 and its sums.

These bands fill the entire square, and Figure 5 shows an example of what we've constructed when $n = 9$ (the boldened entries are the centre band and the projection band). We are now ready to prove that our square has the antimagic property.

Theorem 12 *If* $L_n = C \cup T \cup P$, $n = 4k + 1$, $n \geq 5$, *then* L_n *is a general antimagic square.*

Proof: We have already calculated r_i, c_j, and d_k for each of the bands, all that remains is to combine them into one formula.

$$\begin{aligned} r_i(L_n) &= r_i(C) + r_i(T) + r_i(P) \\ &= nc + \begin{cases} c - 1 & \text{if } i = 0 \\ -c - 2 - i & \text{if } i \in [1, c-1] \\ -1 & \text{if } i = c \\ 3c - i & \text{if } i \in [c+1, 2c] \end{cases} \end{aligned}$$

$$\begin{aligned} c_j(L_n) &= c_j(C) + c_j(T) + c_j(P) \\ &= nc + \begin{cases} -c - 2 & \text{if } j = 0 \\ c - 1 - j & \text{if } j \in [1, c-2] \\ 3c - j & \text{if } j \in [c-1, c] \\ c - 1 - j & \text{if } j \in [c+1, 2c] \end{cases} \end{aligned}$$

$$\begin{aligned} d_1(L_n) &= nc + 0 \\ d_2(L_n) &= nc - (2c + 2) \end{aligned}$$

Let $a+S = \{a+s|s \in S\}$. The row sums are then $nc+[\{c-1,-1\}\cup[-2c-1,-c-3]\cup[c,2c-1]$, the column sums are $nc+[\{-c-2,2c,2c+1\}\cup[1,c-2]\cup[-c-1,-2]$, and the diagonal sums are $nc+\{0,-2c-2\}$. It is now easy to check that $S(L_n) = nc+[-2c-2,2c+2] = nc+[-n-1,n]$, hence L_n is a general antimagic square. □

The remaining odd orders are covered in the next section.

6.4 The construction of L_n for $n = 4k+3$

This type is a modification of the previous odd case. The center band is now three elements thick, but designed in such a way as to put the same sums on the rows and columns and diagonals as C did. We define band C' as follows:

$$\begin{aligned} C'(c+t,c+t) &= t, \\ C'(c-t,c-t-1) &= t, \\ C'(c-t-1,c-t) &= t, \quad \text{for } 0 \le t < n. \end{aligned}$$

One readily verifies

$$r_i(C') = \begin{cases} 3c-i & \text{if } i \in [0,c] \\ 5c+1-i & \text{if } i \in [c+1,2c] \end{cases}$$

$$c_i(C') = \begin{cases} 3c-i & \text{if } i \in [0,c] \\ 5c+1-i & \text{if } i \in [c+1,2c] \end{cases}$$

$$\begin{aligned} d_1(C') &= nc \\ d_2(C') &= 2c \end{aligned}$$

Let T' be defined as:

$$T' = \bigcup_{d=1}^{\frac{n-7}{4}} \left(S^-_{d,c+1,0} \cup S^+_{-d,c-2d,n-2} \right)$$

Applying Lemma 10, we find that T' contributes $(n-7)c$ to the row and column sums. Also it's back diagonal contribution is $d_2(T') = (n-7)c$ and its main diagonal contribution is 0.

Finally we come to the construction of P', where there is a departure from the structure of P in the $4k+1$ case. Define the fragment G by

$$\begin{aligned} G(t,c-1+t) &= |n-1+t|_n \\ G(t,c+t) &= |n-1+t|_n \quad \text{for } 0 \le t < n. \end{aligned}$$

The G fragment occupies the same positions as a staircase of the form $S^{+}_{-\frac{1}{2}(c+1),*,*}$ (or of the form $S^{-}_{\frac{1}{2}(c-1),*,*}$). We find that

$$r_i(G) = \begin{cases} 2c+1+2i & \text{if } i \in [0, c-1] \\ 2c & \text{if } i = c \\ 2i-1-2c & \text{if } i \in [c+1, 2c] \end{cases}$$

$$c_i(G) = \begin{cases} 4c & \text{if } i = 0 \\ 2i-2 & \text{if } i \in [1, 2c] \end{cases}$$

$$\begin{aligned} d_1(G) &= 0 \\ d_2(G) &= 2c-1 \end{aligned}$$

Now define B to be $[S^{-}_{\frac{1}{2}(c+1),\frac{1}{2}(c-1),n-1}]^t$, where the transpose $[]^t$ is reflection through the backdiagonal (as opposed to the usual transpose through the main diagonal). We may calculate row and column sums for B by observing that $r_i(B) = c_{n-1-i}(S^{-}_{\frac{1}{2}(c+1),\frac{1}{2}(c-1),n-1})$, and $c_i(B) = r_{n-1-i}(S^{-}_{\frac{1}{2}(c+1),\frac{1}{2}(c-1),n-1})$. We find that

$$r_i(B) = 2c \text{ if } i \in [0, 2c]$$

$$c_i(B) = \begin{cases} n-2 & \text{if } i \in [0, 2c-1] \\ 2n-2 & \text{if } i = 2c \end{cases}$$

$$\begin{aligned} d_1(G) &= 0 \\ d_2(G) &= c-1 \end{aligned}$$

Now we set $P' = B \cup G$, and upon combining the formulas for B and G above we have

$$r_i(P') = \begin{cases} 6c+1 & \text{if } i = 0 \\ 4c+2i & \text{if } i \in [0, c-1] \\ 4c-1 & \text{if } i = c \\ 2i-2 & \text{if } i \in [c+1, 2c] \end{cases}$$

$$c_i(P') = \begin{cases} 6c & \text{if } i = 0 \\ 2i + 2c - 2 & \text{if } i \in [1, 2c] \end{cases}$$

$$\begin{aligned} d_1(P') &= 0 \\ d_2(P') &= 3c - 2 \end{aligned}$$

Finally we set $L_n = C' \cup T' \cup P'$ and the formulae we have for calculating the contributions of C', P' and T' give the following sums for the completed square L_n:

$$\begin{aligned} r_i(L_n) &= r_i(C') + r_i(T') + r_i(P') \\ &= nc + \begin{cases} 2c + 1 & \text{if } i = 0 \\ i & \text{if } i \in [1, c - 1] \\ -c - 1 & \text{if } i = c \\ -2c + i - 1 & \text{if } i \in [c + 1, 2c] \end{cases} \end{aligned}$$

$$\begin{aligned} c_i(L_n) &= c_j(C') + c_j(T') + c_j(P') \\ &= nc + \begin{cases} 2c & \text{if } i = 0 \\ -2c + i - 2 & \text{if } j \in [1, c] \\ i - 1 & \text{if } j \in [c + 1, 2c] \end{cases} \end{aligned}$$

$$\begin{aligned} d_1(L_n) &= nc + 0 + 0 = nc \\ d_2(L_n) &= (n - 7)c + 2c + 3c - 2 = nc - 2c - 2 \end{aligned}$$

From these it is easy to see that the $2n + 2$ sums of L_n are precisely $nc + [-2c - 2, 2c + 1]$, as desired. The construction even works in the case $n = 7$, where the transparent band T' is empty and hence still contributes $(n - 7)c$ to rows and columns. We have proved

Theorem 13 *If $L_n = C' \cup T' \cup P'$, $n = 4k + 3$, $n \geq 7$, then L_n is a general antimagic square.*

An example for $n = 11$ is shown in figure 6.

6.5 Summary

Theorem 14 *An* AMS(n) *exists for all odd* $n \geq 5$

											43
6	4	10	10	10	10	5	6	0	0	5	66
4	7	3	0	9	0	9	6	7	10	1	56
2	3	8	2	1	8	1	8	7	8	9	57
8	3	2	9	1	2	7	2	7	8	9	58
10	7	4	1	10	0	3	6	3	6	9	59
10	0	6	5	0	0	10	4	5	4	5	49
4	0	1	5	6	10	1	9	5	4	5	50
6	3	1	2	4	7	9	2	8	6	3	51
2	7	2	2	3	3	8	8	3	7	7	52
8	1	8	1	3	4	2	9	7	4	6	53
5	9	0	9	0	4	5	1	10	6	5	54
65	44	45	46	47	48	60	61	62	63	64	55

Figure 6: L_{11} and its sums.

Proof: Theorems 12 and 13 show that for all odd $n \geq 5$ we can form the general antimagic square L_n which is also orthogonal to H_n by lemma 11. Theorem 9 shows how to juxtapose H_n and L_n to form an AMS(n). □

While the proof only gives one AMS(n) for each odd n, we can observe ways to produce variants.

Recall that in H_n, the only property we need is that broken diagonals parallel to the main must each consist of the same element, except for the main diagonal which must be all $\frac{1}{2}n(n-1)$. This means that we could relabel the remaining $(n-1)$ elements without disturbing the antimagic property of the final square. There are $(n-1)!$ ways to do this.

Also, in L_n, notice that we could reflect P and C along the back diagonal and still have a general antimagic low-order square. There are two choices for this reflection.

Finally, each "magic" band of four staircases can be reflected along the back diagonal or not, which yields $2^{\frac{n-5}{4}}$ choices which preserve the antimagic property of the square. In addition to reflection, these groups of four of four can be slid parallel to the main diagonal. This action does not disturb the row and column sums, but can have an impact on the diagonal of either zero, plus n, or minus n. There appears to be $\Pi_{i=0}^{\frac{n-9}{4}}\left(4i+3\right)$ ways to yield zero diagonal effect (in the same way as described in the construction), not to mention all the ways one could compose zeros with combinations of plus and minus n.

Therefore, we conclude that this construction yields *at least*:

$$(n-1)! \cdot 2^{\frac{n-1}{4}} \cdot \prod_{i=0}^{\frac{n-9}{4}} \left(4i+3\right)$$

AMS(n)'s. Of course, this is a tiny fraction of all order n antimagic squares.

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