Enumerating Cyclic 3-Cube Decompositions of K_{25}

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ABSTRACT. Below we prove that there are exactly 244 nonisomorphic cyclic decompositions of the complete graph K_{25} into cubes. The full list of such decompositions is given in the Appendix.

In [7] R.M.Wilson studied the decompositions of complete graphs into isomorphic subgraphs. He had obtained the famous result on the asymptotic existence of such decompositions.

A.Kotzig [4] had proved that, for every d, the graph K_m can be decomposed into subgraphs isomorphic to the d-dimensional cube Q_d, if $m \equiv 1 mod(d \cdot 2^d)$. As a consequence we obtain the following: K_{25} is decomposable into cubes Q_3. The problem of the existence of the decompositions of K_n into 3-cubes is settled in [2]. Here we pay attention to the enumeration aspect of the problem and deal with the decompositions of K_{25} into cubes Q_3. Our main result is the following

Theorem 1. *There are exactly 244 pairwise nonisomorphic cyclic decompositions of K_{25} into cubes.*

1.Preliminaries. Let K_{25} have the vertex set $\{1, 2, ..., 25\}$. We denote by $abcdefgh$ the 3-cube with the vertex set {a,b,c,d,e,f,g,h} as it is drawn on Fig.1. We will call $abcdefgh$ the code of the cube.

Fig.1.The cube $abcdefgh$

A decomposition of the complete graph K_{25} into cubes is a partition of the edge set of K_{25} into subgraphs each of which is a cube. Such a decomposition is cyclic when it posseses a cycle of length 25 as an automorphism.

Here we construct the complete list of pairwise nonisomorphic cyclic decompositions of K_{25} into cubes.

2.Construction. At first, we fix the automorphism $\alpha = \{123...25\}$. Then every cyclic decomposition into cubes can be defined by one of its own cubes called the base cube.

The following well known notions will be used further.

Let's consider a circle with 25 points on it dividing the circle into equal arcs. Number the points by 1,2,...,25 and inscribe a cube in the circle as in Fig.2. The configuration obtained is called a projection of the cube on the circle corresponding to α.

Fig.2.Projection of the cube 1 2 4 7 5 17 9 16 on the circle corresponding to α

The *length* of the edge i, j is the number $d(i,j) = min(|i-j|, 25-|i-j|)$. A cube generates a cyclic decomposition of K_{25} under the action of α if and only if the lengths of its edges are pairwise different.

It is evident that every cyclic decomposition contains a cube $12cdefgh$, so we can take such a cube as a base cube. Moreover we can additionally require $c < f$ because the codes $12cdefgh$ and $12fedcgh$ denote the same cube.

With computer aid we have generated, in lexicographic order, the list of all the cubes satisfying the above mentioned requirements.

3. Isomorph rejection . A further shortening of the list was made using the following two principles.

First. Looking through the list, for every cube Q and every $i(i = 1, 2, ..., 25;$ not $i \equiv 0 mod 5)$ we construct the projection Q_i of Q which corresponds to α^i. Then we change the numbering of the circle points according to α , preserving the inscribed cube unchanged, and turning the obtained projection Q_i' we reduce it to the form $12cdefgh(c < f)$ and then compare with Q. If, for some i, the code Q_i' is lexicographically less than Q we reject Q from the list. This is justified by the fact that Q and Q_i' define isomorphic decompositions.

Second. For every $Q = abcdefgh$ from the list we construct the symmetric cube $Q' = a'b'c'd'e'f'g'h'$ where $i' \equiv 1 + (26 - i) mod 25$. Reducing Q' to the form $12cdefgh(c < f)$ we reject Q from the list in case the obtained code is lexicographically less than the code Q.

Proceeding in this way we obtained a list containing 244 basic cubes. The list is given in the Appendix. With the help of the invariant and its subinvariant described below we proved that the 244 decompositions generated by those basic cubes are pairwise nonisomorphic. Note that a similar distinguishing technique was used in [5],[6].

4.Distinguishing nonisomorphic decompositions. Let R be a cube decomposition of K_{25}, and let Q and P be two different cubes from R. We distinguish 15 types of mutual placement of Q and P according to the character of their common part. Let k express the number of common vertices for Q and P. If $k = 0$ we say

Fig.3. Types for the cases $k = 2$ and $k = 3$

that it is type 1. If $k = 1$, then it's type 2. If $k = 2$ then there are types 3,4,5,6,7, and if $k = 3$ then there are types 8,9,10,11,12 as it is shown on Fig.3. The types 13,14,15 correspond accordingly to the cases $k = 4, k = 5$ and $k > 5$.

We call the *index*of the cube Q in R the vector $(s_1, s_2, ..., s_{15})$ where s_i is the number of cubes in R forming type i with Q. Note that $s_1 + s_2 + ... + s_{15} = 24$ and for a cyclic decomposition all s_i are even. Consider the table

$$T(R) = \left| \begin{array}{cccc|c} s(1,1) & s(1,2) & \dots & s(1,15) & m(1) \\ & & \dots & & \dots \\ s(q,1) & s(q,2) & \dots & s(q,15) & m(q) \end{array} \right| ,$$

where $m(j)$ is the number of cubes in R having the index $(s(j, 1), s(j, 2), ..., s(j, 15))$, the indices are different and lexicographically arranged. It is easy to see that $T(R)$ is an isomorphism invariant in the set of cube decompositions of K_{25}.

For a cyclic decomposition R all cubes have the same indices, therefore $T(R)$ has $q = 1$ line and the index may be used instead of $T(R)$.

For example, the decomposition 1 in our Appendix gives

$T(1) = (044024042002020)$, and the decomposition 2 gives $T(2) = (044220062004000)$.

Since $T(1) \neq (T(2)$, the two decompositions are nonisomorphic.

The invariant T does not distinguish all nonisomorphic decompositions in our list. This makes us use a subinvariant of the following kind.

Let R_1 and R_2 be cyclic decompositions and $T(R_1) = T(R_2)$. Let t be the type occurring in both decompositions, let Q_j be a cube of R_j, and $V(R_j)$ be the set of cubes in R_j, forming type t with Q_j, $j = 1, 2$. Denote by $S(R_j)$ the multiset of types formed by the pairs of cubes from $V(R_j)$, $j = 1, 2$. Evidently $S(R_j)$ is invariant under isomorphism in the set of cyclic decompositions with the prescribed value of T.

For example, the decompositions 1 and 191 from the Appendix have equal values of invariant T. As $s_5 = 2 > 0$, take $t = 5$. In the decomposition 1 two cubes forming type 5 with an arbitrary chosen cube form type 9, so $S(1) = \{9\}$, and in decomposition 191 the corresponding cubes form type 6, and $S(191) = \{6\}$. $S(1) \neq S(191)$ yields nonisomorphism of the decompositions.

The decompositions not distinguished by the invariant T (fortunately the cases are not numerous) are distinguished by the subinvariant.

So the Theorem is proved.

Note that the similar technique may be used to enumerate cyclic decompositions of K_{25} into the subgraphs isomorphic to g where g stands for a cubic graph of order 8 different from a cube. It is well known(see, for instance, [3]) that such decompositions exist for every g.

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Appendix. The full list of nonisomorphic cyclic decompositions of K_{25} into 3-dimensional cubes

1.1 2 4 7 5 12 24 16
2.1 2 4 7 5 13 22 17
3.1 2 4 7 5 17 9 16
4.1 2 4 7 5 17 9 23
5.1 2 4 7 5 17 12 23
6.1 2 4 7 5 17 24 16
7.1 2 4 7 5 22 11 20
8.1 2 4 7 6 13 25 15
9.1 2 4 7 6 15 25 14
10.1 2 4 7 6 18 25 17
11.1 2 4 7 6 19 15 22
12.1 2 4 7 6 20 8 16
13.1 2 4 7 6 20 8 23
14.1 2 4 7 6 23 11 22
15.1 2 4 7 6 23 13 20
16.1 2 4 7 8 13 25 17
17.1 2 4 7 10 15 8 18
18.1 2 4 7 10 15 25 18
19.1 2 4 7 10 22 8 25
20.1 2 4 7 10 22 14 3
21.1 2 4 7 10 23 9 17
22.1 2 4 7 10 23 12 17
23.1 2 4 7 11 6 13 24
24.1 2 4 7 11 20 15 3
25.1 2 4 7 11 20 15 19
26.1 2 4 7 12 17 8 25
27.1 2 4 7 12 17 13 20
28.1 2 4 7 12 20 8 17
29.1 2 4 7 13 18 14 21
30.1 2 4 7 13 22 11 3
31.1 2 4 7 14 9 13 24
32.1 2 4 7 14 9 18 22
33.1 2 4 7 14 9 20 24
34.1 2 4 7 14 10 20 25
35.1 2 4 7 15 20 8 23
36.1 2 4 7 15 22 12 3
37.1 2 4 7 15 23 13 20
38.1 2 4 7 17 6 19 24
39.1 2 4 7 22 9 20 12
40.1 2 4 7 22 11 24 14
41.1 2 4 8 6 16 13 25
42.1 2 4 8 6 18 15 25
43.1 2 4 8 7 15 18 23
44.1 2 4 8 9 14 20 23
45.1 2 4 8 9 15 20 23
46.1 2 4 8 9 24 15 3
47.1 2 4 8 11 14 20 25
48.1 2 4 8 14 5 10 25
49.1 2 4 8 15 18 10 20
50.1 2 4 8 16 19 10 21
51.1 2 4 8 16 24 15 21
52.1 2 4 8 18 5 19 24
53.1 2 4 8 18 24 15 3
54.1 2 4 8 20 5 16 25
55.1 2 4 8 20 7 21 11
56.1 2 4 8 20 10 24 11

57.1 2 4 8 23 10 24 14
58.1 2 4 9 5 12 18 21
59.1 2 4 9 5 24 11 20
60.1 2 4 9 7 11 14 21
61.1 2 4 9 7 11 22 19
62.1 2 4 9 7 18 22 19
63.1 2 4 9 7 18 25 22
64.1 2 4 9 8 11 23 19
65.1 2 4 9 12 8 11 21
66.1 2 4 9 12 18 25 22
67.1 2 4 9 14 17 13 20
68.1 2 4 9 15 11 14 21
69.1 2 4 9 15 18 14 21
70.1 2 4 9 15 24 11 5
71.1 2 4 9 17 14 10 24
72.1 2 4 9 17 14 18 24
73.1 2 4 9 17 20 8 23
74.1 2 4 9 17 20 10 21
75.1 2 4 9 19 15 18 3
76.1 2 4 10 5 12 7 18
77.1 2 4 10 5 12 24 13
78.1 2 4 10 5 17 24 13
79.1 2 4 10 5 19 7 25
80.1 2 4 10 5 19 14 17
81.1 2 4 10 5 20 15 18
82.1 2 4 10 6 9 17 21
83.1 2 4 10 6 14 18 3
84.1 2 4 10 9 6 11 21
85.1 2 4 10 9 6 11 24
86.1 2 4 10 12 20 25 22
87.1 2 4 10 13 9 24 21
88.1 2 4 10 13 24 9 17
89.1 2 4 10 15 22 19 23
90.1 2 4 10 16 19 8 3
91.1 2 4 10 16 24 17 21
92.1 2 4 10 18 13 25 3
93.1 2 4 10 19 6 21 24
94.1 2 4 10 19 16 21 6
95.1 2 4 10 19 22 12 23
96.1 2 4 10 23 6 16 5
97.1 2 4 10 23 6 19 5
98.1 2 4 10 23 12 24 6
99.1 2 4 10 23 12 25 18
100.1 2 4 10 23 13 25 18
101.1 2 4 11 5 8 16 25
102.1 2 4 11 5 8 17 22
103.1 2 4 11 5 10 13 24
104.1 2 4 11 5 21 16 19
105.1 2 4 11 5 24 16 25
106.1 2 4 11 6 10 23 20
107.1 2 4 11 6 18 21 17
108.1 2 4 11 6 19 23 20
109.1 2 4 11 6 19 25 22
110.1 2 4 11 7 10 24 20
111.1 2 4 11 9 21 24 20
112.1 2 4 11 10 15 21 7
113.1 2 4 11 10 24 12 16
114.1 2 4 11 12 21 16 8
115.1 2 4 11 13 8 25 22
116.1 2 4 11 13 18 15 7
117.1 2 4 11 13 18 21 7
118.1 2 4 11 14 5 9 3
119.1 2 4 11 14 5 24 3
120.1 2 4 11 14 10 24 8
121.1 2 4 11 14 19 8 5
122.1 2 4 11 14 19 8 17
123.1 2 4 11 14 23 12 17
124.1 2 4 11 15 6 9 3
125.1 2 4 11 17 5 9 3
126.1 2 4 11 18 21 16 7
127.1 2 4 11 20 16 24 8
128.1 2 4 11 20 24 16 25
129.1 2 4 11 23 14 25 6
130.1 2 4 12 5 21 14 17
131.1 2 4 12 13 7 25 22
132.1 2 4 12 14 17 13 19
133.1 2 4 12 16 11 23 19
134.1 2 4 12 16 20 23 3
135.1 2 4 12 16 23 10 7
136.1 2 4 12 19 7 13 9
137.1 2 4 12 20 17 24 8
138.1 2 4 12 20 23 14 7
139.1 2 4 12 21 5 23 8
140.1 2 4 14 6 10 13 20
141.1 2 4 14 6 24 18 22
142.1 2 4 14 7 24 20 25
143.1 2 4 14 8 19 25 5
144.1 2 4 14 9 6 11 20
145.1 2 4 14 9 20 23 18
146.1 2 4 14 10 16 9 6
147.1 2 4 14 10 16 12 17
148.1 2 4 14 10 16 21 17
149.1 2 4 14 10 21 24 6
150.1 2 4 14 12 19 13 17
151.1 2 4 14 17 10 15 11
152.1 2 4 14 17 10 24 20

153.1 2 4 14 17 20 25 6	154.1 2 4 14 17 21 24 6
155.1 2 4 14 18 24 20 25	156.1 2 4 14 19 24 18 10
157.1 2 4 14 22 8 11 6	158.1 2 4 14 22 16 9 6
159.1 2 4 16 5 13 7 25	160.1 2 4 16 6 9 20 12
161.1 2 4 16 12 5 24 8	162.1 2 4 16 15 10 13 9
163.1 2 4 16 17 9 15 12	164.1 2 4 16 17 9 15 20
165.1 2 4 16 18 11 15 21	166.1 2 4 16 18 24 15 11
167.1 2 4 16 21 13 7 25	168.1 2 4 16 23 19 10 5
169.1 2 4 17 5 8 18 10	170.1 2 4 17 6 20 10 14
171.1 2 4 17 8 16 19 13	172.1 2 4 17 9 20 10 14
173.1 2 4 17 11 7 10 3	174.1 2 4 17 12 19 25 22
175.1 2 4 17 16 8 15 20	176.1 2 4 17 19 8 12 22
177.1 2 4 17 20 16 19 12	178.1 2 4 17 20 23 18 10
179.1 2 4 17 22 5 23 12	180.1 2 4 18 6 12 16 9
181.1 2 4 18 23 14 10 3	182.1 2 4 19 5 8 21 10
183.1 2 4 19 5 11 16 8	184.1 2 4 19 5 11 24 16
185.1 2 4 19 5 13 16 10	186.1 2 4 19 5 13 16 25
187.1 2 4 19 5 18 7 24	188.1 2 4 19 5 18 24 16
189.1 2 4 19 5 21 16 8	190.1 2 4 19 21 13 16 25
191.1 2 4 19 21 18 12 8	192.1 2 4 19 22 5 18 13
193.1 2 4 19 22 5 24 10	194.1 2 4 19 22 11 24 16
195.1 2 4 20 5 17 12 23	196.1 2 4 20 15 5 25 7
197.1 2 4 20 19 22 12 8	198.1 2 4 20 23 6 18 13
199.1 2 4 20 23 6 24 10	200.1 2 4 21 5 11 18 8
201.1 2 4 21 5 15 18 12	202.1 2 4 21 14 17 13 7
203.1 2 4 21 16 23 10 7	204.1 2 4 21 17 14 18 11
205.1 2 4 22 6 12 20 9	206.1 2 4 22 7 12 21 10
207.1 2 4 22 15 10 16 25	208.1 2 4 22 17 14 19 11
209.1 2 4 23 5 12 21 10	210.1 2 4 23 5 14 24 16
211.1 2 4 23 8 16 25 13	212.1 2 4 23 14 10 15 5
213.1 2 4 23 17 10 25 12	214.1 2 4 23 18 6 20 13
215.1 2 4 23 18 11 15 3	216.1 2 4 23 18 11 25 13
217.1 2 5 7 6 14 21 17	218.1 2 5 7 8 17 25 21
219.1 2 5 7 12 20 15 3	220.1 2 5 7 17 10 15 3
221.1 2 5 7 18 11 16 3	222.1 2 5 9 7 12 14 21
223.1 2 5 9 12 17 23 21	224.1 2 5 9 15 17 12 21
225.1 2 5 9 15 20 18 24	226.1 2 5 10 3 16 12 18
227.1 2 5 10 12 6 23 25	228.1 2 5 10 14 16 12 20
229.1 2 5 10 16 20 18 24	230.1 2 5 16 7 20 3 12
231.1 2 5 16 19 11 17 21	232.1 2 5 18 11 20 24 13
233.1 2 5 18 20 6 15 13	234.1 2 5 18 22 8 15 13
235.1 2 5 20 15 10 12 24	236.1 2 5 21 7 20 3 17
237.1 2 5 21 16 20 18 10	238.1 2 5 21 20 10 3 7
239.1 2 5 22 8 14 3 13	240.1 2 5 22 15 8 18 13
241.1 2 6 3 7 14 22 17	242.1 2 6 4 8 13 23 17
243.1 2 6 11 20 13 15 23	244.1 2 6 12 3 20 15 25

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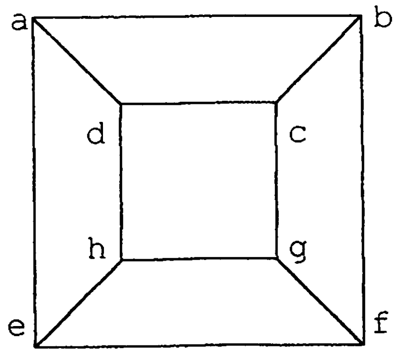

Fig.1. Cube abcdefgh

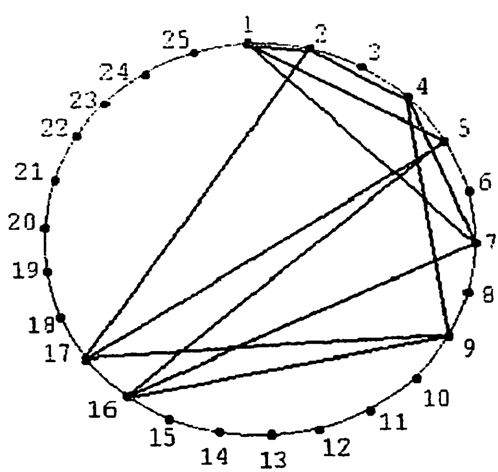

Fig.2. Projection of the cube 1 2 4 7 - 5 17 9 16 in the circle which corresponds to α

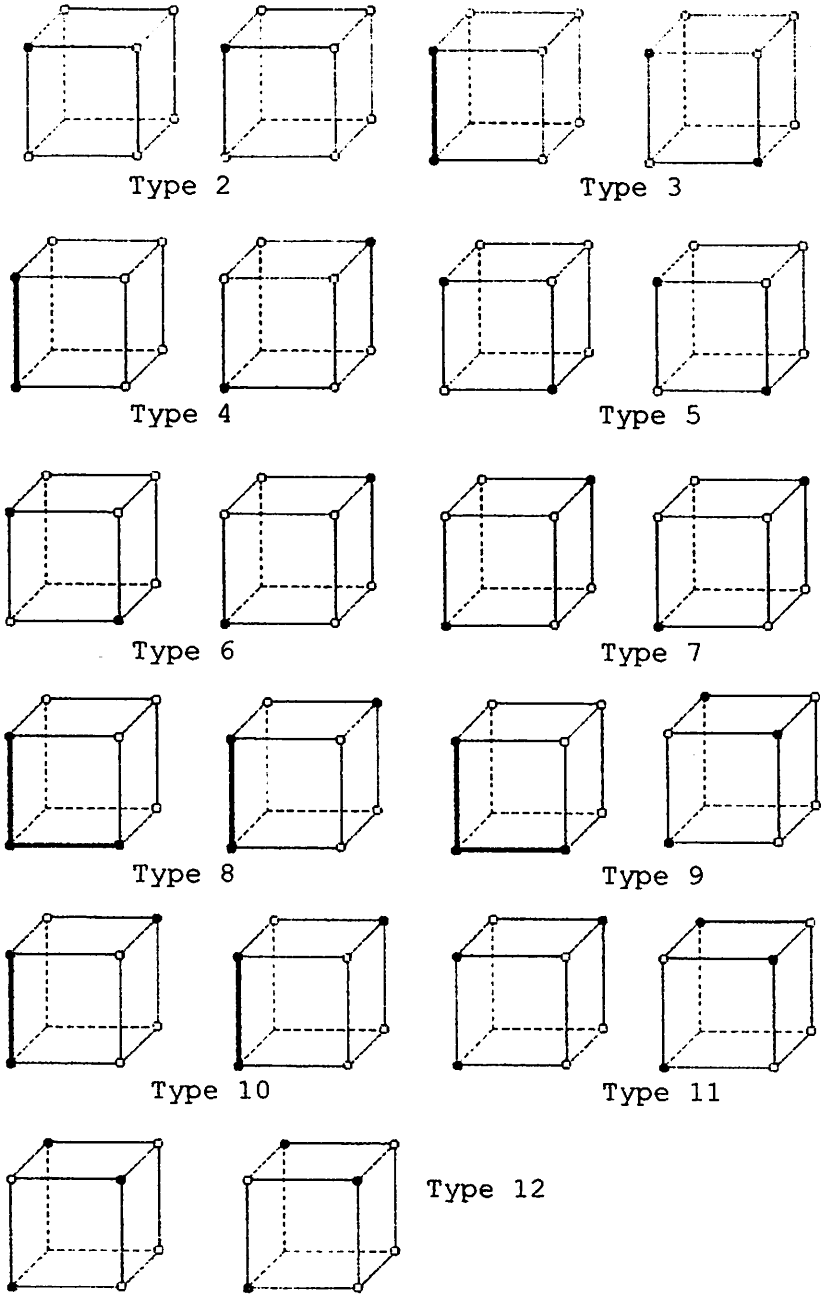

Fig.3. Types for the cases k=2 and k=3