Valuations of a certain class of convex polytopes

Martin BAČA

Department of Applied Mathematics
Technical University, Košice, Slovak Republic
e-mail: hollbaca@ccsun.tuke.sk

Mirka MILLER

Department of Computer Science and Software Engineering
The University of Newcastle, Australia
e-mail: mirka@cs.newcastle.edu.au

Abstract

This paper deals with the problem of labeling the edges of a plane graph in such a way that the weight of a face is the sum of the labels of the edges surrounding that face. The paper describes (a, d)-face antimagic labeling of a certain class of convex polytopes.

1 Introduction

G is a finite connected graph without loops and multiple edges, $V(G)$ is its vertex-set and $E(G)$ is its edge-set. A graph $G = (V, E, F)$ is said to be *plane* if it is drawn on the Euclidean plane in such a way that edges do not cross each other except at vertices of the graph. Let $F(G)$ be the face-set of G and $|V(G)| = p$, $|E(G)| = r$ and $|F(G)| = s$ be the number of vertices, edges and faces of G.

Hartsfield and Ringel [6] introduced the concept of an antimagic graph. An *antimagic graph* is a graph whose edges can be labeled with the integers $1, 2, ..., r$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is no two vertices have the same sum. Hartsfield and Ringel conjecture that every tree different from

K_2 is antimagic, and, more strongly, that every connected graph other than K_2 is antimagic.

Bodendiek and Walther [3] defined the concept of an (a,d)-antimagic graph as a special case of an antimagic graph.

The *weight* $w(f)$ of a face $f \in F(G)$ under an edge labeling $g : E(G) \to \{1,2,...,r\}$ is the sum of the labels of the edges surrounding that face.

A connected plane graph $G = (V,E,F)$ is said to be (a,d)-*face antimagic* if there exist positive integers a and d, and a bijection $g : E(G) \to \{1,2,...,r\}$ such that the induced mapping $g^* : F(G) \to W$ is also a bijection, where $W = \{w(f) : f \in F(G)\} = \{a, a+d, a+2d, ..., a+(s-1)d\}$ is the set of weights of faces.

If $G = (V,E,F)$ is (a,d)-face antimagic and $g : E(G) \to \{1,2,...,r\}$ is a corresponding bijective mapping of G then g is said to be an (a,d)-*face antimagic labeling* of G.

Note that (a,d)-face antimagic labeling of a convex polytope G is equivalent to (a,d)-antimagic labeling of its dual graph G^*. (a,d)-antimagic labelings of the special graphs called parachutes are described in [4] and [5].

2 Construction of plane graph $\mathbf{Q}_n^m$

The *antiprism* Q_n, $n \geq 3$, is a 4-regular graph (Archimedean convex polytope) and for $n = 3$ it is the octahedron.

Let $I = \{1,2,...,n\}$ and $J = \{1,2,...,m\}$ be index sets. For $n \geq 3$ and $m \geq 1$ we denote by Q_n^m the plane graph of a convex polytope, which is obtained as a combination of m antiprisms Q_n. Let us denote the vertex set of Q_n^m by $V(Q_n^m) = \{y_{j,i} : i \in I \text{ and } j \in J \cup \{m+1\}\}$ and the edge set by $E(Q_n^m) = \{y_{j,i}y_{j,i+1} : i \in I \text{ and } j \in J \cup \{m+1\}\} \cup \{y_{j,i}y_{j+1,i} : i \in I \text{ and } j \in J\} \cup \{y_{j,i+1}y_{j+1,i} : i \in I \text{ and } j \in J,\ j \text{ odd}\} \cup \{y_{j,i}y_{j+1,i+1} : i \in I \text{ and } j \in J,\ j \text{ even}\}$.

We make the convention that $y_{j,n+1} = y_{j,1}$ for $j \in J \cup \{m+1\}$.

The face set $F(Q_n^m)$ contains $2mn$ 3-sided faces, an internal n-sided face and an external n-sided face. We insert exactly one vertex x (z) into the internal (external) n-sided face of Q_n^m and connect the vertex x (z) with the vertices $y_{1,i}$ ($y_{m+1,i}$), $i \in I$. Thus, we obtain the plane graph $\mathbf{Q}_n^m$ (labeled as in Figure 1), consisting of 3-sided faces with the vertex set $V(\mathbf{Q}_n^m) = V(Q_n^m) \cup \{x,z\}$ and the edge set $E(\mathbf{Q}_n^m) = E(Q_n^m) \cup \{xy_{1,i} : i \in$

$I\} \cup \{y_{m+1,i}z : i \in I\}$ where $|V(\mathbf{Q}_n^m)| = (m+1)n+2$, $|E(\mathbf{Q}_n^m)| = 3n(m+1)$ and $|F(\mathbf{Q}_n^m)| = 2n(m+1)$.

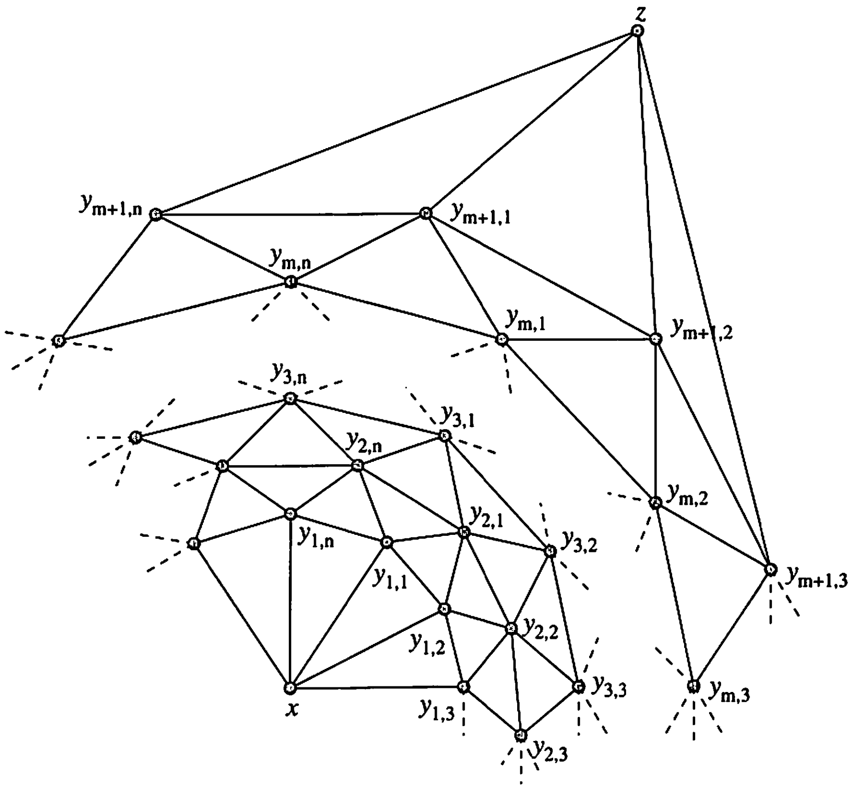

Figure 1: Plane graph $\mathbf{Q}_n^m$

The paper [1] describes $(7n+2,1)$-face antimagic labeling for the plane graph $\mathbf{Q}_n^1$. $\left(\frac{21n}{2}+2,1\right)$-face antimagic labeling for the plane graph $\mathbf{Q}_n^2$ is given in [2].

In this paper we construct $(a,1)$-face antimagic labelings of $\mathbf{Q}_n^m$ for $m \geq 3$, $n \geq 3$. For the other possible values of the parameter d $(2 \leq d \leq 4)$ we propose several conjectures.

3 Necessary conditions

Assume that $\mathbf{Q}_n^m$ is (a,d)-face antimagic and $W = \{w(f) : f \in F(\mathbf{Q}_n^m)\} = \{a, a+d, a+2d, ..., a+[2n(m+1)-1]d\}$ is the set of weights of faces.

$$\sum_{f \in F(\mathbf{Q}_n^m)} w(f) = n(m+1)[2a + d(2nm + 2n - 1)]. \tag{1}$$

Since the edges of $\mathbf{Q}_n^m$ are labeled by the set of integers $\{1, 2, ..., 3n(m+1)\}$ and since each of these labels is used twice in the computation of the weights of faces, the sum of all the edge labels used to calculate the weights of faces is equal to

$$2 \sum_{e \in E(\mathbf{Q}_n^m)} g(e) = 3n(m+1)(3nm + 3n + 1)\ . \tag{2}$$

Thus the following equation holds

$$2 \sum_{e \in E(\mathbf{Q}_n^m)} g(e) = \sum_{f \in F(\mathbf{Q}_n^m)} w(f) \tag{3}$$

which is obviously equivalent to the equation

$$9n(m+1) + 3 = 2a + d[2nm + 2n - 1]\ . \tag{4}$$

If $a = 6$ is the minimal value of weight which can be assigned to a 3-sided face then from (4) we get that

$$d = \frac{9n(m+1) + 3 - 2a}{2n(m+1) - 1} \le \frac{9}{2} - \frac{9}{2n(m+1) - 1} \le 4\ . \tag{5}$$

¿From (5) it follows that

(i) if n is even, $n \geq 4$ and $m \geq 1$, or if n is odd, $n \geq 3$ and m is odd, $m \geq 1$, then d is odd, and we obtain exactly two solutions (a, d) of the Diophantine equation (4):

$(a,d) = \left(\frac{7n(m+1)}{2} + 2, 1\right)$ and $(a,d) = \left(\frac{3n(m+1)}{2} + 3, 3\right)$.

(ii) if n is odd, $n \geq 3$ and m is even, $m \geq 2$, then d is even, and this means that the equation (4) has exactly two solutions,

$(a,d) = \left(\frac{5n(m+1)+5}{2}, 2\right)$ and $(a,d) = \left(\frac{n(m+1)+7}{2}, 4\right)$.

4 Face antimagic labelings

In the sequel we shall use the functions

$$\phi(x) = \begin{cases} 1 & \text{if } x \equiv 1 \pmod 2 \\ 0 & \text{if } x \equiv 0 \pmod 2 \end{cases} \tag{6}$$

$$\rho(x,y,z) = \begin{cases} 1 & \text{if } x \leq y \leq z \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

to simplify later notations.

Let us denote the weights of the 3-sided faces of $\mathbf{Q}_n^m$ (under an edge labeling g) by

$w_{1,i}^1(g) = g(xy_{1,i}) + g(xy_{1,i+1}) + g(y_{1,i}y_{1,i+1})$ for $i \in I$.

$w_{j,i}^2(g) = g(y_{j,i}y_{j,i+1}) + g(y_{j,i}y_{j+1,i}) + g(y_{j,i+1}y_{j+1,i})$ for $i \in I$ and $j \in J$, j odd.

$w_{j,i}^3(g) = g(y_{j,i+1}y_{j+1,i}) + g(y_{j+1,i}y_{j+1,i+1}) + g(y_{j,i+1}y_{j+1,i+1})$ for $i \in I$ and $j \in J$, j odd.

$w_{j,i}^4(g) = g(y_{j,i}y_{j,i+1}) + g(y_{j,i+1}y_{j+1,i+1}) + g(y_{j,i}y_{j+1,i+1})$ for $i \in I$ and $j \in J$, j even.

$w_{j,i}^5(g) = g(y_{j,i}y_{j+1,i}) + g(y_{j+1,i}y_{j+1,i+1}) + g(y_{j,i}y_{j+1,i+1})$ for $i \in I$ and $j \in J$, j even.

$w_{m+1,i}^6(g) = g(y_{m+1,i}z) + g(y_{m+1,i}y_{m+1,i+1}) + g(y_{m+1,i+1}z)$ for $i \in I$.

Let $W_1(g) = \{w_{j,i}^2(g) : i \in I \text{ and } j \text{ odd}, j \in J\} \cup \{w_{j,i}^3(g) : i \in I \text{ and } j \text{ odd}, j \in J\} \cup \{w_{j,i}^4(g) : i \in I \text{ and } j \text{ even}, j \in J\} \cup \{w_{j,i}^5(g) : i \in I \text{ and } j \text{ even}, j \in J\}$,

$W_2(g) = \{w^1_{1,i}(g) : i \in I\}$ and

$W_3(g) = \{w^6_{m+1,i}(g) : i \in I\}$

be the sets of weights.

If $m \equiv 3 \pmod 4$, $m \geq 3$ and $n \geq 3$, we construct an edge labeling g_1 and if $m \equiv 1 \pmod 4$, $m \geq 5$ and $n \geq 3$, we construct an edge labeling g_2 of $\mathbf{Q}^m_n$ in the following way.

$g_1(y_{j,i+1}y_{j+1,i}) = [(j-1)n+i][\rho(1,j,\frac{m-1}{2}) + \rho(\frac{m+7}{2},j,m)]\phi(j) + [(j-2)n+i]\rho(\frac{m+3}{2},j,\frac{m+3}{2})$,

$g_2(y_{j,i+1}y_{j+1,i}) = [(j-1)n+i]\phi(j)$,

$g_1(y_{j,i}y_{j+1,i+1}) = [(m-j)n+1]\rho(1,i,1) + [(m+1-j)n+2-i]\rho(2,i,n)$

for $i \in I$ and j even, $2 \leq j \leq \frac{m-3}{2}$,

$g_1(y_{j,i,}y_{j+1,i+1}) = (nj+1)\rho(1,i,1) + [n(j+1)+2-i]\rho(2,i,n)$

for $i \in I$ and $j = \frac{m+1}{2}$,

$g_1(y_{j,i}y_{j+1,i+1}) = [(m-j)n+1]\rho(1,i,1) + [(m+1-j)n+2-i]\rho(2,i,n)$

for $i \in I$ and j even, $\frac{m+5}{2} \leq j \leq m-1$.

$g_2(y_{j,i}y_{j+1,i+1} = [(m-j)n+1]\rho(1,i,1) + [(m-j+1)n-i+2]\rho(2,i,n)$

for $i \in I$ and j even, $j \in J$.

$g_1(y_{j,i}y_{j+1,i}) = g_2(y_{j,i}y_{j+1,i}) = ([(3m-j+4)n+2-i]\rho(2,i,n) + [(3m-j+3)n+1]\rho(1,i,1))\phi(j) + ([(3m-j+3)n+1+i]\rho(1,i,n-1) + [(3m-j+3)n+1]\rho(n,i,n))\phi(j+1)$

for $i \in I$ and $j \in J$.

$g_1(y_{j,i}y_{j,i+1}) = ((3m+2j+1)\frac{n}{2}\rho(1,i,1) + [(3m+2j-1)\frac{n}{2}+i-1]\rho(2,i,n))\rho(1,j,\frac{m-1}{2}) + (\frac{n}{2}(m+2j-1)\rho(1,i,1) + [\frac{n}{2}(m+2j-3)+i-1]\rho(2,i,n))$

$\rho(\frac{m+3}{2}, j, m)$

for $i \in I$ and j odd, $j \in J$,

$g_1(y_{j,i}y_{j,i+1}) = ([(3m+2j+1)\frac{n}{2} - 1 - i]\rho(1,i,n-2) + [(3m+2j+3)\frac{n}{2} - 1 - i]\rho(n-1,i,n))\rho(2,j,\frac{m+1}{2}) + ([\frac{n}{2}(m+2j-1) - 1 - i]\rho(1,i,n-2) + [\frac{n}{2}(m+2j+1) - 1 - i]\rho(n-1,i,n))\rho(\frac{m+5}{2}, j, m-1)$

for $i \in I$ and j even, $j \in J$,

$g_1(y_{j,i}y_{j,i+1}) = [\frac{n}{2}(m+2j-3)+3-i]\rho(1,i,2) + [\frac{n}{2}(m+2j-1)+3-i]\rho(3,i,n)$

for $i \in I$ and $j = m+1$.

$g_2(y_{j,i}y_{j,i+1}) = ((3m+2j+1)\frac{n}{2}\rho(1,i,1) + [(3m+2j-1)\frac{n}{2}+i-1]\rho(2,i,n))$ $\rho(1,j,\frac{m+1}{2}) + (\frac{n}{2}(m+2j-1)\rho(1,i,1) + [\frac{n}{2}(m+2j-3)+i-1]\rho(2,i,n))$ $\rho(\frac{m+5}{2}, j, m)$

for $i \in I$ and j odd, $j \in J$,

$g_2(y_{j,i}y_{j,i+1}) = ([(3m+2j+1)\frac{n}{2} - 1 - i]\rho(1,i,n-2) + [(3m+2j+3)\frac{n}{2} - 1 - i]\rho(n-1,i,n))\rho(2,j,\frac{m-1}{2}) + ([\frac{n}{2}(m+2j-1) - 1 - i]\rho(1,i,n-2) + [\frac{n}{2}(m+2j+1) - 1 - i]\rho(n-1,i,n))\rho(\frac{m+3}{2}, j, m-1)$

for $i \in I$ and j even, $j \in J$,

$g_2(y_{j,i}y_{j,i+1}) = [\frac{n}{2}(m+2j-3)+3-i]\rho(1,i,2) + [\frac{n}{2}(m+2j-1)+3-i]\rho(3,i,n)$

for $i \in I$ and $j = m+1$.

$g_1(xy_{1,i}) = g_2(xy_{1,i}) = [(2m+1)n+1]\rho(1,i,1) + [2(m+1)n-i+2]\rho(2,i,n)$,

$g_1(y_{m+1,i}z) = g_2(y_{m+1,i}z) = [(2m+3)n+i-2]\rho(1,i,2) + [2(m+1)n + i - 2]\rho(3,i,n)$

for $i \in I$.

Theorem 1. If m is odd, $m \geq 3$, $n \geq 3$, then the plane graph $\mathbf{Q}_n^m$ has

$\left(\frac{7n(m+1)}{2}+2,1\right)$-face antimagic labeling.

Proof. If $m \equiv 3 \pmod 4$, $m \geq 3$ and $n \geq 3$, we label the edges of the plane graph $\mathbf{Q}_n^m$ by the edge labeling g_1 and if $m \equiv 1 \pmod 4$, $m \geq 5$ and $n \geq 3$, we label the edges of $\mathbf{Q}_n^m$ by the labeling g_2.

It is easy to verify that the labeling g_1 (respectively, g_2) uses each integer $1, 2, ..., 3n(m+1)$ exactly once. This implies that the labelings g_1 and g_2 are bijections from the edge set $E(\mathbf{Q}_n^m)$ onto the set $\{1, 2, ..., 3n(m+1)\}$.

By direct computation we obtain that the weights of 3-sided faces under the labelings g_1 and g_2 constitute the sets of consecutive integers:

$$W_1(g_1) = W_1(g_2) = \{\tfrac{7n(m+1)}{2}+2, \tfrac{7n(m+1)}{2}+3, ..., \tfrac{11nm+7n}{2}+1\},$$

$$W_2(g_1) = W_2(g_2) = \{\tfrac{11nm+7n}{2}+1+i : i \in I\},$$

$$W_3(g_1) = W_3(g_2) = \{\tfrac{11nm+9n}{2}+1+i : i \in I\}.$$

It is not difficult to check that $\bigcap_{k=1}^{3} W_k(g_1) = \emptyset$, $\bigcap_{k=1}^{3} W_k(g_2) = \emptyset$ and further that the sets $\bigcup_{k=1}^{3} W_k(g_1)$, $\bigcup_{k=1}^{3} W_k(g_2)$ consist of consecutive integers. This proves that the edge labelings g_1 and g_2 are $(\frac{7n(m+1)}{2}+2,1)$-face antimagic.

□

Now, we define an edge labeling g_3 (if n is even, $n \geq 4$ and $m \equiv 0 \pmod 4$, $m \geq 4$) and edge labeling g_4 (if n is even, $n \geq 4$ and $m \equiv 2 \pmod 4$, $m \geq 6$) of $\mathbf{Q}_n^m$ as follows, where again the functions $\phi(x)$ and $\rho(x,y,z)$, defined in (6) and (7), are used:

$$g_3(y_{j,i+1}y_{j+1,i}) = [\tfrac{n}{2}(m+2j+1)+i]\rho(1,j,\tfrac{m}{2}-1) + [\tfrac{n}{2}(2j-m-1)+i]\rho(\tfrac{m}{2}+1,j,m-1)$$

for $i \in I$ and j odd, $j \in J$.

$$g_4(y_{j,i+1}y_{j+1,i}) = [\tfrac{n}{2}(2m-4j-1)+i]\rho(1,j,\tfrac{m}{2}-2) + [\tfrac{n}{2}(2m-1)+i]\rho(\tfrac{m}{2},j,\tfrac{m}{2}) + [\tfrac{n}{2}(4m-4j+1)+i]\rho(\tfrac{m}{2}+2,j,m-1)$$

for $i \in I$ and j odd, $j \in J$.

$g_3(y_{j,i}y_{j+1,i+1}) = ([\frac{n}{2}(2j-1)+1]\rho(1,i,1)+[\frac{n}{2}(2j+1)-i+2]\rho(2,i,n))\rho(2,j,\frac{m}{2}) + ([\frac{n}{2}(2j-3)+1]\rho(1,i,1)+[\frac{n}{2}(2j-1)-i+2]\rho(2,i,n))\rho(\frac{m}{2}+2,j,m)$

for $i \in I$ and j even, $j \in J$.

$g_4(y_{j,i}y_{j+1,i+1}) = ([\frac{n}{2}(2m-4j-1)+1]\rho(1,i,1)+[\frac{n}{2}(2m-4j+1)-i+2]\rho(2,i,n))\rho(2,j,\frac{m}{2}-1) + ([\frac{n}{2}(4m-4j+1)+1]\rho(1,i,1)+[\frac{n}{2}(4m-4j+3)-i+2]\rho(2,i,n))\rho(\frac{m}{2}+1,j,m)$

for $i \in I$ and j even, $j \in J$.

$g_3(y_{j,i}y_{j+1,i}) = ([\frac{n}{2}(6m+5-2j)+1]\rho(1,i,1)+[\frac{n}{2}(6m+7-2j)-i+2]\rho(2,i,n))\rho(1,j,\frac{m}{2}-1) + ([\frac{n}{2}(6m+3-2j)+1]\rho(1,i,1)+[\frac{n}{2}(6m+5-2j)-i+2]\rho(2,i,n))\rho(\frac{m}{2}+1,j,m-1)$

for $i \in I$ and j odd, $j \in J$.

$g_4(y_{j,i}y_{j+1,i}) = [\frac{n}{2}(6m+5-2j)+1]\rho(1,i,1)+[\frac{n}{2}(6m+7-2j)-i+2]\rho(2,i,n)$

for $i \in I$ and j odd, $j \in J$.

$g_3(y_{j,i}y_{j,i+1}) = \frac{n}{2}(4m+3)\rho(1,i,1) + ([\frac{n}{2}(4m+1)+\frac{i}{2}]\phi(i+1) + [\frac{n}{2}(4m+2)+\frac{i-1}{2}]\phi(i))\rho(2,i,n)$

for $i \in I$ and $j = 1$.

$g_3(y_{j,i}y_{j,i+1}) = (\frac{n}{2}(4m+5-2j)\rho(1,i,1)+[\frac{n}{2}(4m+3-2j)+i-1]\rho(2,i,n))\phi(j) + ([\frac{n}{2}(4m+5-2j)-1-i]\rho(1,i,n-2)+[\frac{n}{2}(4m+7-2j)-1-i]\rho(n-1,i,n))\phi(j+1)$

for $i \in I$ and $2 \leq j \leq m+1$.

$g_4(y_{j,i}y_{j,i+1}) = \frac{n}{2}(3m+5)\rho(1,i,1) + ([\frac{n}{2}(3m+3)+\frac{i}{2}]\phi(i+1) + [\frac{n}{2}(3m+4)+\frac{i-1}{2}]\phi(i))\rho(2,i,n)$

for $i \in I$ and $j = 1$.

$g_4(y_{j,i}y_{j,i+1}) = (\frac{n}{2}(3m+3+2j)\rho(1,i,1)+[\frac{n}{2}(3m+1+2j)+i-1]\rho(2,i,n))\rho(1,j,\frac{m}{2}) + (\frac{n}{2}(m+1+2j)\rho(1,i,1)+[\frac{n}{2}(m-1+2j)+i-1]\rho(2,i,n))\rho(\frac{m}{2}+2,j,m+1)$

for $i \in I$ and j odd, $2 \leq j \leq m+1$.

$g_4(y_{j,i}y_{j,i+1}) = ([\frac{n}{2}(3m+3+2j) - i - 1]\rho(1,i,n-2) + [\frac{n}{2}(3m+5+2j) - i - 1]\rho(n-1,i,n))\rho(2,j,\frac{m}{2}-1) + ([\frac{n}{2}(m+1+2j) - i - 1]\rho(1,i,n-2) + [\frac{n}{2}(m+3+2j) - i - 1]\rho(n-1,i,n))\rho(\frac{m}{2}+1,j,m)$

for $i \in I$ and j even, $j \in J$.

$g_3(y_{j,i}y_{j+1,i}) = ([\frac{n}{2}(6m+5-2j) + 1]\rho(n,i,n) + [\frac{n}{2}(6m+5-2j) + 1 + i]\rho(1,i,n-1))\rho(2,j,\frac{m}{2}) + ([\frac{n}{2}(6m+7-2j) + 1 + i]\rho(1,i,n-1) + [\frac{n}{2}(6m+7-2j) + 1]\rho(n,i,n))\rho(\frac{m}{2}+2,j,m)$

for $i \in I$ and j even, $j \in J$.

$g_4(y_{j,i}y_{j+1,i}) = [\frac{n}{2}(6m+5-2j) + 1 + i]\rho(1,i,n-1) + [\frac{n}{2}(6m+5-2j) + 1]\rho(n,i,n)$

for $i \in I$ and j even, $j \in J$.

$g_3(xy_{1,i}) = g_4(xy_{1,i}) = \frac{i}{2}\phi(i+1) + [\frac{n}{2}(6m+5) + 1]\rho(1,i,1) + [3n(m+1) - \frac{i-3}{2}]\phi(i)$,

$g_3(y_{m+1,i}z) = g_4(y_{m+1,i}z) = [(4m+3)\frac{n}{2} + 3 - i]\rho(1,i,2) + [(4m+5)\frac{n}{2} + 3 - i]\rho(3,i,n)$

for $i \in I$.

Theorem 2. If n and m are even, $n \geq 4$, $m \geq 4$, then the graph of the convex polytope $\mathbf{Q}_n^m$ has $(\frac{7n(m+1)}{2} + 2, 1)$-face antimagic labeling.

Proof. Label the edges of $\mathbf{Q}_n^m$ by the edge labeling g_3 (for $n \equiv 0 \pmod 2$, $n \geq 4$, and $m \equiv 0 \pmod 4$, $m \geq 4$) and by the edge labeling g_4 (for $n \equiv 0 \pmod 2$, $n \geq 4$ and $m \equiv 2 \pmod 4$, $m \geq 6$).

It is a matter of routine checking to see that the edge labelings g_3 and g_4 are the bijections from the edge set $E(\mathbf{Q}_n^m)$ onto the set $\{1, 2, ..., |E(\mathbf{Q}_n^m)|\}$.

Thus, weights of 3-sided faces

(i) under the labeling g_3 constitute the sets of consecutive integers

$W_1(g_3) = \{\frac{7n(m+1)}{2}+2, \frac{7n(m+1)}{2}+3, ..., \frac{10nm+7n}{2}+1\} \cup \{\frac{10nm+11n}{2}+2, \frac{10nm+11n}{2}+3, ..., \frac{11n(m+1)}{2}+1\}$,

$W_2(g_3) = \{\frac{10nm+7n}{2}+2, \frac{10nm+7n}{2}+3, ..., \frac{10nm+9n}{2}+1\}$,

$W_3(g_3) = \{\frac{10nm+9n}{2}+2, \frac{10nm+9n}{2}+3, ..., \frac{10nm+11n}{2}+1\}$ and

(ii) under the labeling g_4 constitute the sets of consecutive integers

$W_1(g_4) = \{\frac{7n(m+1)}{2}+2, \frac{7n(m+1)}{2}+3, ..., \frac{9n(m+1)}{2}+1\} \cup \{\frac{9nm+11n}{2}+2, \frac{9nm+11n}{2}+3, ..., \frac{11nm+9n}{2}+1\}$,

$W_2(g_4) = \{\frac{9n(m+1)}{2}+2, \frac{9n(m+1)}{2}+3, ..., \frac{9nm+11n}{2}+1\}$,

$W_3(g_4) = \{\frac{11nm+9n}{2}+2, \frac{11nm+9n}{2}+3, ..., \frac{11n(m+1)}{2}+1\}$.

Each face of $\mathbf{Q}_n^m$ receives exactly one label of weight from $\bigcup_{k=1}^{3} W_k(g_3)$ ($\bigcup_{k=1}^{3} W_k(g_4)$) and each number from the set $\bigcup_{k=1}^{3} W_k(g_3)$ ($\bigcup_{k=1}^{3} W_k(g_4)$) is used exactly once as a label of weight.

It can be seen that the induced mappings $g_3^* : F(\mathbf{Q}_n^m) \to \bigcup_{k=1}^{3} W_k(g_3)$ and $g_4^* : F(\mathbf{Q}_n^m) \to \bigcup_{k=1}^{3} W_k(g_4)$ are bijections. □

5 Conclusion

In this paper we have proved that for $n \geq 3$ and $m \geq 3$ there exist $\left(\frac{7n(m+1)}{2}+2, 1\right)$-face antimagic labelings for graphs of the convex polytopes $\mathbf{Q}_n^m$.

We suggest the following

Conjecture 1 If n is odd, $n \geq 3$, and m is even, $m \geq 2$, then the plane graph $\mathbf{Q}_n^m$ has $\left(\frac{5n(m+1)+5}{2}, 2\right)$-face antimagic labeling and $\left(\frac{n(m+1)+7}{2}, 4\right)$-face antimagic labeling.

Conjecture 2. If n is even, $n \geq 4$, and $m \geq 1$, or if n is odd, $n \geq 3$, and m is odd, $m \geq 1$, then the graph of the convex polytope $\mathbf{Q}_n^m$ has $\left(\frac{3n(m+1)}{2} + 3, 3\right)$-face antimagic labeling.

REFERENCES

[1] M. Bača, On face antimagic labelings of plane graphs, *Acta Math. Applic. Sinica*, submitted.

[2] M. Bača, Face antimagic labelings of convex polytopes, *Utilitas Math.* **55** (1999), 221 - 226.

[3] R. Bodendiek and G. Walther, Arithmetisch antimagische graphen, In: K. Wagner and R. Bodendiek, *Graphentheorie III*, BI-Wiss. Verl., Mannheim, 1993.

[4] R. Bodendiek and G. Walther, On (a, d)-antimagic parachutes, *Ars Combin.* **42** (1996), 129 - 149.

[5] R. Bodendiek and G. Walther, (a, d)-antimagic parachutes II, *Ars Combin.* **46** (1997), 33 - 63.

[6] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, Boston - San Diego - New York - London, 1990.