Total domination in claw-free cubic graphs

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Abstract

We prove that the total domination number of an n-vertex claw-free cubic graph is at most $n/2$.

1 Introduction

The open neighbourhood of a vertex v of a graph $G = (V, E)$ is $N(v) = \{u \in V : uv \in E\}$. Whenever necessary we write $V = V(G)$ to indicate the graph concerned. For $S \subseteq V$, the *open neighbourhood* of S is defined by $N(S) = \cup_{v \in S} N(v)$. A set $S \subseteq V$ is a *total dominating set*, abbreviated *TDS*, if every vertex in V is adjacent to a vertex in S. (That is, $N(S) = V$.) Every graph without isolated vertices has a TDS, since $S = V$ is such a set. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality among its total dominating sets. A TDS of G of cardinality $\gamma_t(G)$ is called a *γ_t-set* of G. Total domination in graphs was introduced by Cockayne, Dawes and Hedetniemi [1] and is now well studied in graph theory – see [5, 6].

Several upper bounds for γ_t are given in [5, pp. 160-161]. More recently, Henning [2] proved:

Theorem 1 *If G is a connected graph of order n with minimum degree $\delta(G) \geq 2$ and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then $\gamma_t(G) \leq 4n/7$.*

Further, Favaron, Henning, Mynhardt and Puech [4] proved:

Theorem 2 *If G has order n and $\delta(G) \geq 3$, then $\gamma_t(G) \leq 7n/13$.*

They also posed the following conjecture.

Conjecture 1. If G has order n and $\delta(G) \geq 3$, then $\gamma_t(G) \leq n/2$.

Work on this conjecture motivated this paper which establishes the bound for claw-free cubic graphs.

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2 The bound

Let S be a TDS of a cubic graph G. The subsets of vertices of S which have degree one, two and three in $G[S]$ are denoted by S_1, S_2 and S_3 respectively. Note that $G[S]$ has no isolated vertices. Further, let W_1, W_2 and W_3 respectively denote the subsets of $V - S$ which send one, two and three edges to S. Since S is total dominating, $V - S = W_1 \cup W_2 \cup W_3$. Since G is cubic, W_3 is independent and no edge of G joins W_3 to $W_1 \cup W_2$. If $w \in W_1$ and s is the (unique) neighbour of w in S, then w is called an *external private neighbour* (*epn*) of s. A vertex of S may have no, one, or two epns.

We require the following minimality condition obtained by Cockayne, Dawes and Hedetniemi [1].

Proposition 3 *In any graph, a TDS S is minimal if and only if for each $s \in S$,*

$$\begin{array}{ll} \text{(i)} & s \text{ is adjacent to a vertex of } S_1, \text{ or} \\ \text{(ii)} & s \text{ has an external private neighbour.} \end{array} \tag{1}$$

In particular, each γ_t-set has Property (1) which is also known as open-open irredundance (*cf.* [7]).

A graph is *claw-free* if it has no induced subgraph isomorphic to the claw graph $K_{1,3}$. We now show that every claw-free cubic graph has a γ_t-set which satisfies certain properties.

Proposition 4 *Any claw-free cubic graph G has a γ_t-set S such that*

(i) *$S_3 = \emptyset$ (i.e., each component of $G[S]$ is a path or a cycle), and*

(ii) *if s has degree two in a component P_k of $G[S]$, where $k \geq 4$, then s has a unique epn.*

Proof. Among all γ_t-sets of G, choose S so that the number of edges in $G[S]$ is minimum.

(i) Let $s \in S_3$. By Proposition 3, s (having no epn) is adjacent to some $u \in S_1$. The same proposition implies that u has an epn u'. If s is adjacent to two vertices of $S_2 \cup S_3$, then $S' = (S - \{s\}) \cup \{u'\}$ is a TDS of G and $G[S']$ has fewer edges than $G[S]$, a contradiction. Therefore s is adjacent to at least two vertices of S_1, which contradicts the claw-free property.

(ii) Let $G[S]$ have a component P_k, where $k \geq 4$, and let s be a vertex of degree two in this component. If s is not adjacent to an endvertex of the component, then s has an epn by Proposition 3. Now suppose that s is adjacent (in P_k) to the endvertex t of P_k, but s has no epn. By Proposition 3, t has an epn t'. Then $S' = (S - \{s\}) \cup \{t'\}$ is total dominating and $G[S']$ has one edge less than $G[S]$, a contradiction. Finally, since $s \in S_2$ and G is cubic, s may have at most one epn. ■

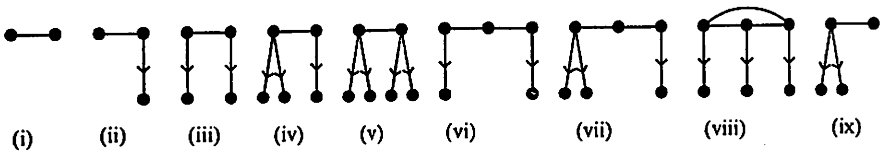

Figure 1: Types of components of $G[S]$

The purpose of this paper is to establish the following result.

Theorem 5 *If G is an n-vertex claw-free cubic graph, then $\gamma_t(G) \leq n/2$.*

Proof. Let S be a γ_t-set satisfying the hypothesis of Proposition 4. Observe that Proposition 3 implies that each endvertex in a component of $G[S]$ isomorphic to P_k ($k \geq 3$) has an epn.

Define the *weight* of an edge from S to W_i ($i \in \{1,2,3\}$) to be $1/i$. For a subset $T \subseteq S$, $\eta(T)$ is defined to be the difference of the sum of weights of all edges from T to $V - S$ and the cardinality of T. The proof will use a detailed analysis of the components of $G[S]$ to show that $\eta(S) \geq 0$, which is equivalent to $|V - S| \geq |S|$ and hence implies that $\gamma_t \leq n/2$.

If a component of $G[S]$ contains an induced subpath of three vertices, each of which has an epn, then the closed neighbourhood of the central vertex is a claw. This observation, together with Propositions 3 and 4, disqualifies C_k and P_k ($k \geq 4$) and P_3 where the central vertex has an epn, from being components of $G[S]$. The situation of a P_3 component in which each endvertex has two epns and the central vertex has no epn, is similarly eliminated since the graph is cubic and claw-free. Hence all possible types of components of $G[S]$ and epns of their vertices are those depicted in Figure 1. Edges from S to W_1 are arrowed.

Since G is claw-free, each $w \in W_3$ joins adjacent vertices of a component $F(w)$ of $G[S]$ and one more vertex $g(w)$ of a component $G(w)$. It is possible that $G(w) = F(w)$, in which case the component is a P_3 of type (vi).

Consider the digraph D whose vertex set is $\mathcal{C}$, the set of components of $G[S]$, and for $C_1, C_2 \in \mathcal{C}$, C_1C_2 is an arc (*i.e.*, directed edge) of D if for some $w \in W_3$, $C_1 = F(w)$ and $g(w) \in V(C_2)$. We say that the *degree* of C in D, abbreviated deg(C), is (p,q) if its indegree and outdegree are p and q respectively.

Each of the types of components C of $G[S]$, except types (v) and (viii), may send edges to $W_2 \cup W_3$ in one of several ways which determines deg(C) and $\eta(V(C))$ (which we abbreviate to $\eta(C)$). The following result will be used repeatedly to reduce the number of cases.

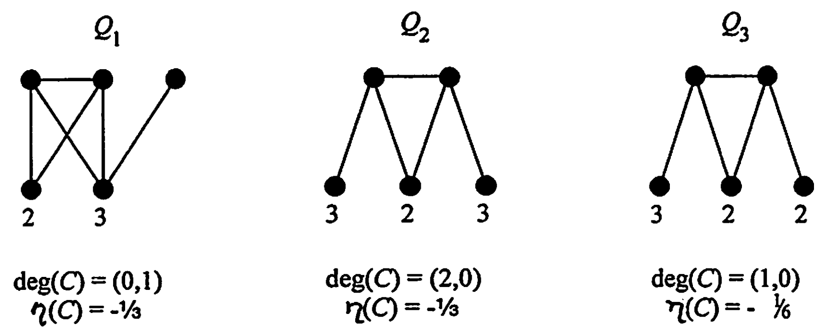

Figure 2: Type (i) components

Lemma 5.1 *Let C_1, C_2 be distinct components of $G[S]$, where $C_1 = F(w)$ and $g(w) \in V(C_2)$ for $w \in W_3$. Then $g(w)$ has no epn.*

Proof. If $g(w)$ has an epn, then $N[g(w)]$ is a claw. □

We now present the catalogue of components of $G[S]$ under four headings.

Components which do not join W_3

Each such component C has $\deg(C) = (0,0)$ (*i.e.*, C is an isolated vertex of D). The weights of edges from these components are either 1 or $\frac{1}{2}$ and hence it is easy to verify that $\eta(C) \geq 0$. This situation includes components of types (v), (viii) and (iv) (which do not join W_3 by Lemma 5.1) and type (ix) (which do not join W_3 by the claw-free property).

Type (i)

Suppose that C is a type (i) component with $V(C) = \{v_1, v_2\}$. If $C = F(w)$ for $w \in W_3$, then there exists $w' \in W_2$ adjacent to both v_1 and v_2 (otherwise $(S - \{v_1, v_2\}) \cup \{w\}$ is total dominating, a contradiction). Thus $N(V(C)) \cap W_3 = \{w\}$ and $\deg(C) = (0,1)$. If C joins W_3 but $C \neq F(w)$ for any $w \in W_3$, then the claw-free property implies that neither v_1 nor v_2 is adjacent to two vertices of W_3. The three possibilities for type (i) components are depicted in Figure 2. In this figure as well as Figures 3 and 4 below, the labels 1, 2 and 3 on vertices of $V - S$ signify that the vertex is an element of W_1, W_2 and W_3 respectively.

Type (ii)

Let C be a type (ii) component which joins $w \in W_3$, where $V(C) = \{v_1, v_2\}$ and v_2 has the epn. If $C = F(w)$, then there are two choices for the third edge incident with v_1. If $C \neq F(w)$, then (by Lemma 5.1) $v_1 = g(w)$ and the claw-free property implies the existence of $w' \in W_2$ adjacent to both v_1 and v_2. The possibilities are depicted in Figure 3.

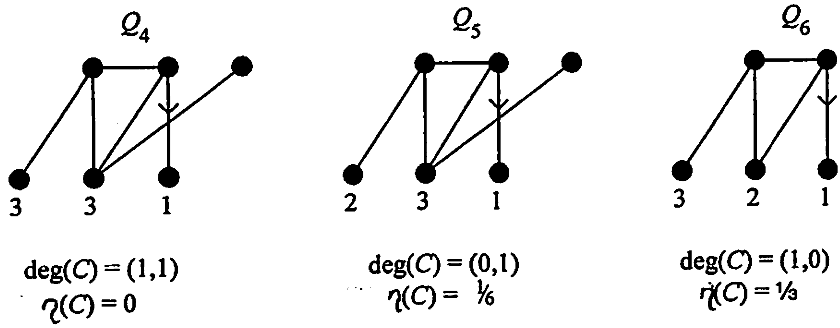

Figure 3: Type (ii) components

Types (iii), (vi) and (vii)
If C is a component of type (iii), (vi) or (vii) which joins W_3, then by the claw-free property, some $w \in W_3$ joins adjacent vertices of C. This fact and Lemma 5.1 eliminate all but the possibilities in Figure 4.

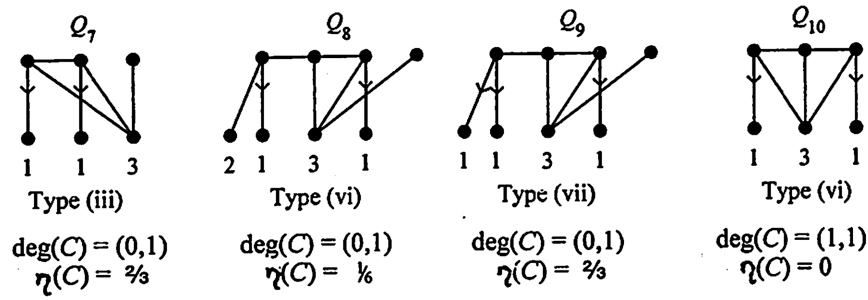

Figure 4: Components of Types (iii), (vi) and (vii)

A component of $G[S]$ which is connected to $V-S$ as shown in a diagram labelled Q_i in Figure 2, 3 or 4 will be called a *Q_i-component*, and an arc of D from a Q_i-component to a Q_j-component will be termed a *Q_iQ_j-arc*.

Lemma 5.2 *If* $(i,j) \in \{(1,2),(4,2),(1,3),(4,3),(5,2)\}$, *then D has no Q_iQ_j-arc.*

Proof. Suppose the contrary and that v_1, v_2 are the vertices of the Q_i-component of $G[S]$, where v_1 has no epn. There exists a vertex $w \in W_3$ adjacent to v_1 and v_2. Suppose that v_3, v_4 are the vertices of the Q_j-component where $g(w) = v_3$. Then $(S-\{v_1,v_4\}) \cup \{w\}$ is total dominating, a contradiction. □

Let $\mathcal{X}$ be the set of components of the digraph D. For $X \in \mathcal{X}$, let the vertices of X be the subset $\{C_1, ..., C_t\}$ of components of $G[S]$ and let $U(X) = \cup_{i=1}^{t} V(C_i)$. Each component of $G[S]$ is a vertex of precisely one component of D and hence $\{U(X) : X \in \mathcal{X}\}$ is a partition of S. The proof of the theorem will be completed by showing that $\eta(U(X)) \geq 0$ for each $X \in \mathcal{X}$, which will imply that $\eta(S) = \Sigma_{X \in \mathcal{X}} \eta(U(X)) \geq 0$, as required.

Observe that $\eta(C) \geq 0$ for each Q_i-component C, where $i \geq 4$. Hence $\eta(U(X)) \geq 0$ unless X has a vertex corresponding to a Q_i-component for $i = 1, 2$ or 3 (*i.e.*, some type (i) component). Thus it is sufficient to consider X having this property and degrees in $\{(1,0), (0,1), (2,0), (1,1)\}$.

Suppose that X has a vertex C corresponding to a Q_2-component. Since $\deg(C) = (2,0)$, C is adjacent from two vertices C_j, $j = 1, 2$, which have outdegree one. By Lemma 5.2, if C_j is a Q_i-component, then $i \notin \{1, 4, 5\}$. Furhter, $i \neq 10$ since a Q_{10}-component is an isolated loop of D. Hence $i \in \{7, 8, 9\}$ and so C_j has indegree zero. Then $V(X) = \{C, C_1, C_2\}$ and for all choices of C_1 and C_2, $\eta(U(X)) = \eta(C) + \eta(C_1) + \eta(C_2) \geq 0$.

Next, let X have a vertex C corresponding to a Q_3-component. Since $\deg(C) = (1,0)$, C is adjacent from C_1 which has outdegree one. By Lemma 5.2, if C_1 is a Q_i-component, then $i \notin \{1, 4\}$ and so $i \in \{5, 7, 8, 9\}$. Each choice of C_1 has indegree zero, hence $V(X) = \{C, C_1\}$ and $\eta(U(X)) = \eta(C) + \eta(C_1) \geq 0$.

Finally, suppose that X has a vertex C corresponding to a Q_1-component and no vertex corresponding to a Q_2- or Q_3-component. All other vertices of X have degrees in $\{(1,1), (0,1), (1,0)\}$. In this case X is a directed path with vertex sequence $C, C_1, ..., C_s, C'$, where $s \geq 0$, each C_j, $j = 1, ..., s$, is a Q_4-component, and C' is a Q_6-component. For all such paths $\eta(U(X)) = 0$. ■

Acknowledgement
This paper was written while E. J. Cockayne was visiting the Department of Mathematics at the University of South Africa. Grants from NSERC (Canada), the NRF (South Africa) and the Franco/South African Agreement for Co-operation in Science and Engineering are gratefully acknowledged.

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