A Lower Bound For Domination Numbers Of The Queen's Graph

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Abstract

The queen's graph Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are adjacent if they are in the same row, column, or diagonal. Let $\gamma(Q_n)$ be the minimum size of a dominating set of Q_n. Spencer proved that $\gamma(Q_n) \geq (n-1)/2$ for all n, and the author showed $\gamma(Q_n) = (n-1)/2$ implies $n \equiv 3 \pmod 4$ and any minimum dominating set of Q_n is independent.

Define a sequence by $n_1 = 3$, $n_2 = 11$, and for $i > 2$, $n_i = 4n_{i-1} - n_{i-2} - 2$. We show that if $\gamma(Q_n) = (n-1)/2$ then n is a member of the sequence other than $n_3 = 39$, and (counting from the center) the rows and columns occupied by any minimum dominating set of Q_n are exactly the even-numbered ones. This improvement in the lower bound enables us to find the exact value of $\gamma(Q_n)$ for several n; $\gamma(Q_n) = (n+1)/2$ is shown here for $n = 23, 39$, and elsewhere for $n = 27, 71, 91, 115, 131$.

Keywords: dominating set, queen domination, queen's graph.

The queen's graph Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are adjacent if they are in the same row, column, or diagonal. A set D of squares of Q_n is a *dominating set* for Q_n if every square of Q_n is either in D or adjacent to a square in D. If no two squares of a set I are adjacent then I is an *independent set*. Let $\gamma(Q_n)$ denote the minimum size of a dominating set for Q_n; a dominating set of this size is a *minimum* dominating set. Let $i(Q_n)$ denote the minimum size of an independent dominating set for Q_n.

The task of determining values of $\gamma(Q_n)$ and $i(Q_n)$ appears as Problem C18 in the recent collection [11] of unsolved problems. However, finding these values has interested mathematicians for at least 140 years [1, 8, 13].

The general method of establishing the value of $\gamma(Q_n)$ or $i(Q_n)$ for a specific n is to construct a dominating set of Q_n whose size equals a proven lower bound. As minimum dominating sets for even n seem to be closely related to those for the odd values $n-1$ or $n+1$, we shall focus on odd n.

The earliest non-trivial lower bound established for the domination numbers of Q_n is

$$\gamma(Q_n) \geq (n-1)/2, \tag{1}$$

due to Spencer [7, 14]. Well known dominating sets showed that equality held in (1) for $n = 3, 11$, but no other such n have been found, so it seemed that (1) could be bettered. As $\gamma(Q_n) \leq i(Q_n)$ for all n and $i(Q_n) \leq \lceil n/2 \rceil + 1$ is shown in [10] for $n \leq 120$ (and conjectured for all n), even small improvements in (1) give values of $\gamma(Q_n)$ and $i(Q_n)$ for more n; see below.

The author proved (see Theorem 2 below) several facts about dominating sets of size $(n-1)/2$ for Q_n, in particular that they are independent sets and that $n \equiv 3 \pmod 4$ is necessary. Thus $\gamma(Q_{4k+1}) \geq 2k+1$ for all k, and a combined effort [2, 4, 9, 12, 14] has shown that equality holds for $k \leq 21$; this is extended to $k \leq 32$ in [10]. Burger and Mynhardt built on Theorem 2 in [3], and were able to show with the aid of a computer search that $\gamma(Q_{4k+3}) > 2k+1$ for $3 \leq k \leq 7$. They then established the values $\gamma(Q_{19}) = 10$ and $\gamma(Q_{31}) = 16$. Several of their conjectures will be discussed after Theorem 4.

Define a sequence by $n_1 = 3$, $n_2 = 11$, and for $i > 2$, $n_i = 4n_{i-1} - n_{i-2} - 2$. In this paper, we use the Parallelogram Law (Theorem 1) and other new techniques to prove (Theorems 3 and 4) that if $\gamma(Q_n) = (n-1)/2$ then n is a member of the sequence other than $n_3 = 39$, and any minimum dominating set D of Q_n occupies the even-numbered rows and columns, counting from the center. We show $\gamma(Q_n) = (n+1)/2$ for $n = 23, 39$; in [10] this is shown for $n = 27, 71, 91, 115, 131$.

We will identify the $n \times n$ chessboard with a square of side length n in the Cartesian plane, having sides parallel to the coordinate axes. The origin of the coordinate system may be placed at any point convenient for the task at hand; to study a minimum dominating set D of Q_n when $\gamma(Q_n) = (n-1)/2$, we will take the origin at the center of the sub-board U specified in Theorem 2 below. We refer to board squares by the coordinates of their centers; the square (x,y) is in *column* x and *row* y. The square (x,y) is *even* if $x+y$ is even, *odd* if $x+y$ is odd. If n is odd, we divide the even squares of Q_n into two classes: (x,y) is *even-even* if both x and y are even, *odd-odd* if both are odd.

Columns and rows will be referred to collectively as *orthogonals*. The *difference diagonal* (respectively *sum diagonal*) through square (x,y) is the set of all board squares with centers on the line of slope +1 (respectively −1) through the point (x,y). The value of $y-x$ is the same for each square (x,y) on a difference diagonal, and we will refer to the diagonal by this value. Similarly, the value of $x+y$ is the same for each square on a sum diagonal, and we associate this value to the diagonal. We refer to orthogonals and diagonals collectively as *lines* of Q_n.

In several kinds of problems involving Q_n, one wishes to choose a set of queen squares so that a specified set of lines is occupied. The following elementary result has not previously been applied to these problems; we will see in the proofs of Theorems 3 and 4 that it greatly restricts the possibilities for line sets.

Theorem 1 (Parallelogram Law) *Let D be a set of k squares of Q_n that occupies columns numbered $(c_i)_{i=1}^k$, rows $(r_i)_{i=1}^k$, difference diagonals $(d_i)_{i=1}^k$, and sum diagonals $(s_i)_{i=1}^k$. Then*

$$2\sum_{i=1}^{k} c_i^2 + 2\sum_{i=1}^{k} r_i^2 = \sum_{i=1}^{k} d_i^2 + \sum_{i=1}^{k} s_i^2. \tag{2}$$

Proof. Let $D = \{(x_i, y_i) : 1 \le i \le k\}$. Then there are permutations e, f, g, h such that for each i we have $x_i = c_{e(i)}$, $y_i = r_{f(i)}$, $y_i - x_i = d_{g(i)}$, and $y_i + x_i = s_{h(i)}$. The conclusion follows from the identity $2x_i^2 + 2y_i^2 = (y_i - x_i)^2 + (y_i + x_i)^2$ and summation. □

The following is Theorem 2 of [14], together with some facts established in its proof.

Theorem 2 *Let n be a positive integer such that $\gamma(Q_n) = (n-1)/2$. Then $n \equiv 3 \pmod 4$). Let D be a dominating set of $(n-1)/2$ squares of Q_n. Then D is independent, so $i(Q_n) = (n-1)/2$, and there is an odd integer j, $3(n+1)/4 \le j \le n$, such that there is a $j \times j$ sub-board U of Q_n satisfying:*

(a) each edge square of U is attacked exactly once;

(b) each orthogonal of Q_n that does not meet U contains exactly one square of D, and the orthogonals of Q_n that extend the edge orthogonals of U contain no squares of D.

(c) difference diagonal 0 of U contains exactly one square of D, and exactly half the remaining squares of D lie above the extension of that diagonal to Q_n, and half below; similarly for sum diagonal 0 of U;

(d) for every square of D, the absolute value of each diagonal number is strictly less than $j-1$, and thus at least one of its orthogonals meets U.

We adopt the notation of Theorem 2, and will make frequent use of its conclusions, especially (a). To this end, let E be the set of edge squares of U, and let E_c, E_r, E_d, and E_s be the subsets of E consisting of those squares that are covered along their column, row, difference diagonal, and sum diagonal, respectively. Then Theorem 2(a) says these four sets form a partition of E.

A line containing a square of D is *occupied*; a square where four occupied lines meet is *eligible*. Clearly each occupied line contains at least one eligible square.

Let m be an odd positive integer. Let s_1, s_2 be squares of the same edge row (respectively edge column) of U whose column (respectively row) numbers differ by m. If s_1 and s_2 are dominated along diagonals that cross inside U, we say that they form an *m-crosspair*. The *mean* of an m-crosspair of the top edge of U is the mean value of the column numbers of its two squares.

A diagonally covered square of E that is not in any m-crosspair is an *m-singleton*. If the type of diagonal is important, we refer to *difference* m-singletons and *sum* m-singletons.

Set $\ell = (j-1)/2$. For integers $m > 0$ and i, let $S(i,m) = \{(x,\ell) : -\ell \leq x \leq \ell \text{ and } x \equiv i \pmod{m}\}$.

Thus $\{S(i,m) : 0 \leq i < m\}$ is a partition of the top edge of U.

Let i be an integer, $|i| \leq \ell$. Then $\text{Orth}(i)$ denotes a set of lines: columns $\pm i$ and rows $\pm i$ of Q_n. If $i \neq 0$ then $\text{Orth}(i)$ contains four lines. Similarly, let $\text{Diag}(i)$ denote the set of sum and difference diagonals of Q_n numbered $\pm i$. (For $|i| > j$, some of these diagonals may be "off the board".) Say $\text{Orth}(i)$ is *full* if all of its lines are occupied, *empty* if none are occupied; similarly for $\text{Diag}(i)$. Thus Theorem 2(b) says in part that $\text{Orth}(\ell)$ is empty, Theorem 2(c) says in part that $\text{Diag}(0)$ is full, and Theorem 2(d) says that if $|i| \geq j$ then $\text{Diag}(i)$ is empty.

Let L denote the set of lines that meet U. The eight-element group of symmetries of a square will be denoted D_4. The natural action of D_4 on U gives an action of D_4 on L.

Theorem 3 *Let n be a positive integer such that $\gamma(Q_n) = (n-1)/2$, and let D be a dominating set of size $(n-1)/2$. Then $U = Q_n$ and every square of D has both coordinates even.*

Proof. The unique minimum dominating set $\{(0,0)\}$ of Q_3 satisfies the desired conclusion, and $\gamma(Q_7) = 4$ was shown in Corollary 3 of [14], so Theorem 2 allows us to assume $n \geq 11$, and then that $j \geq 9$.

First we must prove that $\text{Orth}(\ell - m)$ is full for each odd m, $1 \leq m < \ell$. The proof is by induction on m; we may assume for each odd i, $1 \leq i < m$, that $\text{Orth}(\ell - i)$ is full. (This is vacuous for $m = 1$, but the argument for

$m = 1$ is essentially similar to that for the inductive step, so we treat it here.) For purposes of contradiction, we may suppose that some line of Orth$(\ell - m)$ is unoccupied; by rotating the board we may take that line to be row $\ell - m$. We break the argument into a sequence of lemmas.

Lemma A *Orth(0) is full.*

Proof. Suppose that row 0 contains no square of D. Then Theorem 2(b) implies the square $(\ell, 0)$ is not covered orthogonally, so one of its diagonals is occupied. This implies that one of the squares $(0, \pm\ell)$ is attacked diagonally. Since these are squares of E, Theorem 2(a) implies that column 0 is empty.

We next show for each i, $1 \leq i < \ell$, that row i is occupied if and only if row $i - \ell$ is occupied. Suppose that row i is occupied. By Theorem 2(a), the squares $(\pm\ell, i)$ are not diagonally attacked, so both diagonals through the square $(0, i - \ell)$ are empty. Since column 0 is empty, this square must be covered along row $i - \ell$. Conversely, if row $i - \ell$ is occupied then squares $(\pm\ell, i - \ell)$ are not diagonally attacked, so $(0, i)$ is not diagonally attacked, and since column 0 is empty we see row i is occupied.

Since rows ℓ, 0, $-\ell$ are empty and any occupied rows with numbers among $\pm1, \ldots, \pm(\ell - 1)$ come in pairs of the form $\{i, i - \ell\}$, the number of occupied rows meeting U is even. By Theorem 2(b), every row not meeting U is occupied, and the number of these is $n - j$, which is even. Thus the number of occupied rows is even, and since D is independent, each occupied row contains just one square of D, so $|D|$ is even. But Theorem 2 says $n \equiv 3 \pmod 4$, and then $|D| = (n - 1)/2$ implies $|D|$ is odd. This contradiction shows that row 0 is occupied, and similarly column 0 is occupied; Lemma A is proved.

Lemma B *There is a non-negative integer t such that the rightmost t squares of $S(\ell, m)$ are difference m-singletons, the leftmost t squares of $S(-\ell, m)$ are sum m-singletons, and all other diagonally covered squares of row ℓ of U are in m-crosspairs.*

Proof. The following fact will be used several times.

Claim B1. The difference and sum diagonals numbered $-(m-2), -(m-4), \ldots, -1, 1, \ldots, m-4, m-2$ are empty.

By hypothesis, the edge squares $(\pm\ell, \pm(\ell-2k+1))$ and $(\pm(\ell-2k+1), \pm\ell)$ of U are orthogonally covered for $1 \leq k \leq (m-1)/2$, so by Theorem 2(a), their diagonals are empty: Claim B1 is proved.

There are $m - 1$ diagonal numbers named in Claim B1, none is divisible by m, and (since m is odd) no two are congruent modulo m. It follows that for each u, $1 \leq u \leq m - 1$, exactly one of these numbers is congruent to u modulo m.

Suppose there is a difference m-singleton (x, ℓ); then there is x_1 such that (x_1, ℓ) is a difference m-singleton while $(x_1 - m, \ell)$ is not. We will prove the following.

Claim B2. The sum diagonal of square $(x_1, \ell - m)$ is not occupied.

If $x_1 - m \geq -\ell$ then $(x_1 - m, \ell)$ is a square of row ℓ of U and (by definition of m-crosspair) is not covered along sum diagonal $x_1 - m + \ell$, so also square $(x_1, \ell - m)$ is not covered along this diagonal. If $x_1 - m < -\ell$ then $x_1 < -\ell + m$, so since column x_1 is empty, we have $x_1 = -\ell + a$ for some even integer a, $0 \leq a < m$. From Theorem 2(c), $(-\ell, \ell) \notin E_s$, so Theorem 2(a) implies $a \neq 0$. Then $(x_1, \ell - m) = (-\ell + a, \ell - m)$ is on sum diagonal $a - m$, which by Claim B1 is empty, since $-m < a - m < 0$ and $a - m$ is odd. This establishes Claim B2.

By our assumptions, $(x_1, \ell - m)$ is not orthogonally covered, so Claim B2 implies its difference diagonal $-x_1 + \ell - m$ is occupied.

If $x_1 + m \leq \ell$ this diagonal meets row ℓ of U at the square $(x_1 + m, \ell)$, and then since (x_1, ℓ) is not covered along its sum diagonal, we may repeat the argument to show that difference diagonal $-x_1 + \ell - 2m$ is occupied. Continuing in this way, we obtain a run of occupied difference diagonals numbered $-x_1 + \ell - im$ for $0 \leq i \leq t$, where t is the least integer such that $-x_1 + \ell - tm < 0$.

If $x_1 \not\equiv \ell \pmod{m}$ then for each i, $-x_1 + \ell - im \not\equiv 0 \pmod{m}$ and we have a contradiction, as by Claim B1, one of this run of diagonals must be empty. Thus $x_1 \equiv \ell \pmod{m}$, and from our choice of x_1, it follows that the only squares of row ℓ of U that are difference m-singletons are the t squares of $S(\ell, m) = S(x_1, m)$ at or to the right of (x_1, ℓ).

Reflecting in column 0 gives a proof that among the sets $S(i, m)$, only $S(-\ell, m)$ can contain sum m-singletons, and that any such squares come at the left end; say there are t' of these. By Theorem 2(c) and (d), row ℓ of U contains $(n+1)/4$ squares of E_d and $(n+1)/4$ squares of E_s. Since each m-crosspair contains one of each, we see $t = t'$, which completes the proof of Lemma B.

Lemma C *Orth($\ell - m$) is empty.*

Proof. We are assuming row $\ell - m$ is empty. The eight squares $(\pm\ell, \pm(\ell - m))$, $(\pm(\ell - m, \pm\ell)$ are each uniquely covered. Using this, a short examination of cases shows that if some but not all lines of Orth($\ell - m$) are unoccupied, we may rotate the board so that column and row $-\ell + m$ and sum diagonals $-m$ and $2\ell - m$ are occupied. By Theorem 2(c), the square $(\ell, \ell) \in E_d$, and since sum diagonal $2\ell - m$ is occupied, $(\ell - m, \ell) \in E_s$. Thus (ℓ, ℓ) is in an m-crosspair of the top edge of U, so by Lemma B, the top edge of U has no difference m-singletons, and therefore no sum m-singletons.

But $(-\ell + m, \ell) \in E_c$ implies that $(-\ell, \ell)$ is a sum m-singleton. This contradiction establishes Lemma C.

Lemma D *The sub-board U has the same center square as Q_n, and the set L' of occupied lines has D_4 as its group of symmetries.*

Proof. By Lemma C we can apply the argument for Lemma B to each of the four edges of U, establishing the following.

Claim D1. The number of difference m-singletons on each edge of U equals the number of sum m-singletons on that edge. If this number is not zero, the m-singletons of the edge are spaced every m squares, starting at each end of the edge.

Let t_r, t_b, t_l be the numbers of difference m-singletons on the right, bottom, and left edges of U, respectively. We begin by showing $t = t_r$.

If $t \neq t_r$, it suffices by the symmetry of U across difference diagonal 0 to consider the case $t > t_r$. Let p be the largest integer such that sum diagonal im is occupied for $0 \leq i \leq p$. By Theorem 2(c), $p \geq 0$. Then either $t = p$ and square $s_1 = (-\ell + (p+1)m, \ell) \in E_d$ (so $\{(-\ell + tm, \ell), s_1\}$ is an m-crosspair), or $t = p + 1$ and $s_1 \in E_c$. Making a similar analysis of the right edge of U, and using the fact that a sum diagonal meets the right edge of U if and only if it meets the top edge of U, we see that $t > t_r$ can only occur if $t = p + 1$, $s_1 \in E_c$, and the square $s_2 = (\ell, -\ell + tm) \in E_d$, giving $t_r = t - 1$. This argument and the rotational symmetry of U imply the following.

Claim D2. The values of t and t_l can differ by at most one.

Moving along the difference diagonal of s_2, we see that $s_3 = (\ell - tm, -\ell) \in E_d$, and then that the bottom edge of U has at most $t - 1$ sum m-singletons (since if square $(\ell - (t-1)m, -\ell) \in E_s$, it forms an m-crosspair with s_3). Thus $t_b \leq t - 1$.

Moving along the column of s_1, we reach $s_4 = (-\ell + tm, -\ell)$, and since $s_1 \in E_c$, also $s_4 \in E_c$. Now consider $s_5 = (-\ell + (t-1)m, -\ell)$. If $s_5 \in E_d$, then since $s_4 \in E_c$, we see s_5 is a difference m-singleton, and then by Claim D1 that the bottom edge of U has t difference m-singletons, which contradicts $t_b \leq t - 1$, so $s_5 \notin E_d$. Since $(-\ell, -\ell) \in E_d$ by Theorem 2(c), this implies $t > 1$. The column of s_5 meets the top edge of U at a square in the occupied sum diagonal $(t-1)m$, so also $s_5 \notin E_c$, and thus $s_5 \in E_s$. The sum diagonal of s_5 meets the left edge of U at $s_6 = (-\ell, -\ell + (t-1)m)$, so $s_6 \in E_s$. If the square $(-\ell, -\ell + (t-2)m) \in E_d$, it forms an m-crosspair with s_6, so there are at most $t - 2$ difference m-singletons of the left edge of U. But then $t_l \leq t - 2$, which violates Claim D2, so $t = t_r$.

Then the rotational symmetry of U implies $t_r = t_b$ and $t_b = t_l$. So all edges of U have the same t-value, which together with Claim D1 implies the following useful fact.

Claim D3. If s, s' are diagonally covered edge squares of U that share a line and are not on the same edge of U, then either both are m-singletons or both are in m-crosspairs.

If there are no m-crosspairs, all edge squares of U that are not m-singletons are orthogonally covered and we are done. If there are m-crosspairs, by rotating the board we may assume the top edge of U contains at least as many m-crosspairs as any other edge of U, say k of them. Label these m-crosspairs $P_1, \ldots, P_k$ so that their means satisfy $|\mu_1| \geq \ldots \geq |\mu_k|$. The square $(\mu_1 - \frac{m}{2}, \ell)$ of P_1 is on sum diagonal $\mu_1 - \frac{m}{2} + \ell$, which by Claim D3 contains a square of another m-crosspair, namely $(\ell, \mu_1 - \frac{m}{2})$, implying $\{(\ell, \mu_1 - \frac{m}{2}), (\ell, \mu_1 + \frac{m}{2})\}$ is an m-crosspair. Moving along the difference diagonal from $(\ell, \mu_1 + \frac{m}{2})$ we see that Claim D3 implies $\{(-\mu_1 - \frac{m}{2}, -\ell), (-\mu_1 + \frac{m}{2}, -\ell)\}$ is an m-crosspair. Then moving vertically, the squares $(-\mu_1 - \frac{m}{2}, \ell)$ and $(-\mu_1 + \frac{m}{2}, \ell)$ are in either one or two m-crosspairs. If they are in two, one of these m-crosspairs has mean of greater absolute value than $|\mu_1|$, a contradiction. So those squares form one m-crosspair, and it follows that all eight images of P_1 under the action of D_4 are m-crosspairs; in particular, $\mu_2 = -\mu_1$. Continuing inductively, we see that every m-crosspair of the top edge has the property that all eight of its images under the action of D_4 are m-crosspairs, and then that the same is true for every m-crosspair. Since by Theorem 2(d) every occupied diagonal meets U, the set of occupied diagonals is closed under the action of D_4. This fact has two consequences.

First, the set of edge squares of U that are not diagonally covered is closed under the action of D_4, which implies that the set of those occupied orthogonals that meet U is also closed under the action of D_4. If $U = Q_n$, Lemma D is proved. Otherwise, the symmetry implies

$$\sum_{(x,y)\in D, |x|\leq \ell} x = 0, \text{ and } \sum_{(x,y)\in D, |y|\leq \ell} y = 0.$$

Second, since by Theorem 2(d) every occupied diagonal meets U, the symmetry implies $\sum_{(x,y)\in D}(y - x) = 0$ and $\sum_{(x,y)\in D}(y + x) = 0$. This gives $\sum_{(x,y)\in D} x = 0$ and $\sum_{(x,y)\in D} y = 0$. By the previous paragraph, we have

$$\sum_{(x,y)\in D, |x|> \ell} x = 0, \text{ and } \sum_{(x,y)\in D, |y|> \ell} y = 0. \quad (3)$$

Theorem 2(b) says that every orthogonal that misses U contains exactly one square of D, so there must be integers u, v such that the row numbers in the second sum of (3) are $\ell+1, \ell+2, \ldots, \ell+u$ and $-\ell-1, -\ell-2, \ldots, -\ell-v$. Then (3) implies $u = v$. A similar argument applies to columns, so Lemma D is proved.

Lemma E *$Orth(\ell - 2m)$ is full, and thus t is either 0 or 2.*

Proof. By Lemma D it suffices to show that row $\ell - 2m$ is occupied; we will assume it is not.

First examine the case $m = 1$. If there are any 1-crosspairs, let $\{(d, \ell), (d+1, \ell)\}$ be the one on the top edge of U with the smallest positive mean; by Lemma A we have $d > 0$. If $d = 1$ then by Lemma D, square $(d-2, \ell) = (-1, \ell)$ is covered along its difference diagonal and if $d > 1$ then by Lemma B, $(d-2, \ell)$ is covered along its column. Either way, sum diagonal $\ell + d - 2$ is empty by Theorem 2(a), which since the row and column of $(d, \ell - 2)$ are empty implies this square is covered along difference diagonal $\ell - d - 2$. Since difference diagonal $\ell - d - 1$ is also occupied, we see by Lemma B that any square of the top edge of U to the right of $(d+1, \ell)$ is a 1-singleton. So Lemma D implies that either each edge of U has exactly two 1-crosspairs, or there are no 1-crosspairs.

If there are two 1-crosspairs on each edge, we may conclude that $(x, \ell) \in E_c$ when $|x| < d$, that $(d, \ell) \in E_s$, and that $(x, \ell) \in E_d$ when $d < x \le \ell$. By Lemma D we then know how all edge squares of U are covered. In particular, $\text{Diag}(\ell + d)$ is full, but the squares of U that are in diagonals of $\text{Diag}(\ell + d)$ are in unoccupied orthogonals, so these diagonals contain occupied squares external to U. However, the only squares external to U where any of these diagonals meet another occupied diagonal are $(0, \pm(\ell + d))$ and $(\pm(\ell + d), 0)$. As these four squares are pairwise adjacent and D is independent, at most one can be occupied, which will not suffice to occupy those four diagonals.

So there are no 1-crosspairs; then for $|x| \le \ell - t$ we have $(x, \ell) \in E_c$, and for $\ell - t < x \le \ell$ we have $(x, \ell) \in E_d$, so difference diagonal d is occupied for $0 \le d < t$. By Lemma D we again know which of the lines meeting U are occupied. But by Theorem 2(d), every occupied diagonal meets U, so the occupied diagonals are exactly those of absolute value less than t. Since each $(x, y) \in D$ is the intersection of two occupied diagonals, we see from $|y + x| < t$ and $|y - x| < t$ that $|x| + |y| < t \le \ell$, so (x, y) is inside U. Therefore $U = Q_n$ here, and the size $(n-1)/2$ of D equals the number $2(\ell - t) + 1$ of occupied columns, which along with $n = j$ gives $n = 4t - 1$. Then Theorem 1 implies $8\sum_{i=1}^{t-1} i^2 = 4\sum_{i=0}^{t-1} i^2$, so $t = 1$ and $n = 3$, but we are assuming $n \ge 11$. This finishes the case $m = 1$, so we consider $m \ge 3$.

Claim E1. If v is an integer and $v \not\equiv \ell, -\ell \pmod{m}$, then $S(v, m) \subseteq E_c$.

Let (x_0, ℓ) be the rightmost square of $S(v, m)$; then $\ell - m < x_0 < \ell$. Assume for purposes of contradiction that (x_0, ℓ) is diagonally covered; then $\ell - x_0$ is even, so $\ell - x_0 - m$ is odd, and since $-m < \ell - x_0 - m < 0$ we see by Claim B1 that difference diagonal $\ell - x_0 - m$ is empty. It contains the square $(x_0, \ell - m)$, which by assumption is not orthogonally covered, so $(x_0, \ell - m)$ is covered along its sum diagonal, which it shares with $(x_0 - m, \ell)$.

Then (x_0, ℓ) is not in E_s, as this would make (x_0, ℓ) an m-singleton, which since $v \not\equiv \ell, -\ell \pmod m$ would contradict Lemma B. So we may conclude that (if (x_0, ℓ) is diagonally covered) $\{(x_0 - m, \ell), (x_0, \ell)\}$ is an m-crosspair. Then let d be the largest positive integer such that the rightmost $2d$ squares of $S(v, m)$ form d m-crosspairs. The leftmost of these squares is $s_1 = (x', \ell)$, where $x' = x_0 + m(1 - 2d)$. Our assumptions imply the square $(x', \ell - 2m)$ has its row and column empty. Could it be covered along its difference diagonal? If $d = 1$, this is difference diagonal $\ell - x_0 - m$, which we have seen is empty. If $d > 1$ then this difference diagonal contains the left square of the $(d-1)$st crosspair from the right in $S(v, m)$, which by the definition of m-crosspair is in E_s, so the difference diagonal is empty. Thus we may conclude sum diagonal $x' + \ell - 2m$ is occupied.

If this diagonal meets the top edge of U, then by our choice of v it must do so to the right of $(-\ell, \ell)$. Then square $s_2 = (x' - m, \ell) = (x_0 - 2md, \ell)$ must be covered along its sum diagonal to avoid having an m-singleton in $S(v, m)$, but this gives a $(d+1)$st m-crosspair, contradicting our definition of d. So sum diagonal $x' + \ell - 2m$ does not meet the top edge of U, which implies $x' - 2m < -\ell$. Then either s_1 or s_2 is the leftmost square of $S(v, m)$. Could it be s_1 ? If so, then $x' < -\ell + m$ and since s_1 is the left square of an m-crosspair, its sum diagonal $x_1 + \ell$ is occupied. Then Claim B1 implies $x' + \ell$ is even. But $\ell - x_0$ is even and, since m is odd, $x' + \ell \not\equiv x_0 + \ell \equiv \ell - x_0 \pmod 2$, a contradiction.

Therefore s_2 is the leftmost square of $S(v, m)$, so $-\ell < x' - m$. But the occupied sum diagonal $x' + \ell - 2m$ meets the left edge of U where row $y_1 = x' + 2\ell - 2m$ does, and from $-\ell < x' - m$ we see $\ell - m < y_1$. Since also $\ell - y_1 \equiv \ell - x' \not\equiv \ell - x_0 \pmod 2$, we see $\ell - y_1$ is odd, so by the inductive hypothesis row y_1 is occupied, giving a double cover of an edge square of U.

Therefore (x_0, ℓ) is in E_c. If $S(v, m) \not\subseteq E_c$ then by a symmetrical argument the leftmost square of $S(v, m)$ is in E_c, and some squares of $S(v, m)$ are diagonally covered; by Lemma B, the latter are in m-crosspairs. Then there are positive integers h, c such that the rightmost h squares of $S(v, m)$ are in E_c, the next $2c$ squares of $S(v, m)$ are in m-crosspairs, and at least the next square of $S(v, m)$ is in E_c.

The righthand square of the rightmost m-crosspair of $S(v, m)$ is $(x_0 - hm, \ell)$. By assumption, the orthogonals containing $s_3 = (x_0 - hm, \ell - 2m)$ are empty, and its sum diagonal meets the top edge of U at a square that is either in E_d (if $c > 1$) or in E_c (if $c = 1$). Therefore s_3 is covered along its difference diagonal, which meets row ℓ at $s_4 = (x_0 - (h-2)m, \ell)$. If $h > 1$ then $s_4 \in E_c$, contradicting Theorem 2(a), so $h = 1$. Therefore this diagonal meets column ℓ at (ℓ, y_2), where $y_2 = 2\ell - x_0 - m$. From $\ell - m < x_0 < \ell$ we see both that $y_2 < \ell$, so (ℓ, y_2) is in the right edge of U, and that $\ell - y_2 < m$. Claim B1 implies $y_2 - \ell = \ell - x_0 - m$ is even, but this contradicts $\ell - x_0$ even and m odd. We have proved Claim E1.

Define ρ to be the proportion of squares of the top edge of U covered along their columns. Since exactly $n-j$ occupied columns miss U, there are $\frac{n-1}{2}-(n-j)$ occupied columns that meet U, so $\rho = 1-\frac{n+1}{2j}$. By the inductive assumption, rows and columns $\pm(\ell-1)$ are occupied. Any squares of these lines that are outside U do not satisfy Theorem 2(d), so the squares of D occupying these lines are inside U. By Theorem 2(d), there are exactly $2(n-j)$ squares of D outside U, so we have $(n-1)/2 > 2(n-j)+1$, which implies $3(n+1)/4 < j \leq n$ and then

$$\frac{1}{3} < \rho < \frac{1}{2}. \tag{4}$$

By the Division Algorithm there are integers q, r satisfying $j = mq + r$ and $0 \leq r < m$. Then of the m sets $S(i,m)$, r sets contain $q+1$ squares and $m-r$ contain q squares. We examine the possibilities for r.

If $r = 0$ then all but two of the sets $S(i,m)$ are contained in E_c, so $(j-2q)/j \leq \rho < 1/2$, which with $j = mq$ implies $m < 4$, so $m = 3$ and $j = 3q$, which implies $\ell \equiv 1 \pmod 3$. As $S(0,3)$ contains exactly $j/3$ squares, from (4) and Lemma D we see that each of $S(1,3)$ and $S(-1,3)$ contains a square of E_c.

Suppose for the moment that column $\ell-2$ is empty. Since $S(0,3) \subseteq E_c$, Lemma D implies rows $\ell-1$ and $\ell-4$ are occupied. Then by Theorem 2(a), the difference diagonals containing squares $(\ell,\ell-1)$ and $(\ell,\ell-4)$ are empty, which means squares $(\ell-2,\ell-3)$ and $(\ell-2,\ell-6)$ are not covered along their difference diagonals. By our assumptions, these squares are in empty orthogonals, so their sum diagonals $2\ell-5$ and $2\ell-8$ are occupied. This implies squares $s_5 = (\ell-8,\ell)$ and $s_6 = (\ell-5,\ell)$ of $S(-1,3)$ are covered along their sum diagonals. Thus $\{s_5, s_6\}$ is not a 3-crosspair, so s_5 is a 3-singleton. Since $S(-1,3) = S(-\ell,3)$, Lemma B implies all squares of $S(-1,3)$ at or left of s_5 are 3-singletons. By assumption, $(\ell-2,\ell)$ is diagonally covered, so no square of $S(-1,3)$ is in E_c, a contradiction. Therefore column $\ell-2$ is occupied.

A similar argument then applies to the squares $(\ell-5,\ell-3)$ and $(\ell-5,\ell-6)$; their difference diagonals strike the edge of U at squares of E_c or E_r, thus are empty, so if column $\ell-5$ is empty, then sum diagonals $2\ell-8$ and $2\ell-11$ are occupied, so $(\ell-8,\ell)$ is a 3-singleton, and Lemma B implies all squares of $S(-1,3)$ at or left of $(\ell-8,\ell)$ are 3-singletons.

Continuing, we see that the squares of $S(-1,3)$ are covered as follows: starting at the right, we have one or more squares vertically covered, then a diagonally covered square, and the remaining squares of $S(-1,3)$ are in E_s. By Lemma D, reflecting across column 0 gives the coverage of $S(1,3)$. Thus either $S(-1,3)\cap E_d$ and $S(-1,3)\cap E_s$ are both empty, or each has just one member, with their members being $(\pm k,\ell)$ for some positive integer k.

Column $\ell-2$ is occupied, so by Lemma D, row $\ell-2$ must be. By Theorem 2(d), the occupied square $(x,\ell-2)$ must satisfy $|x| \leq \ell+1$, and $|x| = \ell$ is not possible by the definition of U. It follows that if $x \equiv 1 \pmod 3$, then $(x,\ell-2)$ attacks the square $(x+2,\ell)$ of $S(0,3)$ along its difference diagonal, but $S(0,3) \subseteq E_c$ so we have a contradiction of Theorem 2(a). Similarly, if $x \equiv -1 \pmod 3$, then $(x,\ell-2)$ attacks the square $(x-2,\ell)$ of $S(0,3)$ along its sum diagonal, contradicting Theorem 2(a). Therefore $x \equiv 0 \pmod 3$. Since $(\pm\ell,\ell-1) \in E_r$, we cannot have $x = \pm(\ell-1)$, so $(x-2,\ell) \in S(1,3) \cap E_s$ and $(x+2,\ell) \in S(-1,3) \cap E_d$. By the last sentence of the previous paragraph, this gives $x = 0$. Applying a similar argument to the bottom edge of U shows the squares $(0, \pm(\ell-2))$ are occupied, which since $j \geq 9$ contradicts the independence of D.

If $r = 1$ then $\ell \equiv -\ell \pmod m$ so at least $j-(q+1)$ squares of the top edge of U are in E_c. Then $(j-(q+1))/j \leq \rho < 1/2$, which gives $(m-2)q < 1$, impossible for $m \geq 3$.

Otherwise $r \geq 2$, and from $(j-2(q+1))/j \leq \rho < 1/2$ we get $(m-4)q < 4-r \leq 2$. If $m \geq 5$ then this implies $m = 5$ and $q = 1$ and $r = 2$, so $j = 7$, contrary to $j \geq 9$. So $m = 3$ and then $2 \leq r < m$ gives $j = 3q+2$. By the inductive assumption, columns and rows numbered $\pm(\ell-1)$ are occupied. As $(\ell-2,\ell) \in S(0,3)$, we see column $\ell-2$ is occupied, so $(\ell,\ell-2) \in E_r$. Then the difference diagonals of $(\ell-4,\ell-3)$ and $(\ell-4,\ell-6)$ are empty, so if column $\ell-4$ is empty then their sum diagonals are occupied, giving a run to $(-\ell,\ell)$. Continuing as in the case $r = 0$, we see $S(1,3)$ has at most one square in E_d and symmetrically $S(-1,3)$ has at most one square in E_s. For some x with $|x| < \ell$, the square $(x,\ell-1)$ is occupied. To avoid double attacks on squares of $S(0,3)$ we need $x \equiv 0 \pmod 3$. Then $(x-1,\ell) \in S(-1,3) \cap E_s$ and $(x+1,\ell) \in S(1,3) \cap E_d$ together imply $x = 0$, but applying the same argument to the bottom edge of U implies squares $(0, \pm(\ell-1))$ are occupied, which since $j \geq 9$ contradicts the independence of D.

Therefore any value of r leads to a contradiction of the assumption that row $\ell-2m$ is empty, and then Lemma D implies Orth$(\ell-2m)$ is full. As column $\ell-2m$ is occupied, $t \leq 2$, and since column $\ell-m$ is empty we see $(\ell-m,\ell)$ is diagonally covered, so either $t = 0$ or $t = 2$. This completes the proof of Lemma E.

Let S denote the set of m-crosspairs of row ℓ of U. By Lemma D and the fact that m is odd, reflecting any member of S across column 0 gives a different member of S, so $|S|$ is even. Examining the cases $t = 0, 2$ shows there are $2-t$ members of S that involve corner squares, so for some integer b there are $2b$ members of S that do not involve corners. By Theorem 2(c) and (d), exactly $(n+1)/2$ squares of row ℓ of U are covered diagonally. Then counting squares of m-crosspairs and m-singletons, $2(2b+(2-t))+2t =$

$(n+1)/2$, giving $b=(n-7)/8$.

Moving left from $(\ell-1,\ell)$ along row ℓ, let $c_1, c_2, \ldots, c_b$ be the numbers of the difference diagonals containing the right-hand squares of the first b m-crosspairs encountered (thus not including the corner m-crosspair if there is one). Let $\mathcal{C}^+$ be the set of these numbers. If $t=0$, set $\mathcal{C}=\mathcal{C}^+\cup\{0\}$, associating $c_0=0$ to the corner m-crosspair; if $t=2$ then set $\mathcal{C}=\mathcal{C}^+$.

Lemma F *We have* $|\mathcal{C}^+|=(n-7)/8$, $|\mathcal{C}|=(n+1-4t)/8$, *and* $n\equiv -1 \pmod{32}$.

Proof. The first statement has just been proved, and the second follows easily upon examination of the cases $t=0,2$. We use the Parallelogram Law (Theorem 1) to prove the last statement.

By Theorem 2(d), every occupied diagonal meets the sub-board U. So the symmetry of occupied lines established in Lemma D implies that we can find both sides of equation (2) from the set $\mathcal{C}$.

It is easily seen that the occupied diagonals with positive numbers meet the top edge of U. The top edge m-crosspair associated to $c\in\mathcal{C}$ has occupied difference diagonal c, which passes through square $s_1(c)=(\ell-c,\ell)$, and so this m-crosspair also contains the square $s_2(c)=(\ell-c-m,\ell)$, which means sum diagonal $2\ell-m-c$ is occupied. If $t=2$, the four diagonals numbered 0 or m are occupied but not in m-crosspairs. Then using Lemma D, the right side of equation (2) here becomes

$$4\left(\frac{t}{2}m^2+\sum_{c\in\mathcal{C}}\left[c^2+(2\ell-m-c)^2\right]\right). \tag{5}$$

By Theorem 2(a), the members of $\{1,2,\ldots,(n-1)/2\}$ that are not numbers of occupied columns are the column numbers of the squares $s_1(c)$, $s_2(c)$ defined above, for all $c\in\mathcal{C}$; if $t=2$, ℓ and $\ell-m$ are numbers of unoccupied columns that do not contain squares associated to any top edge m-crosspair. By Lemma D, the left side of (2) here becomes

$$8\left(\sum_{i=1}^{(n-1)/2} i^2-\sum_{c\in\mathcal{C}}\left[(\ell-c)^2+(\ell-c-m)^2\right]-\frac{t}{2}(\ell^2+(\ell-m)^2)\right). \tag{6}$$

Equating (5) and (6) and simplifying yields

$$\frac{n^3-n}{12}=\frac{t}{2}\left(m^2+2\ell^2+2(\ell-m)^2\right)+ \tag{7}$$
$$\sum_{c\in\mathcal{C}}\left[c^2+(2\ell-m-c)^2+2(\ell-c)^2+2(\ell-c-m)^2\right].$$

Since m is odd, each of the quantities m^2 and $\ell^2 + (\ell - m)^2$ is congruent to 1 modulo 4, as are $c^2 + (2\ell - m - c)^2$ and $(\ell - c)^2 + (\ell - c - m)^2$ for each $c \in C$. Thus the right side of (7) is congruent to $-|C| - (t/2)$ modulo 4. Since $n \equiv -1 \pmod 8$, the left side of (7) is divisible by 4, so $0 \equiv -|C| - (t/2) \pmod 4$. Then examining the cases $t = 0, 2$, we see $(n+1)/8 \equiv 0 \pmod 4$, so $n \equiv -1 \pmod{32}$ and Lemma F is established.

For any m-crosspair W, one of its two squares lies on an occupied difference diagonal, say with number w. By Lemma D, there is a unique $c \in C$ such that for some ϕ in D_4, $\phi(W)$ is the m-crosspair from S with difference diagonal c. The function f_m defined below sends w to c; note that if $|w| > \ell - \frac{m+1}{2}$, then $\phi^{-1}(c)$ is the occupied sum diagonal containing a square of W.

$$f_m(w) = \begin{cases} |w| & \text{if } |w| \leq \ell - \frac{m+1}{2}, \\ 2\ell - m - |w| & \text{otherwise.} \end{cases}$$

The following lemma summarizes restrictions on C, including what may be loosely described as closure under a binary operation.

Lemma G *For each $c \in C^+$, $2 \leq c \leq \ell - \frac{m+1}{2}$; if c is odd then $m < c$. The set C contains neither m nor $2m$, and no two members of C differ by m.*

Suppose that $c_i \leq c_k$, with strict inequality if $t = 2$. Then exactly one of the following holds:

(I) Both $c_k - c_i = f_m(c_k - c_i)$ and $f_m(c_k + c_i)$ are in C;

(II) All of $f_m(c_k + c_i - m)$, $f_m(c_k - c_i + m)$, $f_m(c_k + c_i + m)$, and $f_m(c_k - c_i - m)$ are in C;

(III) We have $t = 2$, $c_k - c_i = 2m$, and $\ell > 2m$, and C contains all of $f_m(c_k + c_i - m)$, $f_m(c_k - c_i + m) = f_m(3m)$, and $f_m(c_k + c_i + m)$;

(IV) We have $t = 2$, $c_k + c_i = 2\ell - 3m$, and $\ell > 2m$, and C contains all of $f_m(c_k + c_i - m) = f_m(2\ell - 4m)$, $f_m(c_k - c_i + m)$, and $f_m(c_k - c_i - m)$.

Thus if (I) does not hold, C contains $f_m(c_k - |c_i - m|)$.

Proof. For each odd i, $1 \leq i < m$, Claim B1 implies $i \notin C$. By Lemma E, square $(\ell - 2m, \ell) \in E_c$ so $m \notin C$ and $2m \notin C$. The last two sentences imply $1 \notin C$, so for any $c \in C^+$ we have $2 \leq c$. Since m-crosspair c has positive mean, $c \leq \ell - \frac{m+1}{2}$. Suppose $c, c' \in C$; by Theorem 2(a), the left-hand square of an m-crosspair of the top edge of U cannot be the right-hand square of another m-crosspair, so c and c' do not differ by m.

For the second part of the lemma, we take m-crosspairs with numbers $c_i \leq c_k$. By definition of the m-crosspair c_i and Theorem 2(a), columns $\ell - c_i$ and $\ell - c_i - m$ are empty; then Lemma D implies rows $\ell - c_i$ and $\ell - c_i - m$ are also empty. Likewise columns $\ell - c_k$ and $\ell - c_k - m$ are empty; then the squares of U where these rows and columns meet must be diagonally covered. We name four of these squares below by their relative position (top or bottom, left or right) in the square they make.

Square	Difference diagonal	Sum diagonal
$s_{tl} = (\ell - c_k - m, \ell - c_i)$	$c_k - c_i + m$	$2\ell - m - c_k - c_i$
$s_{tr} = (\ell - c_k, \ell - c_i)$	$c_k - c_i$	$2\ell - c_k - c_i$
$s_{bl} = (\ell - c_k - m, \ell - c_i - m)$	$c_k - c_i$	$2\ell - 2m - c_k - c_i$
$s_{br} = (\ell - c_k, \ell - c_i - m)$	$c_k - c_i - m$	$2\ell - m - c_k - c_i$

If $t = 0$, all occupied diagonals come from m-crosspairs. If $t = 2$, the diagonals numbered $0, \pm m$ are the only occupied diagonals not coming from m-crosspairs. We begin by investigating how these last could be involved in the diagonal domination of $s_{tl}, s_{tr}, s_{bl}, s_{br}$.

Suppose first that $t = 2$ and $c_i < c_k$. By their difference diagonal numbers, $s_{tl}, s_{tr}, s_{bl}, s_{br}$ are strictly above difference diagonal $-m$. Since $c_k \neq c_i + m$, none is on difference diagonal 0, and if any of the four is on difference diagonal m, it must be s_{br}, giving $c_k - c_i = 2m$. By Lemma E, $(\ell - 2m, \ell) \in E_c$ so difference diagonal $2m$ is empty. Thus s_{tr} and s_{bl} are covered along their sum diagonals $2\ell - c_k - c_i$ and $2\ell - 2m - c_k - c_i$. From $c_i + 2m = c_k \leq \ell - \frac{m+1}{2}$ it follows that these diagonal numbers are positive, and the oddness of m implies neither equals m, so each covers the left square of an m-crosspair of the top edge of U. The right-hand squares of these m-crosspairs are in the difference diagonals $c_k + c_i - m$ and $c_k + c_i + m$, so C contains $f_m(c_k + c_i - m)$ and $f_m(c_k + c_i + m)$.

It remains to see how s_{tl} is covered. Its sum diagonal $2\ell - m - c_k - c_i$ cannot be occupied, as it meets the top edge of U where the occupied difference diagonal $c_k + c_i + m$ does. So s_{tl} is covered along difference diagonal $c_k - c_i + m = 3m$, which meets the top edge of U at $(\ell - 3m, \ell)$. Then $f_m(3m) \in C$, and as $(\ell - 2m, \ell) \in E_c$, by Lemma B the square $(\ell - 3m, \ell)$ is not an m-singleton. Thus $(\ell - 4m, \ell) \in E_s$, implying $\ell - 4m > -\ell$ and thus $\ell > 2m$. This shows how (III) arises; a similar argument shows that if $t = 2$ and any of $s_{tl}, s_{tr}, s_{bl}, s_{br}$ lies on any of the occupied sum diagonals $0, \pm m$ then (IV) occurs. Otherwise all occupied diagonals covering the four squares come from m-crosspairs, and the remaining cases (I), (II) can be established by remembering that two m-crosspairs of the top edge of U cannot share a square.

To prove the last statement of the lemma, note that if (I) does not apply to the pair c_i, c_k, then both $f_m(c_k + c_i - m)$ and $f_m(c_k - c_i + m)$ are in C, and one of $c_k + c_i - m$ and $c_k - c_i + m$ equals $c_k - |c_i - m|$.

Definition. Set $v = \min(\{c_1\} \cup \{c_i - c_{i-1} : 2 \leq i \leq (n-7)/8\})$.

Since n and j are odd, there is a non-negative integer z such that $n-j=2z$. From $c_1+\sum_{i=2}^{(n-7)/8}(c_i-c_{i-1})=c_{(n-7)/8}\le \ell-\frac{m+1}{2}$, we have

$$v\le 4+\frac{4(5-2z-m)}{n-7}. \tag{8}$$

Lemma H *We have $v\in\{1,2,3\}$ and $v<m/2$.*

Proof. From Lemma F we have $n\ge 31$, and then (8) gives $v\le(29-2z-m)/6$, so $v\le 4$; later we show $v\ne 4$. For the rest, assume for purposes of contradiction that $v>m/2$.

We first consider the possibility $v=2m$. As $v\le 4$ and m is odd, this implies $v=2$ and $m=1$. Then $2=2m\notin C$ by Lemma G, so $2<c_1$ and $2\le c_2-c_1$, implying $4<c_2$. Choose h with $c_{h+1}-c_h=2$. As $2\notin C$, Lemma G(I) does not apply to the pair c_h, c_{h+1}, so $f_1(3)\in C$, and since $3<c_2\le\ell-1$, here $3=f_1(3)$. Thus $3\in C$, and from $3<c_2$ we have $c_1=3$.

Let k be the largest integer such that $c_i=3i$ for each i, $1\le i\le k$. Since $v\ne 3$, c_{k+1} exists and $c_{k+1}\ne c_k+3$. By Lemma G, one of $c_{k+1}-c_1=c_{k+1}-3$ and $c_{k+1}-c_1+m=c_{k+1}-2$ is in C, which implies $c_{k+1}<c_k+3$, and by the definition of v we have $c_k+2\le c_{k+1}$, so $c_{k+1}=c_k+2$. Now consider the pair c_1, c_{k+1}; as $c_{k+1}-c_1=c_k-1$ and $c_{k+1}-c_1-m=c_k-2$ are not in C, only Lemma G(III) applies, so $c_{k+1}-c_1=2$, giving $k=1$ and $c_2=5$.

Let r largest with $c_i=2i+1$ for each i, $1\le i\le r$. Since $c_r-c_1=2r-2\notin C$, Lemma G(I) does not apply to the pair c_1, c_r, so C contains $f_1(c_r+c_1-1)=f_1(c_r+2)$. By the definition of r, $c_r+2\notin C$, so $f_1(c_r+2)\ne c_r+2$. If $r<(n-7)/8=|C^+|$, then c_{r+1} exists, and by the definition of v we would have $c_r+2<c_{r+1}$, which would give $f_1(c_r+2)=c_r+2$, a contradiction. Thus $r=(n-7)/8$, so all members of C^+ are known, and then for some $s>0$ we have $c_s=f_1(c_r+2)=2\ell-1-(c_r+2)$, which implies c_r+c_s is odd, a contradiction.

Therefore $v\ne 2m$; we will show $c_1=v$. If not, there is h with $c_{h+1}-c_h=v$ and Lemma G(I) does not apply to the pair c_h, c_{h+1}. Since $v\ne 2m$ also (III) does not apply, so either (II) or (IV) holds, and then $f_m(c_{h+1}-c_h-m)\in C$. From $m/2<v\le 4$ we see $|v-m|\le\ell-\frac{m+1}{2}$, so $f_m(c_{h+1}-c_h-m)=|v-m|$. By Lemma G we have $c_{h+1}-c_h\ne m$, and then our assumption $v>m/2$ implies $0<|v-m|<v\le c_1$, which contradicts c_1 being the least positive member of C. Thus $c_1=v$.

Let p largest such that $c_i=vi$ for $1\le i\le p$; we show $p=(n-7)/8$. If not, the definitions of p and v imply $c_p+v<c_{p+1}$. Thus $c_{p+1}-v\notin C$, so Lemma G(I) does not apply to the pair c_1, c_{p+1}, but then the last sentence of Lemma G gives $c_{p+1}-|v-m|=f_m(c_{p+1}-|v-m|)$ in C. As noted previously, $|v-m|<v$ so $c_p<c_{p+1}-|v-m|<c_{p+1}$, but C does not

contain a value between c_p and c_{p+1}. Thus $p = (n-7)/8$, so all of C^+ is known.

By Lemma F we have $n \geq 31$ and then $p \geq 3$, so we can apply Lemma G to the pair c_1, c_p. If (I) does not occur, then C contains $c_p - |v-m| = f_m(c_p - |v-m|)$, which is not possible since $|v-m| < v$ and C contains no values between c_{p-1} and c_p. So (I) holds, giving $f_m(c_p + c_1) \in C$. Since $c_p + c_1 \notin C$, we see that $f_m(c_p+c_1) = 2\ell - m - v(n+1)/8$ is the number of the sum diagonal associated to an m-crosspair W of the top edge of U, and the mean of W is negative. It is easily calculated that the difference diagonal through the right-hand square of W has number $v(n+1)/8$. Therefore v squares to the right of W is the m-crosspair W' whose right-hand square is in difference diagonal $v(n-7)/8$. That is, W' is associated to c_p and has positive mean. It follows that W is the rightmost of the top edge m-crosspairs with negative mean, so $f_m(c_p + c_1) = c_p$. This equation gives

$$j = v(n-3)/4 + m + 1. \tag{9}$$

Since j and m are odd, (9) implies v is odd. We have shown $v \leq 4$, and if $v = 1$ then $c_1 = 1$ which is not possible by Lemma G, so $v = 3$. Our assumption $v > m/2$ implies $m \in \{1, 3, 5\}$. The bound $3(n+1)/4 \leq j$ from Theorem 2 with (9) implies $2 \leq m$, and $m = 3$ conflicts (Lemma G) with $c_1 = v = 3$, so $m = 5$.

Then (9) gives $j = 3(n+5)/4$; since $n \geq 31$ by Lemma F, this means $j < n$. Therefore by Lemma D, row $(\ell + 1)$ is on Q_n, and by Theorem 2(b) this row is occupied, so it contains an eligible square $(x, \ell+1)$. By Lemma D and Theorem 2(d), we may assume $0 \leq x \leq \ell - 2$.

The squares of the top edge of U that are in E_d are those of form $(\ell - 3i, \ell)$, and one of $(\pm(\ell-5), \ell)$. The top edge squares of U in E_s are those of form $(-\ell + 3i, \ell)$, and one of $(\pm(\ell - 5), \ell)$. As the diagonals of $(x, \ell+1)$ are occupied, we have $(x-1, \ell) \in E_d$ and $(x+1, \ell) \in E_s$.

Since $(\ell - 3, \ell) \in E_d$ we see $x \neq \ell - 4$, and since $(\ell - 7, \ell) \notin E_d$ we see $x \neq \ell - 6$. Thus square $(\ell - 5, \ell)$ does not share a diagonal with $(x, \ell + 1)$, which implies $x - 1 = \ell - 3i$ and $x + 1 = -\ell + 3h$ for some integers i, h. But these equations imply $j \equiv -1 \pmod 3$, contrary to $j = 3(n+5)/4$. Thus $v < m/2$. If $v = 4$ then from (8) we have $5 - 2z - m \geq 0$ so $m \leq 5$, counter to $v < m/2$. Thus $v \in \{1, 2, 3\}$. This completes the proof of Lemma H, and also completes our proof that $\mathrm{Orth}(\ell - 1)$ is full.

We will show $v \in C$. If not, we may choose h with $c_{h+1} - c_h = v$. As Lemma G(I), (III) do not apply to the pair c_h, c_{h+1}, the square s_{br} is in the occupied difference diagonal $v - m$; since $-m < v - m < 0$ here, Claim B1 says $v - m$ is even, so v is odd. Therefore $v \in \{1, 3\}$. One of Lemma G(II), (IV) applies to the pair c_h, c_{h+1}, so $f_m(v+m)$ and $f_m(v-m) = f_m(m-v)$ are in C. We need to evaluate these.

Suppose first that $v + m \le \ell - \frac{m+1}{2}$. Here $m + v = f_m(v + m)$ and $m - v = f_m(v - m)$ are in C. By Lemma G, $2m \notin C$, so Lemma G(I) does not apply to the pair $m - v$, $m + v$, but then $(m + v) + (m - v) - m = m$ is in C, which is not possible.

Therefore $v + m > \ell - \frac{m+1}{2}$. Let d be the integer such that $m = \ell - \frac{m+1}{2} - d$. Then $f_m(v + m) = 2\ell - m - (v + m) = m - v + 1 + 2d$ is an odd member of C, so $m - v + 1 + 2d > m$, implying $2d > v - 1$ and thus $d > 0$. Then $m \le \ell - \frac{m+1}{2} - 1$ and $v + m \ge \ell - \frac{m+1}{2} + 1$, which implies $v \ge 2$, so $v = 3$. Then $m \ge \ell - \frac{m+1}{2} - 2$ so $d \in \{1, 2\}$.

If $d = 1$ then $f_m(v+m) = m$ is in C, a contradicting Lemma G. So $d = 2$, giving $2\ell = 3m+5$ and $m+2 = f_m(m+3)$ in C. Since $m-3 = m-v < m+2$, also $m - 3 = f_m(m - v)$ is in C. Apply Lemma G to the pair $c_i = m - 3$, $c_k = m + 2$. By Lemma H we have $m > 2v = 6$, so $c_k - c_i = 5 \notin C$, so (I) does not apply. Then C contains $f_m(c_k - c_i + m) = f_m(m + 5) = 2\ell - m - (m + 5) = m$, which is not possible.

Therefore $v \in C$, so $c_1 = v$. As $v < m/2$ by Lemma H, $c_1 = v$ cannot be odd by Lemma G. Since $v \le 3$ (Lemma H), we have $v = 2$.

Let p largest with $c_i = 2i$ for each i, $1 \le i \le p$. We will show $p = (n - 7)/8$. If not, look at the pair c_1, c_{p+1}. By the definition of p, $c_{p+1} - c_1 \notin C$, so by Lemma G we get $f_m(c_{p+1} + c_1 - m)$ in C. Let $u = c_{p+1} + c_1 - m$.

If $0 < u$ then since $u < c_{p+1} \le \ell - \frac{m+1}{2}$, we get $u \in C$ so $u = 2i$ for some i, $1 \le i \le p$. But then if $i > 1$ we get $c_{p+1} = c_{i-1} + m$ and if $i = 1$ we get $c_{p+1} = m$, neither of which is possible.

Otherwise $-m < u \le 0$, and u is the number of an occupied difference diagonal so u is even, implying c_{p+1} is odd. But also $u \le 0$ implies $c_{p+1} \le m - c_1$ so c_{p+1} is even, a contradiction. Therefore $p = (n - 7)/8$.

Then by Lemma F we have $p > 1$, and we may apply Lemma G to the pair c_1, c_p. If (I) holds then $f_m(c_p + c_1) \in C$, but $c_p + c_1 \notin C$ so $2\ell - m - (c_p + c_1) = 2i$ for some integer i. Then m odd and $c_p + c_1$ even gives a contradiction. So (I) does not hold, implying square s_{tr} is covered along its sum diagonal $2\ell - c_p - 2$. By parity, this cannot be sum diagonal m, so it passes through the left-hand square of an m-crosspair of the top edge of U. Then the right-hand square of that m-crosspair is in the occupied difference diagonal $c_p + c_1 - m$, whence $c_p + c_1 - m > 0$. Since $v = 2$, Lemma H implies $m \ge 5$, so $c_p + c_1 - m < c_p \le \ell - \frac{m+1}{2}$. Then $c_p + c_1 - m = f_m(c_p + c_1 - m)$ is in C, but $c_p + c_1 - m$ is odd and each $c_i = 2i$ is even.

This completes the proof that $\mathrm{Orth}(\ell - m)$ is full for each odd m, $1 \le m < \ell$. In particular, columns $\pm(\ell - m)$ are occupied for these m, and by Lemma A, column 0 is also occupied. This implies at least ℓ squares of the top edge of U are in E_c, giving $\frac{j-1}{2j} \le \rho = 1 - \frac{n+1}{2j}$, which implies $n \le j$, and thus $n = j$. It follows that every square of D has both coordinates even. □

Theorem 4 *Define a sequence of integers by* $n_1 = 3$, $n_2 = 11$, *and* $n_i = 4n_{i-1} - n_{i-2} - 2$ *for* $i > 2$. *If* $\gamma(Q_n) = (n-1)/2$ *then* $n = n_i$ *for some* i.

Proof. Suppose $\gamma(Q_n) = (n-1)/2$ and D is a minimum dominating set of Q_n. By Theorem 2 there is an integer k such that $n = 4k+3$; then $d = 2k+1$ is the cardinality of D. By Theorem 3, each square of D has both coordinates even, and thus the numbers of the occupied rows and columns are

$$0, \pm 2, \pm 4, \dots, \pm 2k. \tag{10}$$

The numbers of the occupied diagonals may then be obtained from Corollary 12 and Theorem 1 of [15], or from Theorems 16, 17 of [3]; for completeness, we give an argument here.

As $n \equiv 3 \pmod 4$, the edge orthogonals are empty, so since D covers the corner squares, sum and difference diagonal 0 are occupied. We may thus define e, f, u to be the largest integers such that difference diagonal $2i$ is occupied for $|i| \le e$, sum diagonal $2i$ is occupied for $|i| \le f$, and difference diagonals $\pm(2e+4i)$ and sum diagonals $\pm(2f+4i)$ are occupied for each i with $1 \le i \le u$. There are then at least $2e+2u+1$ occupied difference diagonals, which gives $e \le k-u$; similarly $f \le k-u$.

By the definitions of e and f, we may choose one of difference diagonals $\pm(2e+2)$ and one of sum diagonals $\pm(2f+2)$ such that both are empty. These diagonals meet, possibly off the n-board, at an even square (x,y); the larger of $|x|$ and $|y|$ is $e+f+2$. If $e \not\equiv f \pmod 2$, then (x,y) is odd-odd, thus is not covered by D, so is off the board; we have $e+f+2 \ge 2k+2$, so $e+f \ge 2k$, which since $e+f$ is odd implies $e+f \ge 2k+1$, contradicting $e, f \le k-u \le k$.

Therefore $e \equiv f \pmod 2$. By the definition of u, either at least one of difference diagonals $\pm(2e+4u+4)$ or at least one of sum diagonals $\pm(2f+4u+4)$ is empty. If the former, we look at the intersection of an unoccupied one of difference diagonals $\pm(2e+4u+4)$ with an unoccupied one of sum diagonals $\pm(2f+2)$; if the latter, we look at the intersection of an unoccupied one of difference diagonals $\pm(2e+2)$ with an unoccupied one of sum diagonals $\pm(2f+4u+4)$. In either case, we obtain a square (x',y') for which the larger of $|x'|$ and $|y'|$ is $e+f+2u+3$. Then x' and y' are odd, so (x',y') is not covered by D, so must be off the board: we have $e+f+2u+3 \ge 2k+2$, and then $e+f+2u \ge 2k-1$, which since $e+f$ is even implies $e+f+2u \ge 2k$. Then $e, f \le k-u$ gives $e = f = k-u$, so the numbers of the occupied sum and difference diagonals are

$$0, \pm 2, \pm 4, \dots, \pm 2e, \pm(2e+4), \pm(2e+8), \dots, \pm(4k-2e). \tag{11}$$

(Conversely, if e, f, u are defined as above for a set D, and $e = f$ and $u = k-e$, then Theorem 1 of [15] implies that D dominates Q_n.)

Thus the values of k and e determine exactly which lines are occupied by D; applying Theorem 1 to (10) and (11) gives the equation

$$2\cdot 2\cdot 2\sum_{i=1}^{k}(2i)^2 = 2\cdot 2\left[\sum_{i=1}^{e}(2i)^2 + \sum_{i=1}^{k-e}(2e+4i)^2\right].$$

This simplifies to $(2k+1)^2+2 = 3(2k+1-2e)^2$, or

$$d^2 - 3(d-2e)^2 = -2. \tag{12}$$

Set $X = d$ and $Y = d-2e$; then $X, Y > 0$ and $X^2 - 3Y^2 = -2$. The positive solutions of this Pell's equation can be found by standard methods (e.g., Chapter 13-5 of [6]); we obtain a sequence $(X_i, Y_i)_{i=1}^{\infty}$ which may be recursively defined by $(X_1, Y_1) = (1,1)$, $(X_2, Y_2) = (5,3)$, and $(X_i, Y_i) = 4(X_{i-1}, Y_{i-1}) - (X_{i-2}, Y_{i-2})$ for $i > 2$. Returning to d, e, we obtain a sequence $(d_i, e_i)_{i=1}^{\infty}$ defined by

$$\begin{aligned}(d_1,e_1) = (1,0), (d_2,e_2) = (5,1), \text{ and}\\ (d_i,e_i) = 4(d_{i-1},e_{i-1}) - (d_{i-2},e_{i-2}) \text{ for } i > 2.\end{aligned} \tag{13}$$

Then using $n_i = 2d_i + 1$ gives the desired sequence. □

Here are the first few values of n_i, d_i, and e_i.

i	1	2	3	4	5	6	7	8
n_i	3	11	39	143	531	1979	7383	27551
d_i	1	5	19	71	265	989	3691	13775
e_i	0	1	4	15	56	209	780	2911

The linear recursion (13) gives

$$\begin{aligned}d_i &= [(\sqrt{3}-1)(2+\sqrt{3})^i - (\sqrt{3}+1)(2-\sqrt{3})^i]/2\\ &= \lfloor(\sqrt{3}-1)(2+\sqrt{3})^i/2\rfloor,\\ e_i &= \lfloor(2+\sqrt{3})^{i-1}/2\sqrt{3}\rfloor, \text{ and}\\ n_i &= \lceil(\sqrt{3}-1)(2+\sqrt{3})^i\rceil\end{aligned}$$

so $\lim_{i\to\infty} n_i/n_{i-1} = \lim_{i\to\infty} d_i/d_{i-1} = \lim_{i\to\infty} e_i/e_{i-1} = 2+\sqrt{3}$.

We now relate our results to some conjectures of Burger and Mynhardt [3]. Theorem 3 implies that a dominating set of size $(n-1)/2$ for Q_n contains no edge squares of Q_n, which is their Conjecture 1. Let k be a non-negative integer. In [3], a set D of $2k+1$ squares of Q_{4k+3} is said to be *edge dominating* if D covers each edge square of Q_{4k+3} exactly once, not necessarily dominating the rest of the board. The *radius* of a square (x, y)

is defined by $R(x,y) = \max\{|x|,|y|\}$. Conjecture 3 of [3] states that if D is an edge dominating set that occupies the even-numbered orthogonals of Q_{4k+3}, then $\sum_{(x,y)\in D} R(x,y) = 8k(k+1)/3$.

From Theorems 2 and 3, we see that if $\gamma(Q_n) = (n-1)/2$, then any minimum dominating set D of Q_n satisfies the hypotheses of Conjecture 3; we verify this case of the conjecture. Note that $R(x,y) = \max\{|x|,|y|\} = (|y-x|+|y+x|)/2$, so $\sum_{(x,y)\in D} R(x,y)$ can be computed from the list (11) of numbers of occupied diagonals. It is then not difficult to show that the desired equation is equivalent to (12).

By assuming their conjectures, and that a dominating set of size $(n-1)/2$ for Q_n occupies the even-numbered orthogonals, Burger and Mynhardt thus derived an equation which gives the sequence (n_i) .

Corollary 5 $\gamma(Q_{23}) = 12$.

Proof. From Theorem 4 we have $\gamma(Q_{23}) \geq 12$, so it suffices to give a dominating set of size 12; we give two related ones, the first of which was also found by Kearse and Gibbons [12]. Let $A = \{\pm(1,-1), \pm(11,3)\}$, $B_1 = \{\pm(3,7), \pm(5,-11), \pm(7,-5), \pm(9,9)\}$, and $B_2 = \{\pm(5,5), \pm(3,-9), \pm(9,-7), \pm(7,11)\}$. Then each $A \cup B_i$ occupies all odd-numbered orthogonals of Q_{23}, so orthogonally covers all but the even-even squares. Each $A \cup B_i$ occupies difference diagonals $0, \pm2, \pm4, \pm8, \pm12, \pm16$ and sum diagonals 0, ±2, ±6, ±10, ±14, ±18, so here $e = 2$, $f = 1$, and $u = 4$. By Theorem 1 of [15], this implies each $A \cup B_i$ is a dominating set of Q_{23}. □

It is interesting that each square of B_1 is two squares distant along its sum diagonal from a square of B_2.

Returning to the question of when $\gamma(Q_n) = (n-1)/2$, we note one further restriction. From (11) we see that each number t of an occupied diagonal satisfies either $t \equiv 2e \pmod 4$ or $t \equiv 2e-2 \pmod 4$, and that there are $d-e$ of the former and e of the latter. Following [3], we refer to the latter as *core* diagonals (both difference and sum). Equation (12) implies $d \approx (3+\sqrt{3})e$, so the core diagonals are outnumbered by the others. Since all numbers of occupied rows and columns are even, a square of D is on a core difference diagonal if and only if it is on a core sum diagonal. It is then easily seen that when $e > 0$, the intersection squares of the core diagonals induce a copy of Q_e, and that this Q_e contains e squares of D, which we will call the *core placement* of D.

Proposition 6 *Suppose that $\gamma(Q_n) = (n-1)/2$, D is a minimum dominating set of Q_n, and C is the core placement of D. There is i with $n = n_i$; choose $r_i \in \{-1,0,1\}$ such that $e_i \equiv r_i \pmod 4$. Let $Z = \{(x,y) : (x,y) \in C \text{ and } x \equiv 0 \pmod 4\}$. Then we have $|Z| = (e_i + r_i)/2$ and $|C - Z| = (e_i - r_i)/2$.*

Proof. From the recursion (13) for (e_i) we see that $e \equiv 2 \pmod 4$ does not occur, so there is such an r_i. We will examine the case $r_i = 1$ (corresponding to $i \equiv 2 \pmod 4$); the others are similar.

For $h, j \in \{0, 2\}$ let $g(h,j) = |\{(x,y) \in D : x \equiv h \pmod 4 \text{ and } y \equiv j \pmod 4\}|$. The recursion of Theorem 4 for (n_i) implies that the sequence (k_i) defined by $k_i = (n_i - 3)/4$ satisfies $k_1 = 0$, $k_2 = 2$, and $k_i = 4k_{i-1} - k_{i-2} + 1$ for $i > 2$. When $i \equiv 2 \pmod 4$, this gives $k_i \equiv 2 \pmod 4$, to go with $e_i \equiv 1 \pmod 4$. Then (10) implies the numbers of occupied rows and columns include exactly $k_i + 1$ that are congruent to 0 modulo 4. Thus $g(0,0) + g(0,2) = k_i + 1 = g(0,0) + g(2,0)$ and $g(2,0) + g(2,2) = k_i$. The core placement gives $g(0,0) + g(2,2) = e_i$. Solving these equations gives $g(0,0) = (e_i + 1)/2$. □

Thus if the core placement were drawn on a normally colored $e_i \times e_i$ board, the numbers of queens on each color would be as nearly equal as possible. It is interesting that if $i \equiv 0 \pmod 4$, so $r_i = -1$, the majority of the core placement squares will be on the minority color.

The necessary condition on n for $\gamma(Q_n) = (n-1)/2$ provided by Theorem 4 is not sufficient: Theorem 3 plus a computer search reported in [3] establish that $\gamma(Q_{39}) \neq 19$. We give here a short direct proof which may generalize to other n_i's (the squares near the ends of row and column 0 seem to merit attention), and show $\gamma(Q_{39}) = 20$.

Proposition 7 $\gamma(Q_{39}) = 20$.

Proof. We first establish $\gamma(Q_{39}) > 19$. If not, then $\gamma(Q_{39}) = 19$ by (1); let D be a dominating set of size 19. From Theorem 4 we have $e = 4$, so the core placement of D consists of four squares in the copy of Q_4 induced inside Q_{39} by the intersection squares of $\mathrm{Diag}(2) \cup \mathrm{Diag}(6)$. There are only two ways to choose a set of four independent squares from Q_4, and the two sets occupy the same lines, so we will take the core placement to be $\{\pm(2,4), \pm(4,-2)\}$. Then from (11) the diagonals remaining to be occupied are those with numbers $4i$ for $|i| \leq 7$, and the orthogonals remaining to be occupied include row 0 and column 0.

Let $S = \{(0, \pm 16), (\pm 16, 0)\}$. If no squares of S are in D, it is not difficult to see that the other eligible squares of the eight difference diagonals ± 16, ± 20, ± 24, ± 28 all lie in the seven sum diagonals $0, \pm 4$, ± 8, ± 12; as D is an independent set, it will not be possible to occupy all eight. Then as the squares of S are pairwise adjacent, exactly one square of S is in D; using symmetry, we may assume it is $(16, 0)$, on difference diagonal -16, and the other seven difference diagonals given above must be occupied by squares of D that also occupy the seven sum diagonals named. But none of these seven squares can lie in column 0, and the seven together with $(16, 0)$

dominate all eligible squares of column 0, which thus contains no square of D, a contradiction.

On the other hand, the set $D_1 = \{\pm(0,8),\pm(2,4),\pm(4,-2),\pm(6,18), \pm(8,-12),\pm(10,10),\pm(12,16),\pm(14,-14),\pm(16,0),\pm(18,-6)\}$ consists of 20 squares that jointly occupy the lines specified in (10) and (11), so by Theorem 1 of [15], D_1 dominates Q_{39}. □

From (1), Theorem 4, and Proposition 7, we have

Corollary 8 *If $n \notin (n_i)_{i=1}^{\infty}$ or $n = 39$, then $\gamma(Q_n) \geq n/2$.*

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