Extremal Triangle-Free Regular Graphs Containing A Cut Vertex

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ABSTRACT. The exact values of $c(n)$ are determined, where $c(n)$ denotes the largest k for which there exists a triangle-free k-regular graph on n vertices containing a cut-vertex. As a corollary, we obtain a lower bound on the densest triangle-free regular graphs of given order that do not have a one-factorization.

1 Introduction

We will be dealing exclusively with finite simple graphs. We will use $V(G)$ and $E(G)$ to denote the set of vertices and edges of G, respectively. For $x \in V(G)$, $N_G(x)$ denotes the set of neighbours of x in G, that is, $N_G(x) = \{y \in V(G)\colon \{x, y\} \in E(G)\}$. The degree of x in G, denoted $\deg_G(x)$, is the number of neighbours of X in G, i.e. $\deg_G(x) = |N_G(x)|$. A graph G is called k-regular if $\deg_G(x) = k$ for all $x \in V(G)$. The complete bipartite graph with bipartition $\{X, Y\}$ will be denoted $K(X, Y)$. The odd girth of a non-bipartite graph G, denoted $\gamma(G)$, is the length of the shortest odd cycle in G. The largest k for which there exists a non-bipartite k-regular graph G on n vertices with $\gamma(G) \geq 5$ is denoted $t(n)$. The following result was proven by Shi [5] (part of it was also proven in [2] and [6]).

Theorem 1.1. *For all $n \geq 5$, $t(n) = 2\lfloor \frac{n}{5} \rceil$ except that $t(6)$ does not exist, $t(8) = 3$, $t(14) = 5$, and $t(24) = 9$.*

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Generalizations and variations of this problem have been discussed in [7] and [8].

Let $F(2m)$ denote the largest k such that there exists a k-regular graph of (even) order $2m$ without a one-factorization. The well-known One-Factorization Conjecture asserts that $F(2m) = 2\lfloor \frac{m-1}{2} \rfloor$; we refer the reader to [4] for further discussion. Let $f(2m)$ denote the largest k for which a triangle-free k regular graph of order $2m$ which is not one-factorizable exists. Since a bipartite regular graph is one-factorizable (see, e.g. [3]), $f(2m)$ is the largest k for which a non-bipartite k-regular graph of order $2m$ and odd girth $\gamma \geq 5$ which is not one-factorizable exists. Hence $f(2m) \leq t(2m)$.

On the other hand, it is easy to see that a (regular) graph containing a cut-vertex is not one-factorizable. Hence $f(2m) \geq c(2m)$, where $c(n)$ denotes the largest k for which there exists a triangle-free k-regular graph of order n containing a cut-vertex.

The purpose of this paper is to evaluate $c(n)$. Let $\lfloor x \rfloor_e$ denote the largest even integer which does not exceed x. We will prove the following result.

Theorem 1.2. *Let $n \geq 16$. Then $c(n) = \lfloor \frac{2n-4}{9} \rfloor + 1$ if $n \in \{24, 32, 42\}$ and $c(n) = \lfloor \frac{2n-4}{9} \rfloor_e + 2$ if $n \in \{17, 19, 27, 37\}$. Otherwise, $c(n) = \lfloor \frac{2n-4}{9} \rfloor$ if n is even and $c(n) = \lfloor \frac{2n-4}{9} \rfloor_e$ if n is odd.*

2 Almost k-Regular Graphs

A graph G will be called *almost k-regular* if one vertex (called the *special vertex*) in G has degree $k-2$ and every other vertex in G has degree k. By a *(k,n)-graph*, we will mean a triangle-free almost k-regular graph of order n.

Let $t'(n)$ denote the largest k for which there exists a (k,n)-graph. Let $S = \{8, 11, 14, 15, 18, 21, 24\}$ and define

$$a(n) = \begin{cases} 4 & \text{if } n = 9, \\ \lfloor \frac{2n-4}{5} \rfloor + 1 & \text{if } n \in S, \\ \lfloor \frac{2n-4}{5} \rfloor - 1 & \text{if } n \equiv 1 \text{ or } 5 \mod 10 \text{ and } n \notin S, \\ \lfloor \frac{2n-4}{5} \rfloor & \text{for all other } n \geq 10. \end{cases}$$

In this section, we will prove that $t'(n) = a(n)$ for all $n \geq 8$; this result will be crucial to proving Theorem 1.2.

Lemma 2.1. *$t'(n) \geq a(n)$ for all $n \geq 8$.*

Proof: We construct an $(a(n),n)$-graph for all $n \geq 8$. If $n \in S \cup \{9\}$, then the corresponding $(a(n),n)$-graph is shown in Figure 2 of the Appendix, while if $n \equiv 2, 3$, or $4 \pmod 5$ and $n \notin S \cup \{9\}$, then the corresponding $(a(n),n)$-graph is shown in Figure 3 of the Appendix. Thus, we may assume that $n \equiv 0, 1 \pmod 5$ and $n \notin S$. Let H be the $2m$-regular graph of order n shown in Figure 4 of the Appendix. Select a vertex $b_i \in B_i$ for $i = 0, 1, 2$; if n is odd (whence $n \geq 25$) select a second vertex $x \in B_1$ (i.e. $x \neq b_1$). Let H' be the subgraph of H obtained by deleting the vertices b_0 and b_2, and also deleting b_1 when n is odd. Let M be a one-factor in H' if n is even, or a 2-factor in H' if n is odd. Then the graph G obtained from H by deleting the set of edges $M \cup \{\{b_0, b_1\}, \{b_1, b_2\}\}$ (together with the edges $\{b_0, x\}$ and $\{b_2, x\}$ if n is odd) is an $(a(n), n)$-graph. □

We work now to show that $t'(n) \leq a(n)$ for all $n \geq 8$. Henceforth, we assume that G is a triangle-free, almost k-regular graph of order $n \geq 8$ containing the special vertex x, where $k > \lfloor \frac{2n-4}{5} \rfloor$. Now if G is bipartite, then since the degree sum over each part in its bipartition must be the same, it is easy to see that $-2 \equiv 0 \pmod k$, forcing $k \leq 2$, a contradiction. Hence G is not bipartite.

We will need the following result of Andrásfai, Erdös, and Sós [1] (the case $\gamma = 5$ was also proven by Sheehan in [6]).

Lemma 2.2. *Let H be a graph of order n and odd girth $\gamma \geq 5$. Then for any odd cycle in H of length γ with degree sequence $\{k_i \colon 1 \leq i \leq \gamma\}$, we have $\sum_{i=1}^{\gamma} k_i \leq 2n$. In particular, $\delta \leq 2n/\gamma$ where, as usual, δ denotes the minimum degree in H.*

Applying Lemma 2.2 to our graph G, we see that $\gamma(G) \cdot k - 2 \leq 2n$; therefore, $k \leq (2n+2)/\gamma(G)$. Since $k > \lfloor \frac{2n-4}{5} \rfloor$, it follows immediately that $\gamma(G) = 5$, whence $k \leq (2n+2)/5$. Thus, either $k = \lfloor \frac{2n-4}{5} \rfloor + 1$ or $k = (2n+2)/5$ and $n \equiv 4 \pmod 5$. Now in the latter case *every* 5-cycle in G contains the special vertex x, whence it is easy to see that $G - x$ must be bipartite, i.e. $G - x$ is a subgraph of some $K(X, Y)$. Since G itself is not bipartite, we see that $N_G(x)$ has non-empty intersection with each of X and Y, and so $n - 1 \geq k - 2 + 2(k-1)$, i.e. $n \geq 3k - 3$. But in this case $n = (5k-2)/2$, which forces $(n, k) = (9, 4) = (9, a(9))$. Now if $k = \lfloor \frac{2n-4}{5} \rfloor + 1$ and $n \equiv 2 \pmod 5$, whence $k = (2n+1)/5$, we again conclude that every 5-cycle in G contains the special vertex x, and again $n \geq 3k-3$. This forces $(n, k) = (12, 5)$, which is easily seen to be impossible (consider $G - (\{x\} \cup N_G(x))$).

Thus far we have established the following.

Lemma 2.3. *If G is a triangle-free, almost k-regular graph of order $n \geq 8$ and $k > \lfloor \frac{2n-4}{5} \rfloor$, then either $(n, k) = (9, a(9))$, or $n \not\equiv 2 \pmod 5$ and $k = \lfloor \frac{2n-4}{5} \rfloor + 1 = \lfloor \frac{2n}{5} \rfloor$.*

Suppose now that $G - x$ is bipartite. Then, as above, we have $n \geq 3k - 3$. Furthermore, it must be that

$$\begin{aligned} &|N_G(x) \cap X|(k-1) + (|X| - |N_G(x) \cap X|)k \\ &= |N_G(x) \cap Y|(k-1) + (|Y| - |N_G(x) \cap Y|)k, \end{aligned}$$

which means that

$$|X|k - |N_G(x) \cap X| = |Y|k - |N_G(x) \cap Y|. \tag{2.1}$$

Thus, $|N_G(x) \cap X| \equiv |N_G(x) \cap Y| \pmod k$. But $|N_G(x)| = k - 2$, since x is the special vertex in G and so in fact it must be that $|N_G(x) \cap X| = |N_G(x) \cap Y|$ and so we conclude from Equation (2.1) that $|X| = |Y|$. Thus, n is odd. Since also $n \geq 3k - 3$ and $k = \lfloor \frac{2n}{5} \rfloor$, we deduce that $(n, k) = (11, 4), (15, 6)$, or $(21, 8)$, i e $(n, k) = (11, a(11)), (15, a(15))$ or $(21, a(21))$. We may henceforth assume therefore that $G - x$ is *not* bipartite, whence G contains a 5-cycle $C = \{\alpha_i \colon 1 \leq i \leq 5\}$ that does *not* contain x, i.e. $x \notin V(C)$, where α_i is adjacent to α_j if and only if $j - i \equiv \pm 1 \pmod 5$. Let $T = V(G) \backslash V(C)$. Then each vertex in T is adjacent to at most two vertices in C since G is triangle-free. So we can write

$$T = X \cup Y \cup Z, \quad X = \cup_{i=1}^{5} X_i, \quad Y = \cup_{i=1}^{5} Y_i,$$

where

$$X_i = \{x \in T \colon N(x) \cap C = \{\alpha_{i-1}, \alpha_{i+1}\}\},$$
$$Y_i = \{y \in T \colon N(y) \cap C = \{\alpha_i\}\},$$

and

$$Z = \{z \in T \colon N(z) \cap C = \emptyset\}.$$

Using lower-case letters to denote the cardinality of the corresponding uppercase letter set, we have

$$x + y + z = t = n - 5. \tag{2.2}$$

Now α_i is adjacent to each vertex in $X_{i-1} \cup X_{i+1} \cup Y_i$; since α_i has degree k, we therefore have

$$x_{i-1} + x_{i+1} + y_i = k - 2 \tag{2.3}$$

for each $i = 1, 2, \ldots, 5$. Summing the five corresponding equations yields

$$2x + y = 5k - 10$$

which, together with Equation (2.2), yields

$$y + 2z = 2n - 5k. \tag{2.4}$$

Now by Lemma 2.3, $k = \lfloor \frac{2n}{5} \rfloor$ and $n \not\equiv 2 \mod 5$, whereupon we can set

$$\epsilon = y + 2z = \begin{cases} 0 & \text{if } n \equiv 0 \mod 5 \\ 2 & \text{if } n \equiv 1 \mod 5 \\ 1 & \text{if } n \equiv 3 \mod 5 \\ 3 & \text{if } n \equiv 4 \mod 5 \end{cases};$$

in particular, this implies that $0 \leq |Y \cup Z| \leq 3$.

Given a collection $W = \{W_i \colon 1 \leq i \leq n\}$ of disjoint nonempty sets, we define a *weighted cycle* with *weight set* W, denoted $C(n, W)$ to be the graph H with $V(H) = \cup_{i=1}^{n} W_i$, in which two vertices x and y are adjacent if and only if $x \in W_i$ and $y \in W_{i+1}$ for some $1 \leq i \leq n$.

Lemma 2.4. *Suppose that G is a spanning subgraph of some weighted cycle $C(5, W)$. Then $(n, k) = (8, 3)$ or $(14, 5)$, i e. $(n, k) = (8, a(8))$ or $(14, a(14))$.*

Proof: Let $w_i = |W_i|$ for $i = 1, 2, \ldots, 5$. Suppose that the special vertex $x \in W_1$. Then since $x \notin C$, we have $w_1 \geq 2$.

For each $i = 1, 2, \ldots, 5$, let

$$w_i + w_{i+2} = k + \epsilon_i. \tag{2.5}$$

Note that $\epsilon_i \geq 0$, since $\deg_G(\alpha_{i+1}) = k$. Summing the five corresponding equations yields

$$2n = 5k + \sum_{i=1}^{5} \epsilon_i$$

which, by Equation (2.4), means that

$$\sum_{i=1}^{5} \epsilon_i = 2n - 5k = \epsilon \leq 3.$$

Therefore, there exist $1 \leq i < j \leq 5$ such that $\epsilon_i = \epsilon_j = 0$. Now let e_i denote the number of edges joining W_i to W_{i+1}. Then, without loss of generality, we may suppose that $e_2 = w_2 w_3$.

Now the system of 5 equations in the 5 variables w_i, given by Equation (2.5), yields the solutions

$$w_i = \frac{1}{2}(k + \epsilon_i - \epsilon_{i+1} - \epsilon_{i+2} + \epsilon_{i+3} + \epsilon_{i+4}), \ 1 \leq i \leq 5 \tag{2.6}$$

Now $e_i + e_{i+1} = w_{i+1} k$, $1 \leq i \leq 4$, and $e_5 + e_1 = w_1 k - 2$. Therefore,

$$e_2 = w_2 w_3 = \frac{1}{2} k(w_2 + w_3 - w_4 + w_5 - w_1) + 1. \tag{2.7}$$

Now by Equation (2.6) we have

$$\begin{aligned} w_2 + w_3 - w_4 + w_5 - w_1 &= \frac{1}{2}(k + \epsilon_1 + \epsilon_2 + \epsilon_3 - 3\epsilon_4 + \epsilon_5) \\ &= \frac{1}{2}(k + \epsilon - 4\epsilon_4), \end{aligned}$$

whence Equation (2.7) yields

$$e_2 = w_2 w_3 = \frac{1}{4}k(k + \epsilon - 4\epsilon_4) + 1. \tag{2.8}$$

On the other hand, Equation (2.6) gives

$$e_2 = w_2 w_3 = \frac{1}{2}(k + \epsilon - 2\epsilon_3 - 2\epsilon_4) \cdot \frac{1}{2}(k + \epsilon - 2\epsilon_4 - 2\epsilon_5). \tag{2.9}$$

From Equations (2.8) and (2.9), we get

$$(k + \epsilon - 2\epsilon_3 - 2\epsilon_4)(k + \epsilon - 2\epsilon_4 - 2\epsilon_5) = k(k + \epsilon - 4\epsilon_4) + 4. \tag{2.10}$$

Now Equation (2.10) implies that $\epsilon \neq 0$. Otherwise, we solve for k to yield

$$k = \frac{4 - (\epsilon - 2\epsilon_3 - 2\epsilon_4)(\epsilon - 2\epsilon_4 - 2\epsilon_5)}{\epsilon - 2\epsilon_3 - 2\epsilon_5}.$$

Thus, for example, if $\epsilon = 1$, then either $\epsilon_3 + \epsilon_4 + \epsilon_5 = 0$ (whence $\epsilon_1 + \epsilon_2 = 1$) or $\epsilon_3 + \epsilon_4 + \epsilon_5 = 1$ (whence $\epsilon_1 = \epsilon_2 = 0$); in the former case we get $k = 3$, $n = 8$, while in the latter case (with $\epsilon_4 = 1$) we again get $k = 3$, $n = 8$. A similar analysis in the cases $\epsilon = 2, 3$ yields only one more solution (with $n \geq 8$), that being when $\epsilon = 3$, with $\epsilon_3 = \epsilon_4 = 1$ and $\epsilon_5 = 0$, yielding $k = 5$, $n = 14$. We leave it to the reader to verify the gruesome details. □

From Lemma 2.4, we can henceforth assume that $Y \cup Z \neq \emptyset$; in particular $\epsilon \neq 0$. Consider now the case $\epsilon = 1$, i.e. $Y = \{v\}$ and $Z = \emptyset$. As we can assume that G is not a subgraph of any weighted cycle $C(5, W)$ it follows that for some $i = 1, 2, \ldots, 5$, v is adjacent to some vertex in X_i and to some vertex in X_{i+1}. (Suppose that v is adjacent to α_1 and to some vertex in X_j, $j \neq 1$. Then $j = 3$ or 4 since G is triangle-free. Now if v is not adjacent to some vertex in each of X_3 and X_4, then it is easy to see that $G \subseteq C(5, W)$.)

Lemma 2.5. *If $\epsilon = 1$ (whence $n \equiv 3 \mod 5$) and G is not a subgraph of any weighted cycle, then $(n, k) = (8, 3)$ or $(18, 7)$, i.e. $(n, k) = (8, a(8))$ or $(18, a(18))$.*

Proof: Let $Y = \{v\}$ where v is adjacent to $\alpha_1 \in V(C)$, and v is also adjacent to $v_3 \in X_3$ and $v_4 \in X_4$. Let $S_i = N_G(v) \cap X_i$ for $i = 1, 2, \ldots, 5$ (whence $S_2 = S_5 = \emptyset$) and let $s_i = |S_i|$. Then

$$x_1 + s_3 + s_4 \geq \deg_G(v) - 1,$$
$$x_2 + (x_4 - s_4) + 1 \geq \deg_G(v_3) - 2,$$

and

$$x_5 + (x_3 - s_3) + 1 \geq \deg_G(v_4) - 2. \quad (G \text{ is triangle-free.})$$

Summing these three inequalities gives us

$$x + 2 \geq 3k - 7 \quad \text{(one of } v, v_3, v_4 \text{ may be special)}$$
$$(n-5) + 1 \geq 3k - 7 \quad \text{(since } x + y + z = n - 5\text{)}$$

whence

$$n \geq 3k - 3.$$

But $k = \lfloor \frac{2n}{5} \rfloor$ and $n \equiv 3 \mod 5$ implies $(n, k) = (8, 3), (13, 5)$, or $(18, 7)$. Now since the degree sum in any graph must be even, no $(5, 13)$-graph can exist. Hence $(n, k) = (8, 3)$ or $(18, 7)$. □

The following lemma summarizes our progress to this point.

Lemma 2.6. *If G is a triangle-free, almost k-regular graph of order $n \geq 8$ with $k > \lfloor \frac{2n-4}{5} \rfloor$, then either $(n, k) = (s, a(s))$ for $s \in \{8, 9, 11, 14, 15, 18, 21\}$, or $n \equiv \pm 1 \mod 5$, $k = \lfloor \frac{2n}{5} \rfloor$, G is not a subgraph of any weighted cycle $C(5, W)$ and $G - x$ is not bipartite, where x is the special vertex in G.*

We consider now the case $\epsilon = 2$, i.e. $n \equiv 1 \mod 5$. We have two subcases, according to whether $|Y| = 0$ and $|Z| = 1$, or $|Y| = 2$ and $|Z| = 0$.

Lemma 2.7. *Suppose that $\epsilon = 2$ (i.e. $n \equiv 1 \mod 5$) and $Z = \{v\}$, $Y = \emptyset$. Then $(n, k) = (11, 4) = (11, a(11))$.*

Proof: Since G is not a subgraph of any $C(5, W)$, we may assume that v is adjacent to some $v_3 \in X_3$ and $v_4 \in X_4$. Let $S_i = N_G(v) \cap X_i$ for $i = 1, 2, \ldots, 5$, and let $s_i = |S_i|$. We consider two possibilities.

(i) $s_2 = s_5 = 0$. Then

$$x_1 + s_3 + s_4 \geq \deg_G(v),$$
$$x_2 + (x_4 - s_4) + 1 \geq \deg_G(v_3) - 2,$$

and

$$x_5 + (x_3 - s_3) + 1 \geq \deg_G(v_4) - 2,$$

whence

$$x + 2 \geq 3k - 6.$$

But $x + y + z = n - 5$, and so we get

$$n - 4 \geq 3k - 6,$$

i.e.

$$n \geq 3k - 2. \tag{2.11}$$

Now $k = \lfloor \frac{2n}{5} \rfloor$ and $n \equiv 1 \mod 5$ implies $(n,k) = (11,4)$ or $(16,6)$. But if $(n,k) = (16,6)$, then Inequality (2.11) becomes an equality as, therefore, do all inequalities preceding it. In particular, $s_1 = x_1$, and one of v, v_3, v_4 is special. Now consider Equation (2.3); we get $x_{i-1} + x_{i+1} = k - 2$ for $i = 1, 2, \ldots, 5$, which system has the unique solution $x_i = \frac{1}{2}k - 1$ for $i = 1, 2, \ldots, 5$. That is, in our particular case, we have $x_1 = x_2 = \cdots = x_5 = 2$. Now each vertex in $X_2 \cup X_5$ has degree 6, which implies that each vertex in X_1 is adjacent to each vertex in $X_2 \cup X_5$. But each vertex in X_1 is also adjacent to α_2 and α_5 and, since $s_1 = x_1$, to v. This brings the degree count of each vertex in X_1 to 7, a contradiction. Hence, $(n,k) \neq (16,6)$.

(ii) $s_2 + s_5 > 0$. Suppose, without loss of generality, that $s_2 > 0$, i.e. that v is adjacent to some $v_2 \in X_2$. Now

$$s_1 + s_2 + s_3 + s_4 + s_5 = \deg_G(v)$$

and

$$(x_{i-1} - s_{i-1}) + (x_{i+1} - s_{i+1}) + 1 \geq \deg_G(v_i) - 2, i = 2, 3, 4.$$

Summing these four inequalities gives us

$$x + (x_3 - s_3) + 3 \geq 4k - 8.$$

Now, as in (i), we have $x_3 = \frac{1}{2}k - 1$, and furthermore $s_3 \geq 1$. Hence, the foregoing inequality gives us

$$x + \frac{1}{2}k + 1 \geq 4k - 8;$$

again $x + 1 = n - 5$, and so we have

$$n \geq \frac{7}{2}k - 3.$$

But $k = \lfloor \frac{2n}{5} \rfloor$ and $n \equiv 1 \mod 5$ forces $(n,k) = (11,4)$.

This completes the proof of Lemma 2.7. □

Lemma 2.8. *Suppose that $\epsilon = 2$ (i.e. $n \equiv 1 \mod 5$) and $Y = \{v, v'\}$, $Z = \emptyset$, and v is adjacent to v'. Suppose further that neither v nor v' is adjacent to both X_i and X_{i+1} for any $i = 1, 2, \ldots, 5$. Then $(n,k) = (11,4) = (11, a(11))$.*

Proof: Since G is not a subgraph of any weighted cycle $C(5, W)$, we may assume that $v \in Y_1$ (i.e. v is adjacent to α_1) and that v is adjacent to some $v_3 \in X_3$. Furthermore, v' is not adjacent to α_1 (G is triangle-free) and v' is not adjacent to α_2 (G is not a subgraph of $C(5, W)$). We therefore have three possibilities to consider.

(i) v' adjacent to α_3 (i.e. $v' \in Y_3$). Then $N_G(v') \subseteq X_1 \cup X_3 \cup X_5 \cup \{\alpha_3, v\}$ and $N_G(v) \subseteq X_1 \cup X_3 \cup \{\alpha_1, v'\}$; now from Equation (2.3) (with $y_1 = 1$, $y_3 = 1$ and $y_2 = y_4 = y_5 = 0$) we quickly deduce that $x_1 = x_3 = x_5 = \frac{1}{2}k - 1$, whence in fact $N_G(v) + N_G(v') \leq |X_1| + |X_3| + |X_5| + 4$ (since G is triangle-free) $= \frac{3}{2}k + 1$. But $N_G(v) + N_G(v') \geq 2k - 2$, whence $\frac{3}{2}k + 1 \geq 2k - 2$, and so $(n, k) = (11, 4)$ or $(16, 6)$. Now if $(n, k) = (16, 6)$, then the foregoing inequalities become equalities. In particular, we deduce that $X_1 \cup \{\alpha_1, v'\} \subseteq N_G(v)$ and $X_5 \cup \{\alpha_3, v\} \subseteq N_G(v')$, since v' is not adjacent to both X_5 and X_1. Then either $X_3 \subseteq N_G(v)$, in which case v has degree 6 and v' is the special vertex with degree 4, or $X_3 \subseteq N_G(v')$ in which case v' has degree 6 and v is the special vertex with degree 4. In either case, an analysis similar to that in possibility (i) of Lemma 2.7 leads us to conclude that each vertex in X_5 has at least 7 neighbours, a contradiction. Hence $(n, k) = (11, 4)$.

(ii) v' adjacent to α_4 (i.e. $v' \in Y_4$). Since G is not a subgraph of $C(5, W)$, we deduce that $N_G(v') \subseteq X_1 \cup X_4 \cup \{\alpha_4, v\}$. But again, $N_G(v) \subseteq X_1 \cup X_3 \cup \{\alpha_1, v'\}$, whence $N_G(v) + N_G(v') \leq |X_1| + |X_4| + |X_3| + 4$. Now by Equation 2.3 (with $y_1 = 1$, $y_4 = 1$ and $y_2 = y_3 = y_5 = 0$) we get $x_1 = x_3 = x_4 = \frac{1}{2}k - 1$ and so $N_G(v) + N_G(v') \leq \frac{3}{2}k + 1$. But $N_G(v) + N_G(v') \geq 2k - 2$ and so as in possibility (i) we have $\frac{3}{2}k + 1 \geq 2k - 2$, whence either $(n, k) = (11, 4)$ or $(16, 6)$. But $(n, k) = (16, 6)$ leads one to conclude that each vertex in X_3 must have degree 7, a contradiction (we use the same analysis as in possibility (i)). Hence $(n, k) = (11, 4)$.

(iii) v' adjacent to α_5 (i.e. $v' \in Y_5$). Then we must have $N_G(v') \subseteq X_3 \cup X_5 \cup \{\alpha_5, v\}$ and $N_G(v) \subseteq X_1 \cup X_3 \cup \{\alpha_1, v'\}$ whereupon $N_G(v') + N_G(v) \leq |X_1| + |X_3| + |X_5| + 4$. Now in this case Equation 2.3 (with $y_1 = 1$, $y_5 = 1$, and $y_2 = y_3 = y_4 = 0$) yields $x_1 = \frac{1}{2}k - 2$, $x_3 = \frac{1}{2}k$, and $x_5 = \frac{1}{2}k - 2$ whereupon $N_G(v) + N_G(v') \leq \frac{3}{2}k$. But again $N_G(v) + N_G(v') \geq 2k - 2$ and so $\frac{3}{2}k \geq 2k - 2$, forcing $(n, k) = (11, 4)$.

This completes the proof of Lemma 2.8. □

For $\epsilon = 2$, it remains to consider what happens when $Y = \{v, v'\}$, $Z = \emptyset$ and at least one of v, v' (say v) is adjacent to both X_i and X_{i+1} for some $i = 1, 2, \ldots, 5$. (Since G is not a subgraph of $C(5, W)$, this case will include the possibility that v and v' are not adjacent.) Thus, we will suppose that v is adjacent to α_1 (i.e. $v \in Y_1$) as well as to some $v_3 \in X_3$ and $v_4 \in X_4$. Let $S_i = N_G(v) \cap X_i$ for $i = 1, 2, \ldots, 5$ and let $s_i = |S_i|$. Let $R_i = N_G(v) \cap Y_i$ and let $r_i = |R_i|$ for $i = 1, 2, \ldots, 5$. Let $a_3 = |N(v_3) \cap Y \backslash Y_1|$ and $a_4 = |N(v_4) \cap Y \backslash Y_1|$. Note that in this case each of r_i, a_3, and a_4 is either 0 or 1.

Now we have

$$x_1 + s_3 + s_4 + r_2 + r_3 + r_4 + r_5 \geq \deg_G(v) - 1$$

($r_1 = 0$ since G is triangle-free)

$$x_2 + (x_4 - s_4) + y_1 + a_3 \geq \deg_G(v_3) - 2 \tag{2.12}$$

and

$$x_5 + (x_3 - s_3) + y_1 + a_4 \geq \deg_G(v_4) - 2. \tag{2.13}$$

Since v_3 is not adjacent to any vertex in $Y_2 \cup Y_4$ and v_4 is not adjacent to any vertex in $Y_3 \cup Y_5$, we have $a_3 + r_3 + r_5 \leq y_3 + y_5$ and $a_4 + r_2 + r_4 \leq y_2 + y_4$, whereupon summing the foregoing three inequalities yields

$$x + y + y_1 \geq 3k - 7.$$

Now since $Z = \emptyset$ we have $x + y = n - 5$. Since $y_1 \leq y = 2$ we therefore deduce that

$$n - 3 \geq 3k - 7,$$
$$n \geq 3k - 4.$$

But $k = \lfloor \frac{2n}{5} \rfloor$ and $n \equiv 1 \mod 5$, whereupon $(n,k) \in \{(11,4),(16,6),(21,8),(26,10)\}$. We can rule out $(n,k) = (26,10)$ as follows. If $(n,k) = (26,10)$, then all of the foregoing inequalities become equalities. In particular, $y_1 = y = 2$ and $y_2 = \cdots = y_5 = 0$. Additionally, each of r_i, a_3, and a_4 is equal to 0 since $Y = Y_1$. We see also that one of v, v_3, v_4 is the special vertex (of degree 8). Now Equation (2.3) (with $y_1 = 2$ and $y_2 = \cdots = y_5 = 0$) yields $x_5 = x_1 = x_2 = \frac{1}{2}k - 2 = 3$ and $x_3 = x_4 = \frac{1}{2}k = 5$. At least one of the two vertices in Y will have degree 10 and, since $|X_1 \cup \{\alpha_1\}| = 3 + 1 = 4$, that vertex will contain at least 6 neighbours in $X_3 \cup X_4$. Since $x_3 = x_4 = 5$, that vertex will contain at least one neighbour in each of X_3 and X_4 and so we may assume that vertex is v. That is, v has degree 10 and so one of v_3, v_4 is special. Suppose that v_3 is special. Then $N_G(v) \cap X_3 = \{v_3\}$ (if there were a second neighbour v_3' of v in X_3, then one of v_3', v_4 would also have to be special, a contradiction). This means that $s_3 = 1$ and $s_4 = 5$, whence Inequality (2.12) becomes

$$3 + (5 - 5) + 2 + 0 \geq 6,$$

a contradiction. If on the other hand v_4 were the special vertex, we would arrive at a similar contradiction with Inequality (2.13). Hence (n,k) cannot be $(26,10)$.

Since $(11,4) = (11, a(11)$ and $(21,8) = (21, a(21))$, we need now only consider the possibility that $(n,k) = (16,6)$. The foregoing series of inequalities implies that, in this case, either $y_1 = 2$, or $y_1 = 1$ and one of v, v_3, v_4 is special. Suppose that $y_1 = 2$. Then Equation (2.3) yields $x_5 = x_1 = x_2 = 1$ and $x_3 = x_4 = 3$, so that, as above, we may assume that the vertex v has degree 6. Now if neither v_3 nor v_4 is special, then the foregoing series of inequalities become equalities; in particular, Inequalities (2.12) and (2.13) become

$$x_2 + (x_4 - s_4) + y_1 + a_3 = \deg_G(v_3) - 2$$

and

$$x_5 + (x_3 - s_3) + y_1 + a_4 = \deg_G(v_4) - 2.$$

Now $x_2 + x_4 = 4$, $y_1 = 2$, $a_3 = 0$ and $\deg_G(v_3) = 6$ implies $s_4 = 2$ and, similarly, $s_3 = 2$. Thus, v has 2 neighbours in each of X_3 and X_4; since we may assume that none of these 4 neighbours of v is special, it is not difficult to see that each of these 4 neighbours of v must also be neighbours to $v' = Y_1 \backslash \{v\}$, whereupon we quickly deduce that G must be 6-regular (i.e. there is no special vertex), a contradiction. Thus, if $y_1 = 2$, one of v_3, v_4 must be special, v_3 say. Since both vertices v, v' in Y_1 have degree 6, it follows from the above that v_3 is adjacent to each of v, v'. But $v_3 \in X_3$, whence v_3 is also adjacent to both α_2 and α_4. Now consider a vertex $c \in X_2$. Since c has degree 6 and c is not adjacent to either of v or v', it must be that $N_G(c) = X_1 \cup X_3 \cup \{\alpha_1, \alpha_3\}$. In particular, c is adjacent to v_3. But this brings the degree count for v_3 to 5, contradicting the assertion that v_3 is special. If, on the other hand, v_4 is the special vertex, then we arrive at a similar contradiction by considering a vertex $c \in X_5$. So y_1 cannot be equal to 2.

Hence $y_1 = 1$ and one of v, v_3, v_4 is special. There are two possible configurations to consider, namely whether $y_2 = 1$ or $y_3 = 1$. (By symmetry, the cases $y_4 = 1$ and $y_5 = 1$ will then have been dealt with.)

Suppose first that $y_2 = 1$, i.e. $Y_2 = \{v'\}$. Then $\deg_G(v') = 6$. From Equation (2.3) (with $y_1 = y_2 = 1$ and $y_3 = y_4 = y_5 = 0$) we obtain $x_1 = x_2 = 1$, $x_3 = x_5 = 2$ and $x_4 = 3$. If v' is not adjacent to v, then v' has at least 4 neighbours in $X_4 \cup X_5$ and therefore has at least one neighbour $v_4' \in X_4$ and one neighbour $v_5' \in X_5$. Moreover, at least one of v', v_4', v_5' is special. This forces $v_4' = v_4$ to be the special vertex. But then $N_G(v') \cap X_4 = \{v_4'\}$ (if there were another neighbour v_4'' of v' in X_4, then one of v', v_4'', v_5' would have to be special as well) and so $|N_G(v')| \leq 5$, a contradiction. Hence v' must be adjacent to v. But then v' is not adjacent to v_4 (else $v'v_4v$ forms a triangle) and so v' has at least 3 neighbours in $(X_4 \backslash \{v_4\}) \cup X_5$ and so has at least one neighbour $v_4' \in X_4 \backslash \{v_4\}$ and one neighbour $v_5' \in X_5$. Moreover, at least one of v', v_4', v_5' is special. But this is impossible, as $\{v', v_4', v_5'\} \cap \{v, v_3, v_4\} = \emptyset$. Hence $y_2 \neq 1$.

Suppose then that $y_3 = 1$, i.e. $Y_3 = \{v'\}$. Then again $\deg_G(v') = 6$. From Equation (2.3) (with $y_1 = y_3 = 1$ and $y_2 = y_4 = y_5 = 0$) we obtain $x_2 = 1$ and $x_3 = x_4 = x_5 = x_1 = 2$. Now if v' is not adjacent to v, then v' has at least 3 neighbours in $X_1 \cup X_5$, while if v' is adjacent to v, then v' is not adjacent to v_3 (else $v'v_3v$ forms a triangle) and so again v' has at least 3 neighbours in $X_1 \cup X_5$. Thus v' has a neighbour $v_1' \in X_1$, and a neighbour $v_5' \in X_5$ and, moreover, one of v', v_1', v_5' is special. But this is impossible, as $\{v', v_1', v_5'\} \cap \{v, v_3, v_4\} = \emptyset$. Hence $y_3 \neq 1$.

This exhausts all cases with $\epsilon = 2$; Lemmas 2.7 and 2.8, together with the discussion following Lemma 2.8, now yield the following.

Lemma 2.9. *If $\epsilon = 2$ (whence $n \equiv 1 \mod 5$), then the graph G hypothesized by Lemma 2.6 satisfies $(n, k) = (11, 4)$ or $(21, 8)$, i.e. $(n, k) = (11, a(11))$ or $(21, a(21))$.*

Finally, we consider the case $\epsilon = 3$ (whence $n \equiv 4 \mod 5$). Now we note that when $n \equiv 9 \mod 10$, $k = \lfloor \frac{2n}{5} \rfloor \equiv 3 \mod 4$, i.e. k is odd and so no (k, n)-graph can exist. As $\{14, 24\} \subseteq S$, we may therefore assume that $n \equiv 4 \mod 10$, $n \geq 34$.

We have two subcases, according to whether $|Y| = 3$ and $|Z| = 0$, or $|Y| = |Z| = 1$.

Lemma 2.10. *Suppose that $\epsilon = 3$ (whence $n \equiv 4 \mod 5$) and $Z = \emptyset$, and suppose further that for some $v \in Y$, v is adjacent to X_i and X_{i+1} for some $i = 1, 2, \ldots, 5$. Then $(n, k) = (14, 5)$ or $(24, 9)$, i.e. $(n, k) = (14, a(14))$ or $(24, a(24))$.*

Proof: Without loss of generality, we take $v \in Y_1$ with v adjacent to some $v_3 \in X_3$ and some $v_4 \in X_4$. Using the analysis following Lemma 2.8 and noting that in this case $y_1 \leq y = 3$, we deduce that $n \geq 3k - 5$. Since $k = \lfloor \frac{2n}{5} \rfloor$ and $n \equiv 4 \mod 5$, it must be that $n \leq 34$. But we can rule out $(n, k) = (34, 13)$, as follows. Since in this case $n = 3k - 5$ we deduce that $y_1 = 3$ (and $y_2 = \cdots = y_5 = 0$) and that one of v, v_3, v_4 is special. Now from Equation (2.3) we obtain $x_5 = x_1 = x_2 = 4$ and $x_3 = x_4 = 7$. Now one of the vertices in Y_1 has degree 13, and since $|X_1 \cup \{\alpha_1\}| = 5$, that vertex will have at least 8 neighbours in $X_3 \cup X_4$ and so contains at least one neighbour in each of X_3 and X_4; thus we may assume that vertex is v, i.e. v has degree 13 and so one of v_3, v_4 is special. If v_3 is special, then $N_G(v) \cap X_3 = \{v_3\}$ (if there were another neighbour v_3' of v in X_3 then one of v_3', v_4 would have to be special), whence $s_3 = 1$ and $s_4 = 7$. But then Inequality (2.12) becomes $4 + (7 - 7) + 3 + 0 \geq 11 - 2$, a contradiction. If on the other hand v_4 is special, we arrive at a similar contradiction in Inequality (2.13). Hence $(n, k) \neq (34, 13)$. □

Lemma 2.11. *Suppose that $\epsilon = 3$ (whence $n \equiv 4 \mod 5$) and $|Y| = |Z| = 1$, and suppose further that for some $v \in Y \cup Z$, v is adjacent to*

X_i and X_{i+1} for some $i = 1, 2, \ldots, 5$. Then $(n,k) = (14,5)$ or $(24,9)$, i.e. $(n,k) = (14, a(14))$ or $(24, a(24))$.

Proof: Suppose first that $v \in Y_1$ with v adjacent to both $v_3 \in X_3$ and $v_4 \in X_4$. Modifying the analysis following Lemma 2.8 by setting $P = N_G(v) \cap Z$, $p = |P|$, $a_3 = |N_G(v_3) \cap Z|$ and $a_4 = |N_G(v_4) \cap Z|$, and noting that all $R_i = \emptyset$, we get the following:

$$x_1 + s_3 + s_4 + p \geq \deg_G(v) - 1$$
$$x_2 + (x_4 - s_4) + 1 + a_3 \geq \deg_G(v_3) - 2$$
$$x_5 + (x_3 - s_3) + 1 + a_4 \geq \deg_G(v_4) - 2$$

Summing these inequalities yields

$$x + 2 + p + a_3 + a_4 \geq 3k - 7.$$

Now $x + 2 = n - 5$, and $p + a_3 + a_4 \leq 2$ since G is triangle-free. So we get

$$n - 3 \geq 3k - 7,$$

i.e.

$$n \geq 3k - 4.$$

Since $k = \lfloor \frac{2n}{5} \rfloor$ and $n \equiv 4 \mod 10$ we get $n \leq 24$, as desired.

Thus let $v \in Z$, i.e. $Z = \{v\}$. Let $Y = \{v'\}$. We may suppose that v' is adjacent to α_1, i.e. $Y = Y_1$, and that v' is adjacent to X_3 but v' is not adjacent to any vertex in X_4. Now from Equations (2.3) (with $y_1 = 1$ and $y_2 = \cdots = y_5 = 0$) we get $x_5 = x_1 = x_2 = \frac{1}{2}(k-1) - 1$ and $x_3 = x_4 = \frac{1}{2}(k-1)$. Suppose, if possible, that v' has degree k. Then $N_G(v') = X_1 \cup X_3 \cup \{\alpha_1, v\}$. Since G is triangle-free, v is not adjacent to any vertex in $X_1 \cup X_3$ and so v is adjacent to some $v_4 \in X_4$ and to some $v_5 \in X_5$. Modifying the analysis in the proof of Lemma 2.7 (case (i)), we obtain

$$x_2 + s_4 + s_5 + 1 \geq \deg_G(v) \quad (v \text{ is adjacent to } v')$$
$$x_3 + (x_5 - s_5) + 1 \geq \deg_G(v_4) - 2$$
$$x_1 + (x_4 - s_4) + 1 \geq \deg_G(v_5) - 2$$

whence

$$x + 3 \geq 3k - 6.$$

But again $x + 2 = n - 5$, and so we get

$$n \geq 3k - 2$$

whereupon $n \leq 14 < 34$, as desired. It follows then that v' is the special vertex. Now v is adjacent to X_i and to X_{i+1} for some $i = 1, 2, \ldots, 5$.

Suppose that v is not adjacent to any vertex in $X_{i-1} \cup X_{i+2}$. Then as above we have

$$x_{i-2} + s_i + s_{i+1} + 1 \geq \deg_G(v)$$
$$x_{i-1} + (x_{i+1} - s_{i+1}) + 2 \geq \deg_G(v_i) - 2$$
$$x_{i+2} + (x_i - s_i) + 2 \geq \deg_G(v_{i+1}) - 2$$

whence

$$x + 5 \geq 3k - 4,$$

since none of $v, v_i \in X_i$, $v_{i+1} \in X_{i+1}$ is special. Again $x + 2 = n - 5$, and so we get $n \geq 3k - 2$, as before.

So v is adjacent to some vertex in $X_{i-1} \cup X_{i+2}$, say v is adjacent to some $v_{i-1} \in X_{i-1}$. Modifying the analysis in the proof of Lemma 2.7 (case (ii)) we obtain

$$s_1 + s_2 + s_3 + s_4 + s_5 + 1 \geq \deg_G(v)$$

and

$$(x_{j-1} - s_{j-1}) + (x_{j+1} - s_{j+1}) + 2 \geq \deg_G(v_j) - 2,$$
$$j = i - 1, i, i + 1.$$

Summing these four inequalities yields

$$x + (x_i - s_i) + 7 \geq 4k - 6$$

since none of v, v_j are special. Now $x_i \leq \frac{1}{2}(k-1)$ and $s_i \geq 1$, whence we get

$$x + \frac{1}{2}(k-1) \geq 4k - 12.$$

Since $x + 2 = n - 5$, we get

$$n \geq 4k - \frac{1}{2}(k-1) - 5$$

which, since $k = \lfloor \frac{2n}{5} \rfloor$ and $n \equiv 4 \mod 10$, forces $n = 14$.

As this exhausts all possibilities, Lemma 2.11 is proved. □

It remains only to consider each of our subcases ($|Y| = 3$ and $|Z| = 0$, or $|Y| = |Z| = 1$) under the assumption that for *no* $v \in Y \cup Z$ is v adjacent to X_i and X_{i+1} for some $i = 1, 2, \ldots, 5$, and that G is not a subgraph of any weighted cycle $C(5, W)$.

Lemma 2.12. *If* $\epsilon = 3$ *(whence* $n \equiv 4 \mod 5$*) and* $|Y| = |Z| = 1$*, then under the foregoing assumptions we have* $(n, k) = (14, 5) = (14, a(14))$.

Proof: We may take $Y = \{v\}$, where v is adjacent to α_1 (i.e. $Y_1 = Y$) and v is adjacent to X_3 but not to any vertex in X_4. Since G is not a subgraph of any $C(5, W)$, it follows immediately that v is adjacent to v', where $Z = \{v'\}$. Moreover, it follows for the same reason that $N_G(v')\backslash\{v\} \not\subseteq X_2 \cup X_4$ and $N_G(v')\backslash\{v\} \not\subseteq X_5 \cup X_2$. Hence, we must consider three cases:

(i) $N_G(v')\backslash\{v\} \subseteq X_1 \cup X_3$. Following the proof of Lemma 2.8, we first deduce from Equation (2.3) (with $y_1 = 1$ and $y_2 = \cdots = y_5 = 0$) that $x_5 = x_1 = x_2 = \frac{1}{2}(k-1) - 1$ and $x_3 = x_4 = \frac{1}{2}(k-1)$. Since G is triangle-free, we must have $N_G(v) + N_G(v') \leq 3 + |X_1| + |X_3| = k+1$. But $N_G(v) + N_G(v') \geq 2k-2$, whence $k \leq 3$, which cannot happen.

(ii) $N_G(v')\backslash\{v\} \subseteq X_3 \cup X_5$. Then $N_G(v) + N_G(v') \leq 3 + |X_1| + |X_3| + |X_5| = k+1+\frac{1}{2}(k-1)-1 = (3k-1)/2$. Again $N_G(v)+N_G(v') \geq 2k-2$, whence $k \leq 3$.

(iii) $N_G(v')\backslash\{v\} \subseteq X_4 \cup X_1$. Then $N_G(v) + N_G(v') \leq 3 + |X_1| + |X_3| + |X_4| = k+1+\frac{1}{2}(k-1) = (3k+1)/2$. Since $N_G(v)+N_G(v') \geq 2k-2$, we must have $k = 5$, i.e. $(n, k) = (14, 5)$, as asserted.

□

Lemma 2.13. *If* $\epsilon = 3$ *(whence* $n \equiv 4 \mod 5$*) and* $|Y| = 3$, $|Z| = 0$, *then under the assumptions preceding Lemma 2.12, we have* $(n, k) = (14, 5) = (14, a(14))$.

Proof: Let $Y = \{v, v', v''\}$. Since G is not a subgraph of any $C(5, W)$, at least one pair of vertices in Y are adjacent. We may assume, then, that v is adjacent to v', v' is not adjacent to v'', and that $v \in Y_1$, i.e. v is adjacent to α_1. Moreover, v is not adjacent to any vertex in X_4. There are now two subcases to consider.

(I) v is not adjacent to v''.

This case is essentially identical to the situation in Lemma 2.8; the difference is that we have a third vertex $v'' \in Y$ whose position (i.e. the particular Y_j for which $v'' \in Y_j$) will determine the distribution of the x_is. Thus, for example, as in Lemma 2.8, v' is not adjacent to α_1 (since G is triangle-free) and v' is not adjacent to α_2 (as G would then be a subgraph of $C(5, W)$ regardless of the position of v''). We must consider the following possibilities.

(i) $v' \in Y_3$. Then (by symmetry) we must consider, in turn, $v'' \in Y_1$, $v'' \in Y_2$, and $v'' \in Y_4$. By Equation (2.3), we get the following:

$$v'' \in Y_1 \Rightarrow x_1 = x_5 = \frac{1}{2}(k-3), x_3 = x_4 = \frac{1}{2}(k-1), x_2 = \frac{1}{2}(k-5).$$

$$v'' \in Y_2 \Rightarrow x_1 = x_3 = \frac{1}{2}(k-3), x_4 = x_5 = \frac{1}{2}(k-1), x_2 = \frac{1}{2}(k-5).$$

$$v'' \in Y_4 \Rightarrow x_1 = \frac{1}{2}(k-1), x_2 = x_3 = x_4 = x_5 = \frac{1}{2}(k-3).$$

Now, in all cases we have

$$\begin{aligned} N_G(v) + N_G(v') &\leq 4 + |X_1| + |X_3| + |X_5|. \\ &= 4 + k - 3 + \frac{1}{2}(k-1). \\ &= \frac{3}{2}k + \frac{1}{2}. \end{aligned}$$

But $N_G(v)+N_G(v') \geq 2k-2$, whence $k \leq 5$, and $(n,k) = (14,5)$.

(ii) $v' \in Y_4$. Here we must consider, in turn, $v'' \in Y_1$, $v'' \in Y_3$, and $v'' \in Y_5$. We leave it to the reader to verify that in all cases $k \leq 5$, i.e. $(n,k) = (14,5)$.

(iii) $v' \in Y_5$. Here we must consider, in turn, $v'' \in Y_1$, $v'' \in Y_2$, and $v'' \in Y_3$. We again leave it to the reader to verify that in all cases $k \leq 5$, i.e. $(n,k) = (14,5)$.

(II) v is adjacent to v''.

Again this case is essentially identical to case (I); note that here v'', like v', is not adjacent to α_1 since G is triangle-free. We must consider the following possibilities.

(i) $v' \in Y_2$. Then we must consider, in turn, $v'' \in Y_3$, $v'' \in Y_4$, and $v'' \in Y_5$ (if $v'' \in Y_2$ then G is a subgraph of $C(5, W)$). By Equation (2.3), we get the following:

$$v'' \in Y_3 \Rightarrow x_1 = x_3 = \frac{1}{2}(k-3), x_4 = x_5 = \frac{1}{2}(k-1), x_2 = \frac{1}{2}(k-5).$$

$$v'' \in Y_4 \Rightarrow x_4 = \frac{1}{2}(k-1), x_5 = x_1 = x_2 = x_3 = \frac{1}{2}(k-3).$$

$$v'' \in Y_5 \Rightarrow x_5 = x_2 = \frac{1}{2}(k-3), x_3 = x_4 = \frac{1}{2}(k-1), x_1 = \frac{1}{2}(k-5).$$

In these cases, we have to consider, respectively,

$$N_G(v) + N_G(v'') \leq 4 + |X_1| + |X_3| + |X_5| = \frac{3}{2}k + \frac{1}{2}.$$

$$N_G(v) + N_G(v'') \leq 4 + |X_1| + |X_3| + |X_4| = \frac{3}{2}k + \frac{1}{2}.$$

$$N_G(v) + N_G(v'') \leq 4 + |X_1| + |X_3| + |X_5| = \frac{3}{2}k - \frac{1}{2}.$$

(Note that in the (second) case $v'' \in Y_4$, v'' cannot be adjacent to any vertex in X_2 since G is not a subgraph of $C(5, W)$. For the same reason, v'' cannot be adjacent to any vertex in X_2 in the (third) case $v'' \in Y_5$.) In all cases, it quickly follows that $k \leq 5$, since $N_G(v) + N_G(v'') \geq 2k - 2$.

(ii), (iii), (iv) $v' \in Y_3$, $v' \in Y_4$, $v' \in Y_5$.

In each of these cases, we proceed as in case (I), considering the sum $N_G(v) + N_G(v')$ and so deducing that in all cases $k \leq 5$. We leave the details for the reader to verify.

□

Collecting all of the results of this section, we have now proved the following result.

Theorem 2.14. *Let $t'(n)$ denote the largest k for which there exists a (k,n)-graph, that is, an almost k-regular triangle-free graph on n vertices. Let $S = \{8, 11, 14, 15, 18, 21, 24\}$ and define*

$$a(n) = \begin{cases} 4 & \text{if } n = 9, \\ \lfloor \frac{2n-4}{5} \rfloor + 1 & \text{if } n \in S, \\ \lfloor \frac{2n-4}{5} \rfloor - 1 & \text{if } n \equiv 1 \text{ or } 5 \mod 10 \text{ and } n \notin S, \\ \lfloor \frac{2n-4}{5} \rfloor & \text{for all other } n \geq 10. \end{cases}$$

Then $t'(n) = a(n)$ for all $n \geq 8$.

Remark 2.15: *For the sake of completeness, we note that $t'(n)$ does not exist for $n = 1, 2, 3, 4$, and that $t'(5) = t'(6) = t'(7) = 2$.*

3 Determining $c(n)$

In this section, we will prove Theorem 1.2. Let

$$\alpha(n) = \begin{cases} \lfloor \frac{2n-4}{9} \rfloor + 1 & \text{if } n \in \{24, 32, 42\}, \\ \lfloor \frac{2n-4}{9} \rfloor_e + 2 & \text{if } n \in \{17, 19, 27, 37\}, \\ \lfloor \frac{2n-4}{9} \rfloor & \text{for all other even } n \geq 16, \\ \lfloor \frac{2n-4}{9} \rfloor_e & \text{for all other odd } n \geq 21. \end{cases}$$

Lemma 3.1. *$c(n) \geq \alpha(n)$ for all $n \geq 16$.*

Proof.: Define

$$m = \begin{cases} \lceil \frac{5\alpha(n)}{2} \rceil_e & \text{if } n \in \{24, 32, 42\}, \\ \lceil \frac{5\alpha(n)}{2} \rceil_e + 2 & \text{for all other even } n \geq 16, \\ \lceil \frac{5\alpha(n)}{2} \rceil_o & \text{if } n \in \{19, 27, 37\}, \text{ or } 9 \text{ if } n = 17, \\ \lceil \frac{5\alpha(n)}{2} \rceil_o + 2 & \text{for all other odd } n \geq 21. \end{cases}$$

It is straightforward to verify that $n - m \geq 2\alpha(n)$ for every $n \geq 16$. Since $n - m$ is even we can, therefore, construct an $\alpha(n)$-regular bipartite graph G_1 of order $n - m$. On the other hand, it is again straightforward to verify that $t'(m) = \alpha(n)$ for every $n \geq 16$, with the single exception that $t'(18) = 7 = \alpha(30)+1$. Now in this exceptional case, we can construct a triangle-free almost 6-regular graph of order 18 by removing the edges of a one-factor in the (7,18)-graph constructed in Lemma 2.1 (see Figure 2 of the Appendix). Hence, we can always construct a triangle-free almost $\alpha(n)$-regular graph G_2 of order m, with special vertex x, such that $V(G_1) \cap V(G_2) = \emptyset$. Now select two adjacent vertices a and b in G_1. We then obtain an $\alpha(n)$-regular graph on $V(G_1) \cup V(G_2)$ by deleting the edge $\{a, b\}$ and adding the new edges $\{x, a\}$ and $\{x, b\}$. This $\alpha(n)$-regular graph is triangle-free and has x as a cut-vertex. Hence $c(n) \geq \alpha(n)$ for all $n \geq 16$, as asserted. □

We must now show that $c(n) \leq \alpha(n)$ for all $n \geq 16$. Thus, we assume that G is a triangle-free, k-regular graph of order n containing a cut-vertex x, where $k > \lfloor \frac{2n-4}{9} \rfloor$. Let $G - x = G_1' \cup G_2'$, with $n_1 = |V(G_1')|$, $n_2 = |V(G_2')|$ and $n_1 + n_2 = n - 1$. Now let $G_1 = G - V(G_2')$ and $G_2 = G - V(G_1')$, and let $s = \deg_{G_2}(x)$, whence $k - s = \deg_{G_1}(x)$. Now if n is odd, then k is even, whence s is even (the degree sum over G_2 must be even). On the other hand, if n is even, then without loss of generality, n_2 is even and so regardless of the parity of k it must be that s is even (again consider the degree sum over G_2). Thus s is even, $s \geq 2$. Now each of G_1 and G_2 has all but one vertex of degree k, with the remaining vertex x of degree $0 < \deg(x) < k$. Hence neither of G_1, G_2 can be bipartite. Applying Lemma 2.2 to G_1 and G_2 in turn we get the following:

$$2(n_1 + 1) \geq (\gamma(G_1) - 1)k + k - s \Rightarrow n_1 \geq 2k + \frac{1}{2}(k - s - 2). \quad (3.1)$$

$$2(n_2 + 1) \geq (\gamma(G_2) - 1)k + s \Rightarrow n_2 \geq 2k + \frac{1}{2}(s - 2). \quad (3.2)$$

Now $n_1 + n_2 = n - 1$, so adding Inequalities (3.1) and (3.2) yields

$$n - 1 \geq \frac{9}{2}k - 2 \Rightarrow k \leq \frac{2n+2}{9}.$$

Hence, $k = \lfloor \frac{2n-4}{9} \rfloor + 1$. Consider now the following cases.

(I) G_1' and G_2' bipartite.

Now G_1' contains $k - s$ vertices of degree $k - 1$ and $n_1 - (k - s)$ vertices of degree k. Since G is triangle-free, no two vertices of degree $k - 1$ are adjacent and so we deduce that $n_1 \geq (k - s) + 2(k - 1)$. Similarly, we have $n_2 \geq s + 2(k - 1)$. Furthermore, both n_1 and n_2 must be even. Therefore, $n = n_1 + n_2 + 1 \geq 5k - 3$, i.e. $k \leq \frac{1}{5}(n + 3)$. Now $\frac{1}{5}(n+3) > \frac{2n-4}{9} \Leftrightarrow n < 47$. Specifically, those odd n with $17 \leq n \leq 45$ and $\lfloor \frac{1}{5}(n + 3) \rfloor = \lfloor \frac{2n-4}{9} \rfloor + 1 \equiv 0 \mod 2$ are $n = 17, 19, 27$ and 37.

(II) One of G_1', G_2' is not bipartite; say G_1' not bipartite.

By Lemma 2.2, we have

$$\gamma(G_1')(k-1) \leq 2n_1 = 2(n - n_2 - 1).$$

From Inequality (3.2) we now deduce that

$$\gamma(G_1')(k-1) \leq 2(n - 2k - \frac{1}{2}(s-2) - 1) \leq 2(n - 2k - 1),$$

whence

$$k \leq (2n - 2 + \gamma(G_1'))/(\gamma(G_1') + 4).$$

But $k = \lfloor \frac{2n-4}{9} \rfloor + 1$ and $n \geq 16$, which forces $\gamma(G_1') = 5$. Applying Lemma 2.2 to G_1, noting that $\gamma(G_1) = \gamma(G_1') = 5$ and that G_1 contains a 5-cycle that does not contain x, we have

$$5k \leq 2(n_1 + 1) = 2(n - n_2) \leq 2(n - 2k - \frac{1}{2}(s-2)).$$

Hence $\frac{1}{2}(s-2) = 0$ or 1 (else $k \leq \frac{2n-4}{9}$), i.e. $s = 2$ or 4. Thus either $s = 2$ and $n_2 = 2k$ or $2k+1$, or $s = 4$ and $n_2 = 2k+1$. This last case cannot occur, as follows. If $s = 4$ then $n_2 = 2k+1$ is odd, and so G_2' is not bipartite. (To see this, note that G_2' has $s = 4$ vertices of degree $k-1$ and $n_2 - 4$ vertices of degree k. Now $s = 4$, $s < k$ and n_2 odd implies that $k \geq 6$, whence $n_2 \geq 13$. It is therefore impossible to partition the vertex set of G_2' into two independent sets X and Y so that the degree sums over each of X and Y are equal.) Moreover, $s = 4$ and $n_2 = 2k+1$ implies that Inequality (3.2) is in fact an equality, which means that $\gamma(G_2) = 5$ and every 5-cycle in G_2 contains the vertex x. But G_2' not bipartite implies (by Lemma 2.2) that

$$\gamma(G_2')(k-1) \leq 2n_2 = 4k + 2,$$

whence

$$(\gamma(G_2') - 4)k \leq \gamma(G_2') + 2,$$

which, since $k > s = 4$, yields $\gamma(G_2') = 5$. This means that G_2 contains a 5-cycle that does not contain the vertex x, a contradiction. Hence $s = 4$ cannot occur and therefore we must have $s = 2$ and $n_2 = 2k$ or $2k+1$; this in turn forces G_1 to be an almost k-regular graph of order $n_1 + 1 = n - 2k$ or $n - 2k - 1$.

Now note that if $k \leq (2(n_1+1) - 4)/5$ then $k \leq (2(n-2k) - 4)/5$, whereupon $k \leq (2n-4)/9$. Therefore $k > (2(n_1+1) - 4)/5$ ($k > (2(n_1 + 1) - 2)/5$ if $n_1 + 1 = n - 2k - 1$), whence $n_1 + 1 \in S \cup \{9\}$. If $n_1 + 1 = 9$ then $k = 4$, whereupon $n = (n_1 + 1) + 2k = 17$ or $n = (n_1 + 1) + 2k + 1 = 18$. Now if $n = 18$ then G_2' must be a triangle-free graph on 9 vertices with

7 vertices of degree 4 and 2 non-adjacent vertices of degree 3; moreover, G_2' has odd girth 5 (Lemma 2.2) and every 5-cycle in G_2' contains the 2 non-adjacent vertices of degree 3. We leave it as an exercise for the reader to verify that no such graph exists, whence $n \neq 18$.

Otherwise, $n_1 + 1 \in S$ and $k = \lfloor \frac{2(n_1+1)-4}{5} \rfloor + 1$. Since $n \geq 16$ and $k = \lfloor \frac{2n-4}{9} \rfloor + 1 \geq 4$, $n_1 + 1 \neq 8$. We summarize the various possibilities in the following table.

$n_1 + 1 (\in S \backslash \{8\})$	k	$n = (n_1 + 1) + 2k$	or	$n = (n_1 + 1) + 2k + 1$
11	4	19		20
14	5	24		25
15	6	27		28
18	7	32		33
21	8	37		38
24	9	42		43

We'll now show that in fact none of the entries in the last column $n = (n_1 + 1) + 2k + 1$ can occur. First of all, for $n_1 + 1 = 14, 18$ or 24, we have k taking an odd value, whence n must be even in order that a k-regular graph on n vertices exists. Now suppose that $n_1 + 1 = 11$ or 21 and $n = (n_1 + 1) + 2k + 1$. Then $n_1 + 1 = n - 2k - 1$ and so by the analysis in the preceding paragraph we must have $k > (2(n_1 + 1) - 2)/5$. Thus, for $n_1 + 1 = 11$ (resp. 21) we would require $k > 4$ (resp 8), contradicting $k = 4$ (resp. 8). Finally, for $n_1 + 1 = 15$ and $n = 28$, it must be that G_2' is a triangle-free graph on 13 vertices with 11 vertices of degree 6 and 2 non-adjacent vertices of degree 5; it can be concluded from Lemma 2.2 that no such graph exists.

Collecting the results of the foregoing discussion now gives us the following.

Lemma 3.2. *$c(n) \leq \alpha(n)$ for all $n \geq 16$.*

Combining Lemmas 3.1 and 3.2 now yields our main result.

Theorem 3.3. *Let $n \geq 16$ and let*

$$\alpha(n) = \begin{cases} \lfloor \frac{2n-4}{9} \rfloor + 1 & \text{if } n \in \{24, 32, 42\}, \\ \lfloor \frac{2n-4}{9} \rfloor_e + 2 & \text{if } n \in \{17, 19, 27, 37\}, \\ \lfloor \frac{2n-4}{9} \rfloor & \text{for all other even } n \geq 16, \\ \lfloor \frac{2n-4}{9} \rfloor_e & \text{for all other odd } n \geq 21. \end{cases}$$

Then $c(n) = \alpha(n)$.

Remark 3.4: *For the sake of completeness, we note that $c(n)$ does not exist for $n = 1, 2, \ldots, 13$ or for $n = 15$, and that $c(14) = 3$ (see Figure 1 in the Appendix).*

4 Conclusion

As we stated in the introduction, a regular graph containing a cut vertex is not one-factorizable, and so $c(2m)$ is a lower bound on $f(2m) = \max\{k$: there exists a triangle-free k regular graph on $2m$ vertices which is not one-factorizable$\}$. Hence for every $m \geq 8$ we have $f(2m) \geq \alpha(2m) = \lfloor \frac{4}{9}(m-1) \rfloor + 1$ if $m = 12, 16$ or 21, or $\lfloor \frac{4}{9}(m-1) \rfloor$ otherwise. Now using Petersen's results that (i) every $2t$-regular graph contains a 2-factor, and (ii) every bridgeless cubic graph contains a 1-factor, it can be shown that $f(2m)$ does not exist for $m \leq 4$ and that $f(10) = 3$ (attained by the Petersen graph), $f(12) = 2$ and $f(14) = 3$. Moreover, there are exactly two non-isomorphic triangle-free cubic graphs of order 14 with no one-factorization (see Figure 1 in the Appendix).

Finally, it is an immediate consequence of Theorem 3.3 and Remark 3.4 that if G is a connected triangle-free k-regular graph on n vertices with $k > 3$ when $n = 14$, or $k > \alpha(n)$ when $n \geq 16$, then G is 2-connected.

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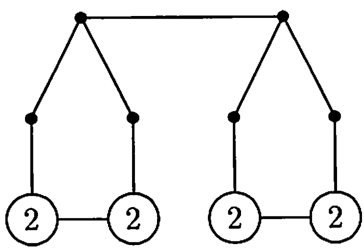
Appendix

We use the following notations. A solid circle (i.e. a dot) denotes a single vertex, while a hollow circle with the number t inside denotes an independent set of t vertices. A solid line between two circles indicates the presence of all possible edges between the corresponding sets of vertices; a dotted line indicates the presence of all possible edges except those of a one-factor between the corresponding sets of vertices, while two dotted lines indicate the presence of all edges except those of two disjoint one-factors.



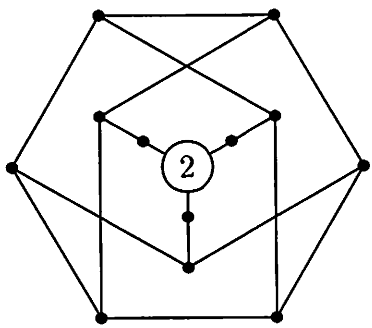


Figure 1

Note the first graph has a cut-vertex, while the second one does not.

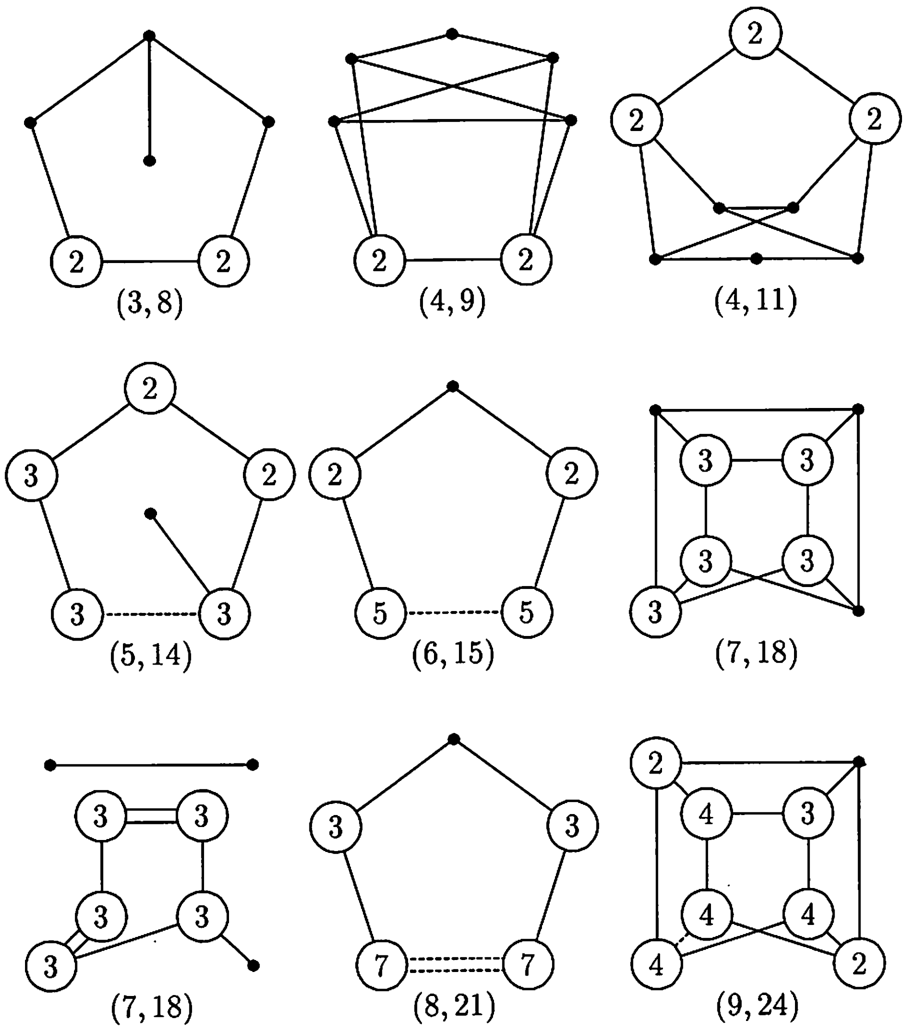


Figure 2

In the second (7, 18) figure, each line represents an edge; the 9 edges form a one-factor in the (7, 18)-graph in the first figure (see the proof of Lemma 3.1).

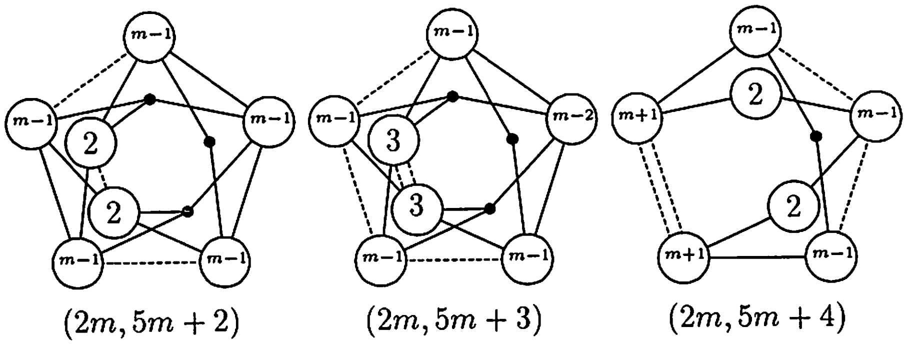


Figure 3

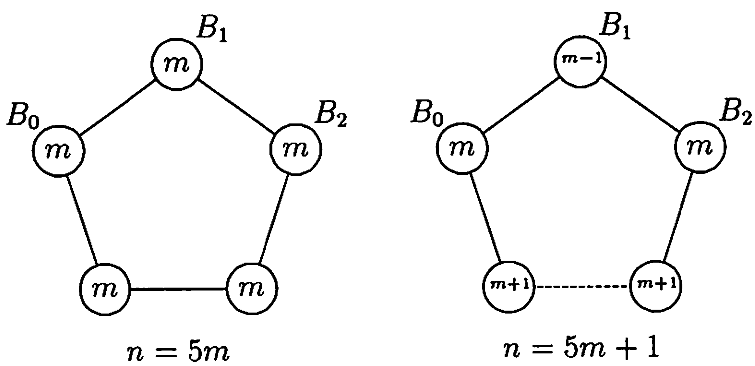


Figure 4