Maximum Toughness Among (n, m)-Graphs

Kevin Ferland
kferland@bloomu.edu
Bloomsburg University, Bloomsburg, PA 17815

Abstract

The maximum possible toughness among graphs with n vertices and m edges is considered. This is an analog of the corresponding problem regarding maximum connectivity solved by Harary. We show that, if $m < \lceil \frac{3n}{2} \rceil$ or $m \geq n(\lfloor \frac{n}{6} \rfloor + \lfloor \frac{n \mod 6}{3} \rfloor)$, then the maximum toughness is half of the maximum conectivity. The same conclusion is obtained if $r = \lfloor \frac{2m}{n} \rfloor \geq 1$ and $\frac{(n-1)(r+1)}{2} \leq m < \frac{n(r+1)}{2}$. However, maximum toughness can be strictly less than half of maximum connectivity. Some values of maximum toughness are computed for $1 \leq n \leq 12$, and some open problems are presented.

1 Terminology

We consider only simple graphs $G = (V, E)$, where V is the set of vertices and $E \subseteq \mathcal{P}_2(V)$ is the set of edges. A graph G is said to be an (n, m)-graph if $n = |V|$ and $m = |E|$. In fact, throughout this paper, the variables n and m are reserved for $|V|$ and $|E|$, respectively. A $K_{1,3}$ subgraph is an induced subgraph that is isomorphic to the complete bipartite graph $K_{1,3}$. The degree 3 vertex in such a subgraph is called a $K_{1,3}$ center. When no $K_{1,3}$ subgraph exists, G is said to be $K_{1,3}$-free. The number of components of the graph obtained from G by removing the vertices of $S \subset V$ is denoted by $\omega(G - S)$. The connectivity of a noncomplete graph G is defined by Whitney [11] as

$$\kappa(G) = \min\{|S| \ : S \subseteq V \text{ and } \omega(G - S) > 1\}.$$

The connectivity of a complete graph K_n is defined to be $n - 1$. A graph G is said to be k-connected if $\kappa(G) \geq k$. The toughness of a non-complete

JCMCC 43 (2002), pp. 43-55

graph G is defined by Chvátal [2] as

$$\tau(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subseteq V \text{ and } \omega(G-S) > 1\}.$$

For a complete graph K_n, we adopt the approach of Pippert [9] and define $\tau(K_n) = \frac{n-1}{2}$. (Chvátal defined $\tau(K_n) = \infty$.) A graph G is said to be t-tough if $\tau(G) \geq t$. For any other standard terminology, the reader is refered to [1] or [7].

For $n \geq 1$ and $0 \leq m \leq \frac{n(n-1)}{2}$, define the maximum connectivity

$$C_n(m) = \max\{\kappa(G) : G \text{ is an } (n,m)\text{-graph}\} \tag{1.1}$$

and the maximum toughness

$$T_n(m) = \max\{\tau(G) : G \text{ is an } (n,m)\text{-graph}\}. \tag{1.2}$$

An (n,m)-graph G is said to be maximally connected if $\kappa(G) = C_n(m)$ and maximally tough if $\tau(G) = T_n(m)$. As we will see, maximally connected graphs need not be maximally tough. For a fixed $n \geq 1$, the behavior of the functions $C_n(m)$ and $T_n(m)$ is best understood by grouping the possible values for m into intervals (see Theorem 3.1 and Section 7). For $0 \leq r \leq n-1$, define the r^{th} interval $I_n(r)$ by $I_n(0) = [0, n-1) \cap \mathbb{Z}$, $I_n(1) = \{n-1\}$, $I_n(n-1) = \{\frac{n(n-1)}{2}\}$, and, for $1 < r < n-1$, $I_n(r) = [\lceil \frac{nr}{2} \rceil, \lceil \frac{n(r+1)}{2} \rceil) \cap \mathbb{Z}$. Note that

$$\left\lceil \frac{nr}{2} \right\rceil \leq m < \left\lceil \frac{n(r+1)}{2} \right\rceil \quad \text{if and only if} \quad r = \left\lfloor \frac{2m}{n} \right\rfloor.$$

Special attention is given to the initial value of each interval. For $r > 1$, an (n,m)-graph is said to be r-sesqui-regular if $n-1$ of the vertices have degree r and the remaining vertex has degree r or $r+1$. That is, the graph is as close to regular as possible given the parity of nr.

2 Introduction

The notion of maximum toughness has been considered by Chvátal [2] and Doty [3]. Maximum connectivity was introduced and computed by Harary [6]. The main reason for discussing connectivity here in addition to toughness comes from the upper bound for toughness given by Chvátal.

Theorem 2.1 ([2]). $\tau(G) \leq \frac{\kappa(G)}{2}$.

Some immediate consequences of Theorem 2.1 and the definitions are listed in the following Proposition for easy reference.

Proposition 2.2. *Denote* $r = \lfloor \frac{2m}{n} \rfloor$.

(a) *For* $1 \le m \le \frac{n(n-1)}{2}$, $T_n(m-1) \le T_n(m)$.

(b) $T_n(m) \le \frac{C_n(m)}{2}$.

(c) *If* $T_n(m) = \frac{r}{2}$ *for some* $m \in I_n(r)$, *then, for all* $m \le k \in I_n(r)$, $T_n(k) = \frac{r}{2}$.

Remark 2.3. *Proposition 2.2(c) is of particular interest in the cases in which* $T_n(\lceil \frac{nr}{2} \rceil) = \frac{r}{2}$ *for some* $r \ge 2$.

For any number of vertices $n \ge 1$, if the number of edges m is very small, then $T_n(m)$ is easy to compute. It is well known that graphs with fewer than $n-1$ edges are disconnected. Further, a graph with $n-1$ edges is connected if and only if it is a tree. Since the toughness of a tree is the reciprocal of its maximum degree [9], the toughest tree is P_n, the path on n vertices, and $\tau(P_n) = \frac{1}{2}$. It is also easy to see that the toughest graph with n edges is the n-cycle C_n, and $\tau(C_n) = 1$ [2]. These observations are summarized in the following theorem.

Theorem 2.4. *Let* $n \ge 1$.

(a) *If* $m \in I_n(0) = [0, n-1) \cap \mathbb{Z}$, *then* $T_n(m) = 0$.

(b) $T_n(n-1) = \frac{1}{2}$. *Recall,* $I_n(1) = \{n-1\}$.

(c) *If* $m \in I_n(2) = [n, \lceil \frac{3n}{2} \rceil) \cap \mathbb{Z}$, *then* $T_n(m) = 1$.

A connection between τ and κ that is stronger than Theorem 2.1 is established by Matthews and Sumner.

Theorem 2.5 ([8]). *If* G *is* $K_{1,3}$*-free, then* $\tau(G) = \frac{\kappa(G)}{2}$.

Theorem 2.5 can be used to show that many of the maximally connected graphs are also maximally tough. As we will see, equality is actually achieved in Proposition 2.2(b) most of the time, but not always. In fact, it is the failures to achieve this upper bound that make the computation of $T_n(m)$ interesting and non-trivial.

3 Maximally Tough Harary Graphs

The relevance of the intervals $I_n(r)$ is made clear by Harary's computation of $C_n(m)$.

Theorem 3.1 **([6]).** *For* $n \geq 1$ *and* $0 \leq m \leq \frac{n(n-1)}{2}$,

$$C_n(m) = \begin{cases} 0 & \text{if } m \leq n-2, \\ \lfloor \frac{2m}{n} \rfloor & \text{if } m \geq n-1. \end{cases}$$

That is, $C_n(m) = r$ *if and only if* $m \in I_n(r)$.

In fact, the very graphs constructed by Harary to prove Theorem 3.1 can also be used to determine some of the values of $T_n(m)$.

3.1 Harary graphs [6]

For $n \geq 3$ and $n \leq m \leq \frac{n(n-1)}{2}$, let $H(n,m)$ denote the Harary graph with n vertices and m edges. Throughout our description, r denotes the value $\lfloor \frac{2m}{n} \rfloor$. Let the vertex set of $H(n,m)$ be given by $V = \{0, 1, \ldots, n-1\}$. The edge set depends on the relationship between the values of n and m and must first be described for the special cases in which $m = \lceil \frac{nr}{2} \rceil$ with $2 \leq r \leq n-1$. That is, we first describe the sesqui-regular Harary graphs.

Case: $m = \frac{nr}{2}$ **for some** $2 \leq r \leq n-1$.

If r is even, then

$$E = \{ \{i,j\} \ : |i-j| \equiv k \mod n \quad \text{for some } 1 \leq k \leq \frac{r}{2}\}.$$

If r is odd and hence n is even, then

$$\begin{aligned} E \quad = \quad & \{ \{i,j\} \ : |i-j| \equiv k \mod n \quad \text{for some } 1 \leq k \leq \frac{r-1}{2}\} \cup \\ & \{ \{i,j\} \ : |i-j| \equiv \frac{n}{2} \mod n\}. \end{aligned}$$

Case: $m = \frac{nr+1}{2}$**, for some** $3 \leq r \leq n-2$.

In this case,

$$\begin{aligned} E \quad = \quad & \{ \{i,j\} \ : |i-j| \equiv k \mod n \quad \text{for some } 1 \leq k \leq \frac{r-1}{2}\} \cup \\ & \{ \{i, i+\frac{n+1}{2}\} \ : 0 \leq i \leq \frac{n-3}{2}\} \cup \{ \{\frac{n-1}{2}, 0\} \}. \end{aligned}$$

General Case.

Harary only uniquely determines $H(n,m)$ for the cases in which $m = \lceil \frac{nr}{2} \rceil$ for some $2 \leq r \leq n-1$. In general, for $m \in I_n(r)$, a Harary graph $H(n,m)$ is obtained by arbitrarily adding $m - \lceil \frac{nr}{2} \rceil$ edges to $H(n, \lceil \frac{nr}{2} \rceil)$. In fact, since it is easy to see that $C_n(m) \leq \lfloor \frac{2m}{n} \rfloor$, Theorem 3.1 is a consequence of the following result.

Theorem 3.2 **([6]).** *For* $r \geq 2$, $\kappa(H(n, \lceil \frac{nr}{2} \rceil)) = r$.

3.2 Conditions implying that $\tau(H(n, \lceil \frac{nr}{2} \rceil)) = \frac{r}{2}$

It turns out that many of the Harary graphs, although only constructed to be maximally connected, are maximally tough as well. Chvátal noted that the following result is easy to see.

Theorem 3.3 **([2]).** *If r is even, then $\tau(C_n^{\frac{r}{2}}) = \frac{r}{2}$. Moreover, if $r = \lfloor \frac{2m}{n} \rfloor$ is even, then $T_n(m) = \frac{r}{2}$. That is, if r even and $m \in I_n(r)$, then $T_n(m) = \frac{r}{2}$.*

Perhaps the easiest way to see Theorem 3.3 is to use Theorems 3.1 and 2.5 together with the following Lemma. Its hypothesis is clearly satisfied by the Harary graph $H(n, \frac{nr}{2}) = C_n^{\frac{r}{2}}$ when r is even.

Lemma 3.4. *If, for every vertex v of G, the subgraph $N(v)$ induced by the neighbors of v is spanned by two complete graphs, then G is $K_{1,3}$-free.*

Proof. For any three distinct vertices in $N(v)$, at least two must be in the same complete subgraph and hence adjacent. □

Theorem 3.3 bounds the behavior of $T_n(m)$ over the intervals $I_n(r)$ and implies a result of particular interest when r is odd.

Corollary 3.5. *For any r, if $m \in I_n(r)$, then $\frac{r-1}{2} \leq T_n(m) \leq \frac{r}{2}$.*

The fact that Harary graphs of even regularity are $K_{1,3}$-free and hence maximally tough is also observed in [5]. Moreover, it is observed that those of odd regularity are not always $K_{1,3}$-free. Here, we explore the odd case in detail. For large odd values of $r = \lfloor \frac{2m}{n} \rfloor$, the Harary graph $H(n, m)$ can also be seen to be maximally tough. In the following theorem, note that

$$\left\lfloor \frac{n \bmod 6}{3} \right\rfloor = \begin{cases} 0 & \text{if } n \equiv 0, 1, \text{or } 2 \mod 6, \\ 1 & \text{if } n \equiv 3, 4, \text{or } 5 \mod 6. \end{cases}$$

Theorem 3.6. *If $m \geq n\left(2\lfloor \frac{n}{6} \rfloor + \lfloor \frac{n \bmod 6}{3} \rfloor\right)$, then $T_n(m) = \frac{C_n(m)}{2}$.*

Proof. Let $r = \lfloor \frac{2m}{n} \rfloor$. The hypothesis of the theorem is equivalent to the condition that

$$r \geq 2\left(2\left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n \bmod 6}{3} \right\rfloor\right). \tag{3.1}$$

Since Theorem 3.3 handles the cases in which r is even, it suffices to assume that r is odd and hence $r \geq 2(2\lfloor \frac{n}{6} \rfloor + \lfloor \frac{n \bmod 3}{6} \rfloor) + 1$. Since the Harary graph $H(n, \lceil \frac{nr}{2} \rceil)$ is r-connected by Theorem 3.2, our desired conclusions will follow from Theorem 2.5 by showing that $H(n, \lceil \frac{nr}{2} \rceil)$ is $K_{1,3}$-free.

To show that no vertex v of $H(n, \lceil \frac{nr}{2} \rceil)$ is a $K_{1,3}$ center, it suffices to consider only the cases in which $v \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Moreover, when n is even, $H(n, \lceil \frac{nr}{2} \rceil)$ is vertex transitive and only $v = 0$ needs to be considered. When n is odd, all of the different cases for v will be essentially the same, except for the case in which $v = 0$, since $v = 0$ is the unique vertex of degree $r+1$. Consequently, to simplify notation, we consider only the case in which $v = 0$, and we remark on any necessary extra considerations arising when n is odd. In fact, our proof is most easily followed by first understanding the case in which n is even and then considering the extra fussiness required when n is odd.

The neighbors of $v = 0$ are naturally broken into three sets

$$V_1 = \{1, \ldots, \frac{r-1}{2}\}, V_2 = \{n - \frac{r-1}{2}, \ldots, n-1\}, \text{ and } V_3 = \{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil\}.$$

Of course, when n is even, V_3 has only one element. This is also the case when n is odd for the appropriate version of V_3 when $v \neq 0$. Clearly each of the sets V_1, V_2, and V_3 induce a complete subgraph of $H(n, \lceil \frac{nr}{2} \rceil)$. Consequently, a trivial graph on 3 vertices among the neighbors of v would have to consist of one vertex from each of V_1, V_2, and V_3. Without loss of generality, we assume that $\lceil \frac{n}{2} \rceil$ is the vertex of V_3. Since, in such a trivial graph, the vertices from V_1 and V_2 cannot be neighbors of $\lceil \frac{n}{2} \rceil$, they must actually come from the subsets

$$V_1' = \{1, \ldots, \left\lceil \frac{n}{2} \right\rceil - \frac{r-1}{2} - 1\} \text{ and } V_2' = \{\left\lceil \frac{n}{2} \right\rceil + \frac{r-1}{2} + 1, \ldots, n-1\}.$$

The situation imposing these restrictions is pictured in Figure 1.

Let i and j be vertices of $V_1' \cup V_2'$, and let $k = |i - j| \mod n$. Clearly,

$$k \leq (n - (\left\lceil \frac{n}{2} \right\rceil + \frac{r-1}{2} + 1)) + (\left\lceil \frac{n}{2} \right\rceil - \frac{r-1}{2} - 1) + 1 = n - r - 1.$$

Our proof is completed by showing that $k \leq \frac{r-1}{2}$ and hence $\{i, j\}$ is an edge of $H(n, \lceil \frac{nr}{2} \rceil)$.

For each of the congruence classes of $n \mod 6$, it is easy to check that $3\lfloor \frac{n \bmod 6}{3} \rfloor + 2 \geq n - 6\lfloor \frac{n}{6} \rfloor$. Therefore, $12\lfloor \frac{n}{6} \rfloor + 6\lfloor \frac{n \bmod 6}{3} \rfloor + 3 \geq 2n - 1$. It follows that $r \geq 4\lfloor \frac{n}{6} \rfloor + 2\lfloor \frac{n \bmod 6}{3} \rfloor + 1 \geq \frac{2n-1}{3}$. Hence, $2n \leq 3r + 1$. Therefore, $k \leq n - r - 1 \leq \frac{r-1}{2}$ as desired. □

Theorem 3.6 tells us that, for large values of n, Harary graphs with roughly $\frac{2}{3}$ or more of the $\frac{n(n-1)}{2}$ possible edges are maximally tough.

4 Maximally Tough Sub-Harary Graphs

The bounds given in Corollary 3.5 raise two natural questions which are central to our work in the cases that r is odd and inequality (3.1) is not

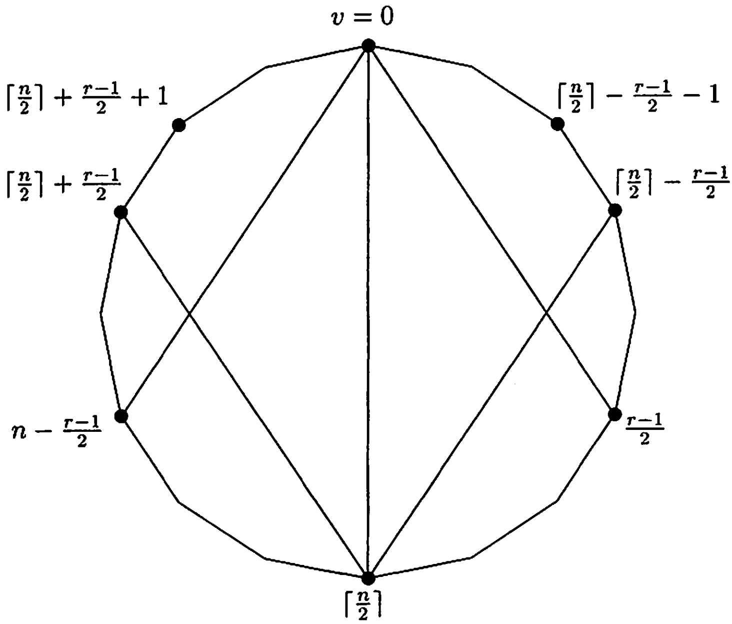

Figure 1: A portion of $H(n, \lceil\frac{nr}{2}\rceil)$

satisfied. Is $T_n(\lceil\frac{nr}{2}\rceil) > \frac{r-1}{2}$? Is there some $m \in I_n(r)$ such that $T_n(m) = \frac{r}{2}$? In Section 8, the answer to the first question is conjectured to be yes. The second question can be answered in the affirmative by considering sub-Harary graphs.

4.1 Sub-Harary Graphs

Define a sub-Harary graph to be a sequi-regular Harary graph with some of its edges removed. Specifically, we construct $H'(n,m)$ by removing a particular set F of c edges from the Harary graph $H(n,q)$, where $q = m+c$. However, we only consider the cases in which $q = \frac{ns}{2}$, for an even integer $s \geq 4$, and $0 < c \leq \frac{s}{2}$. In these cases, the edge set E for $H(n,q)$ is given by

$$E = \{ \{i,j\} : |i-j| \equiv k \mod n, \quad \text{for some } 1 \leq k \leq \frac{s}{2}\}.$$

We choose

$$F = \{ \{i, i+\frac{s}{2}\} : 0 \leq i \leq c-1\}.$$

Note that $H(n,q)$ is s-regular, and $H'(n,m)$ has minimum degree $r = \lfloor\frac{2m}{n}\rfloor = s-1$. For example, when $n = 8$, $m = 14$, and $s = r+1 = 4$, the graph $H'(8,14)$ is pictured in Figure 2.

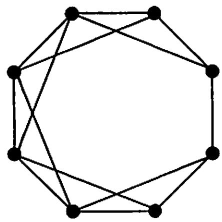

Figure 2: The sub-Harary graph $H'(8,14)$

4.2 Conditions implying that $\tau(H'(n,m)) = \frac{r}{2}$

For odd values of $r = \lfloor \frac{2m}{n} \rfloor \geq 3$, sub-Harary graphs provide examples of maximally tough graphs. Basically, we can remove up to $\frac{r+1}{2}$ well-chosen edges from the Harary graph $H(n, \frac{n(r+1)}{2})$ and still have a maximally tough graph.

Theorem 4.1. *If r is odd and $\frac{(n-1)(r+1)}{2} \leq m < \frac{n(r+1)}{2}$, then $T_n(m) = \tau(H'(n,m)) = \frac{r}{2}$.*

Proof. Let $s = r+1$, $q = \frac{ns}{2}$, and $c = q - m$. Note that $0 < c \leq \frac{s}{2}$ and $H'(n,m)$ is $H(n,q)$ with the set F of c edges removed. As we shall see later in the proof, it suffices to consider the case in which $c = \frac{s}{2}$. By Lemma 3.4, it is easy to see that $H'(n,m)$ is $K_{1,3}$-free. By Theorem 2.5, our desired conclusions follow by showing that $\kappa(H'(n,m)) = r = s-1$.

Since $H'(n,m)$ has minimum degree r, it is clear that $\kappa(H'(n,m)) \leq r$. Let R be a set of r or fewer vertices whose removal disconnects $H'(n,m)$. By Theorem 3.2, $\kappa(H(n,q)) = s$, and hence R does not disconnect $H(n,q)$. Consequently, for some $0 \leq i \leq \frac{s}{2} - 1$, there must be two vertices i and $i + \frac{s}{2}$ that are in different components of $H'(n,m) - R$. For each $1 \leq j \leq \frac{s}{2} - 1$, there is a path $\{i, i+j, i+\frac{s}{2}\}$ in $H'(n,m)$ of length 2 from i to $i + \frac{s}{2}$. Since all of these paths are internally disjoint, R must contain the $\frac{s}{2} - 1$ vertices $\{i+1, \ldots, i+\frac{s}{2}-1\}$. Note that, besides the edge $\{i, i+\frac{s}{2}\}$, it is now irrelevant that the other $\frac{s}{2} - 1$ edges have been removed from $H(n,q)$ to form $H'(n,m)$.

Since $H(n,q)$ is s-connected, there are at least s internally disjoint paths in $H(n,q)$ from i to $i + \frac{s}{2}$. Moreover, since $H(n,q)$ is s-regular, there must be exactly s such paths. Besides the single edge path $\{i, i+\frac{s}{2}\}$ and the $\frac{s}{2} - 1$ paths of length 2 mentioned above, there must be $\frac{s}{2}$ more such paths in $H(n,q)$ and consequently in $H'(n,m)$. Since R disconnects $H'(n,m)$, it must contain at least $\frac{s}{2}$ vertices besides $\{i+1, \ldots, i+\frac{s}{2}-1\}$. Therefore, $|R| \geq (\frac{s}{2} - 1) + \frac{s}{2} = s - 1 = r$, and $\kappa(H'(n,m)) = r$ as claimed. □

5 Doty's Graphs [3]

The task of finding r-regular $\frac{r}{2}$-tough graphs when r is odd was first considered by Chvátal [2], and his results were greatly generalized by Doty [3]. Throughout this section, let r be an odd integer such that $r \geq 3$, and let a be an even integer such that $2 \leq a \leq r-1$. Chvátal showed that $K_a \times K_{r+2-a}$ is an r-regular $\frac{r}{2}$-tough graph.

Theorem 5.1 **([2]).** *Suppose $r \geq 3$ is odd and a is even with $2 \leq a \leq r-1$. Let $n = a(r+2-a)$. Then, for all $m \in I_n(r)$, $T_n(m) = \frac{r}{2}$.*

In [3], Doty defines a class of graphs which provide a generalization of Theorem 5.1. For each $k \geq 1$, Doty constructs an r-regular $\frac{r}{2}$-tough graph on $n = ka(r+2-a)$ vertices [3], which we denote by $D_k(r,a)$. In particular, $D_1(r,a) = K_a \times K_{r+2-a}$.

Theorem 5.2 **([3]).** *Suppose $r \geq 3$ is odd, a is even with $2 \leq a \leq r-1$, and $k \geq 1$. Let $n = ka(r+2-a)$. Then, for all $m \in I_n(r)$, $T_n(m) = \frac{r}{2}$.*

Corollary 5.3. *Let $r \geq 3$ be odd. If either $n = r+1$ or $n \equiv 0 \mod 2r$, then $T_n(\frac{rn}{2}) = \frac{r}{2}$.*

Corollary 5.3 inspires Question 8.2 in Section 8.

Remark 5.4. *In addition to the graphs $D_k(r,a)$, Doty [3] also gives an example of a $\frac{5}{2}$-tough $(11,29)$-graph with 8 vertices of degree 5 and 3 vertices of degree 6.*

6 Computing $T_n(m)$ for small n

The results of the previous sections can be used to determine all of the values of $T_n(m)$ for $n \leq 6$ and many of the values for $7 \leq n \leq 12$. In this section, some of the values not completely determined in previous sections are pinned down. Tables listing these values are presented in Section 7.

Theorem 6.1. $T_7(11) = \frac{4}{3}$.

Proof. All graphs on 7 vertices are listed in [10]. Hence, $T_7(11)$ can be computed directly from the definitions. □

Remark 6.2. *A much better means of computing $T_7(11)$ is given in [4].*

Remark 6.3. *Theorem 6.1 gives the smallest value of n (and its only corresponding value of m) for which $T_n(m) < \frac{C_n(m)}{2}$.*

Theorem 6.4. $T_9(23) = \frac{5}{2}$.

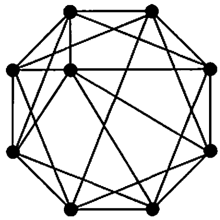

Figure 3: A maximally tough $(9,23)$-graph

Proof. By Corollary 3.5, $T_9(23) \leq \frac{5}{2}$. As verified by computer, the $(9,23)$-graph in Figure 3 is $\frac{5}{2}$-tough. □

Theorem 6.5. $T_{11}(28) \in \{\frac{7}{3}, \frac{5}{2}\}$.

Proof. By Corollary 3.5, $T_{11}(28) \leq \frac{5}{2}$. As verified by computer, the $(11,28)$-graph in Figure 4 is $\frac{7}{3}$-tough. There are no possible fractions $\frac{|S|}{\omega(G-S)}$ strictly

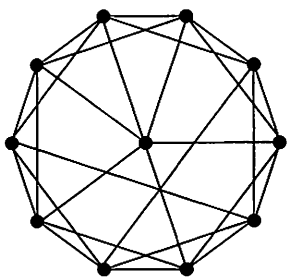

Figure 4: A $\frac{7}{3}$-tough $(11,28)$-graph

between $\frac{7}{3}$ and $\frac{5}{2}$ when there are only 11 vertices. □

Theorem 6.6. $T_{12}(42) = \frac{7}{2}$.

Proof. By Corollary 3.5, $T_{12}(42) \leq \frac{7}{2}$. As verified by computer, the $(12,42)$-graph in Figure 5 is $\frac{5}{2}$-tough. □

7 Values of $T_n(m)$ for small n

Table 1 lists the values of $T_n(m)$ for all $3 \leq n \leq 6$. Next to each value we also give a Theorem which justifies that value. Note however, that Proposition 2.2 is never cited, though often it is used implicitly. For example, the value $T_{10}(6) = \frac{3}{2}$ follows from Proposition 2.2(c) together with the value $T_9(6) = \frac{3}{2}$ given by Theorem 5.1.

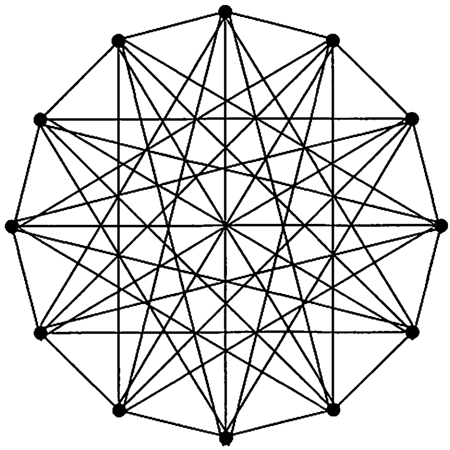

Figure 5: A maximally tough (12, 42)-graph

m	$n = 3$	Thm	$n = 4$	Thm	$n = 5$	Thm	$n = 6$	Thm
0	0	2.4	0	2.4	0	2.4	0	2.4
1	0	2.4	0	2.4	0	2.4	0	2.4
2	$\frac{1}{2}$	2.4	0	2.4	0	2.4	0	2.4
3	1	2.4	$\frac{1}{2}$	2.4	0	2.4	0	2.4
4			1	2.4	$\frac{1}{2}$	2.4	0	2.4
5			1	2.4	1	2.4	$\frac{1}{2}$	2.4
6			$\frac{3}{2}$	3.6	1	2.4	1	2.4
7					1	2.4	1	2.4
8					$\frac{3}{2}$	3.6	1	2.4
9					$\frac{3}{2}$	3.6	$\frac{3}{2}$	5.1
10					2	3.6	$\frac{3}{2}$	5.1
11							$\frac{3}{2}$	5.1
12							2	3.6
13							2	3.6
14							2	3.6
15							$\frac{5}{2}$	3.6

Table 1: Maximum Toughness Values for $3 \leq n \leq 6$

Since Theorems 2.4 and 3.6 specify the values of $T_n(m)$ for small and large values of m, respectively, we exclude these values in subsequent tables. We also exclude the values for $m \in I_n(3)$ as those cases are handled in [4]. Consequently, there are no tables to display for $n = 7$ or 8. The values of $T_n(m)$ for $9 \leq n \leq 12$ are listed in Tables 2 and 3.

m	$n=9$	Thm		m	$n=10$	Thm
$18-22$	2	3.3		19	$\frac{3}{2}$	4.1
23	$\frac{5}{2}$	6.4		$20-24$	2	3.3
$24-26$	$\frac{5}{2}$	4.1		$25-26$	$\frac{5}{2}$	5.1
				$27-29$	$\frac{5}{2}$	4.1,5.1

Table 2: Maximum Toughness Values for $9 \leq n \leq 10$

m	$n=11$	Thm		m	$n=12$	Thm
$22-27$	2	3.3		$24-29$	2	3.3
28	$\frac{7}{3}$ or $\frac{5}{2}$	6.5		$30-35$	$\frac{5}{2}$	5.1
$29-32$	$\frac{5}{2}$	5.4		$36-41$	3	3.3
				$42-47$	$\frac{7}{2}$	6.6

Table 3: Maximum Toughness Values for $11 \leq n \leq 12$

8 Conjectures and Questions

In this section, we list some open problems in the determination of maximum toughness.

Conjecture 8.1. *For $n \geq 1$ and $1 \leq r \leq n-1$, $T_n(\lceil \frac{nr}{2} \rceil) > \frac{r-1}{2}$.*

Question 8.2. *For r odd and n even, does the equality $T_n(\frac{rn}{2}) = \frac{r}{2}$ imply that $n \equiv 0 \bmod a(r+2-a)$ for some $2 \leq a \leq r-1$?*

Question 8.3. *Which is the correct value of $T_{11}(28)$ in Theorem 6.5?*

References

[1] G. Chartrand and L. Lesniak, *Graphs and digraphs*, 1986.

[2] V. Chvátal, *Tough graphs and hamiltonian circuits*, Discrete Math. **5** (1973), 215–228.

[3] L. L. Doty, *A large class of maximally tough graphs*, OR Spektrum **13** (1991), 147–151.

[4] K. K. Ferland, *The maximum toughness of sesqui-cubic graphs*, J. Combin. Math. Combin. Comput. **this Volume**.

[5] W. D. Goddard and H. C. Swart, *On some extremal problems in connectivity*, Graph Theory, Combinatorics, and Applications, vol. 1, Wiley, New York, 1991, pp. 535–551.

[6] F. Harary, *The maximum connectivity of a graph*, Proceedings of the National Academy of Science **48** (1962), 1142–1146.

[7] ———, *Graph theory*, Addison-Wesley, Reading, Mass., 1969.

[8] M. M. Matthews and D. P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs*, Journal of Graph Theory **8** (1984), 139–146.

[9] Raymond E. Pippert, *On the toughness of a graph*, Lecture Notes in Math., vol. 303, Springer, Berlin, 1972, pp. 225–233.

[10] R. C. Read and R. J. Wilson, *An atlas of graphs*, Clarendon Press, Oxford, 1998.

[11] H. Whitney, *Congruent graphs and the connectivity of graphs*, American Journal of Math. **54** (1932), 150–168.