Construction of Minimum Time-Relaxed Broadcasting Communication Networks

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ABSTRACT. A broadcast graph on n vertices is a network in which a message can be broadcast in minimum possible ($= \lceil \log_2 n \rceil$) time from any vertex. Broadcast graphs which have the smallest number of edges are called *Minimum Broadcast Graphs*, and are subjects of intensive study. In this paper, we study how the number of edges in minimum broadcast graphs decreases, as we allow additional time over $\lceil \log_2 n \rceil$.

We improve results obtained by Shastri in [15] and prove a conjecture posed by Shastri in [15, 16].

1 Introduction

Efficient broadcasting is a key component in achieving high performance (throughput) from parallel and distributed processing. The motivation for this work was triggered by our interest in performing an optimal query on distributed database on diverse MIMD multiprocessor architectures [1]. There we investigated how to schedule and evaluate a query in a minimal cost.

We define broadcasting from an *originator(s)* (source(s)) to be the process of passing one (many) unit(s) of information from that source to a set of predefined destinations which are connected via a network. This is accomplished by a series of transmissions over the network. The messages are distributed over the network and spread using the communication network, where each vertex transmits a message to its neighbors upon receiving it regardless of the activities in other vertices (beside the vertex that receives the message that has to be idle). Eventually, broadcasting should logically be viewed as a many-to-many communication.

The broadcasting problem usually is described by the following rules:

1. A processor may send a message to an adjacent processor only.
2. Time is discrete. At a given time each processor will do exactly one of the following:
   (a) receive a message,
   (b) send a message to one neighbor,
   (c) be idle.

More formally, we can view the communication network as a finite, connected, undirected graph on n vertices, where the set of vertices are considered as processors and each edge which connects two vertices assumed to be a direct communication link between these vertices. Then, we define broadcasting from a vertex v (*the originator*) as transmitting a message from v to every vertex in $V\backslash\{v\}$ using the above rules. This problem, that was introduced in [14], is a variation of the gossiping problem [11].

For basic graph theoretical definitions one may see [9] or [17].

We define the *broadcast number* of $v \in V(G)$, $G = G(V, E)$, denoted by $b(v)$, as the minimum time required to broadcast one message from v. The *broadcasting time* of G is defined as: $b(G) = \max\{b(v) \mid v \in V(G)\}$. Let $b(n)$ denote the minimal message broadcast time $b(G)$ over all graphs G with n vertices. A graph G is said to be *minimal broadcast graph* if $b(G) = b(n)$.

The problem of broadcasting in a general graph, namely, the problem of determining $b(v)$ for an arbitrary vertex in an arbitrary graph, was proved by Johnson (see [14]) to be NP-complete. On the other hand, in a tree with equal weights it happens to be linear [14]. Recently a generalization of broadcasting in trees was obtained [2], where, positive weights were assigned to the edges or the vertices of the tree. There are known results in a two and three dimensional grid [18], complete graph and hypercubes (see [3] - [5]). Recently, planar graphs were treated as well [12]. A conjecture concerning minimum broadcasting time starting from a given vertex in a d-dimensional grid ($d \geq 3$), was posed in [18]. This conjecture was recently validated in [13].

The notion of *m-Relaxed Broadcast Graphs (m-RBG)*, as appeared in [15], is a generalization of *1-RBG* that appeared in [8], and was motivated by exploring the sparest possible graphs in which broadcasting can be accomplished in slightly more than the optimal time of $\lceil \log_2 n \rceil$.

Denote by $B_m(n)$ the number of edges in the sparsest possible graph on n vertices in which broadcasting can be accomplished in $\lceil \log_2 n \rceil + m$ steps. Such a graph is called *m-Relaxed Minimum Broadcast Graph (m-RMBG).*

In [6] 0-RMBG graphs were introduced and constructed for all values of $n \leq 15$, and the question of constructing such graphs was posed there. Recently, Kwon and Chwa [5] obtained:

Theorem 1.1.

$$B_0(n) \leq (\lceil \log_2 n \rceil - 1)2^{\lceil \log_2 n \rceil - 1} + n$$

In [15], Shastri obtained the following bounds and posed a conjecture.

Lemma 1.1.

$$B_1(n) \leq 2n - \lceil \log_2 n \rceil - 3,$$

for all n.

This result is a slight improvement upon a former result due to Grigni and Peleg [8].

Lemma 1.2.

$$B_2(n) \leq \frac{3}{2}n - \lceil \log_2 n \rceil,$$

for all $n \geq 8$.

Lemma 1.3.

$$B_m(n) \leq n + 2\sqrt{n} - 1,$$

for $m \geq \frac{1}{2}\lceil \log_2 n \rceil$ and for all n.

Conjecture 1.1:

$$B_m(n) = n - 1,$$

for all n and $m \geq \lceil \frac{1}{2}(\log_2 n + 1) \rceil + 1$.

Conjecture 1.1 coincides with conjecture 1 in [16].

In this paper we improve significantly the bounds of Lemmas 1.1- 1.3 and give an affirmative answer to Conjecture 1.1.

2 Time-Relaxed Broadcasting Graphs

In this section we supply significantly better bounds than those of Lemmas 1.1-1.3 and supply a proof to Conjecture 1.1.

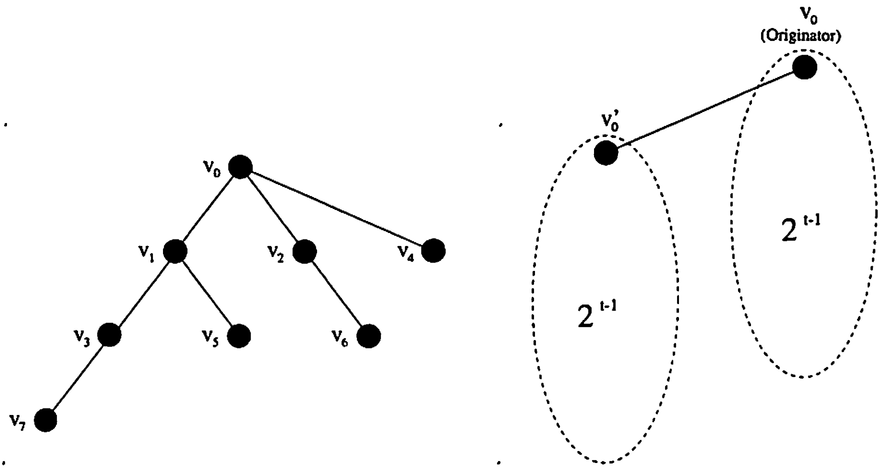
However, before supplying the constructions and proofs we start, in Section 2.1, with some preliminary results which are essential to our constructions. The main results follow in Section 2.2.

2.1 Preliminary Results

One of the main tools in our preceding proofs is the notion of the *Binomial Tree*, which was first introduced in [10].

In this paper we denote $\lceil \log_2 n \rceil$ by t.

Definition 2.1. *The Binomial Tree on $n = 2^t$ vertices, denoted BT_{2^t}, is defined recursively: $BT_1 = (\{v\}, \emptyset)$ and the unique vertex is called* **the root**; *BT_{2^t} is constructed by connecting the roots of two copies of (denoted to the rest of the paper by v_0 and v_0') $BT_{2^{t-1}}$. The root of the new tree is one of the two previous roots (See Figure 1 for example).*



A:2^3-tree B: 2^t-tree

Figure 1: Binomial tree

One can easily observe that $d(v_0) = d(v_0') = \Delta(BT_{2^t}) = t$, where $\Delta(BT_{2^t})$ is the maximum degree of vertices in BT_{2^t} and $d(v)$ is the degree of a vertex v. The vertices v_0 and v_0' are the only vertices in BT_{2^t} having the maximum degree. In addition the diameter of that tree $d(BT_{2^t})$ is $2t - 1$.

As it was observed in [10] $b(v_0) = b(v_0') = t$ and $b(u) \leq t - 1$ for any $u \in V(BT_{2^t})$. Also BT_{2^t} is the maximal .possible tree among all trees while broadcasting from a certain vertex where the total broadcasting time given is t. Furthermore, each non-leaf vertex in BT_{2^t} has exactly one leaf, a fact which yields that their number is exactly 2^{t-1}. Therefore, while broadcasting from v_0 (or v_0') and since in each time unit the number of announced vertices should be doubled, the leaves receive the message at the last time unit (namely, t). Also each non-leaf vertex should continue broadcasting from the moment it received the message until the end of the process.

In view of the above we define for $2^{t-1} < n < 2^t$ the *n-tree*, to be the tree obtained from BT_{2^t}. by deleting $2^t - n$ leaves from BT_{2^t}. Again, the n-tree is an optimal one among all trees on n vertices ($2^{t-1} < n < 2^t$) while broadcasting from a certain vertex. Also we still have $b(v_0) = \lceil \log_2 n \rceil = t$.

Example: In Figure 1A we demonstrate BT_{2^3}, as well as BT_{2^t},where either v_0 is the originator (and either v_0 or v_0' in the general case). Assume v_0 is the originator. Then at $t = 1$ it transmits the message to v_1. Then, both v_0 and v_1 act as originators in their BT_{2^2}-sub-trees, to accomplish broadcasting within at most 2 time steps. The essential properties of BT_{2^t} are presented in the next two lemmas. The proof of the Lemmas is easy and can be done either by induction on t or by labeling each vertex of BT_{2^t} by a t-bit vector identifying how the broadcast message will reach it.

By $d(u, v)$ we denote the shortest path (distance) between the vertices u, v in a connected graph G.

Lemma 2.1. *Let $T = BT_{2^t}$. Define $A \subseteq V(T)$ to be: $A_k = \{u | d(u, v_0) = k\}$. Then,*

$$|A_k| = \binom{t}{k}.$$

Lemma 2.2. *Let $T = BT_{2^t}$. Define $A_{k,d} \subseteq V(T)$, $A_{k,d} = \{u \in A_k | d(u) = d\}$. If $d, k \geq 1$, then,*

$$|A_{k,d}| = \binom{t-d}{k-1}.$$

One of the main tools in our construction is the next lemma proved in [6].

Lemma 2.3.

$$B_0(2^t) = t2^{t-1},$$

and the graph realizes that number is the 2^t-vertex hypercube, which we denote by Q_{2^t}.

2.2 Construction of m-RMBG, $m \geq 1$

In this section we supply in Theorem 2.1, a general upper bound to $B_m(n)$, $m \geq 1$, which yields a significant improvement of the bounds obtained in Lemmas 1.1 - 1.3. We close the section by producing a construction which confirms affirmatively Conjecture 1.1.

Theorem 2.1.

(i) $B_m(n) \leq (1 + 2^{1-m})n - \Theta((\frac{n}{2^m})^{\frac{2}{3}})$, $n > 2^{m+1}$, $n = 2^t$,
and the leading constant depends on t (mod 3) for any fixed m.

(ii) $B_m(n) \le A_{m,n} - \lceil \frac{2^t - n}{2^{t-k}-1} \rceil (2^{t-k-m+1} - t + k + m - 2) - (2^t - n)$, $n > 2^{m+1}$, $2^{t-1} < n < 2^t$, *where $A_{m,n}$ is the bound in (i) (with $n = 2^t$) and $k = \lceil \frac{2}{3}(t-m) \rceil$.*

(iii) $B_m(n) = n - 1$, $n \le 2^{m+1}$.

Proof:

Case 1: $n = 2^t$

Construction of the m-RBG graph G: We construct the required graph G as follows:

We take 2^k copies of $BT_{2^{t-k}}$ trees, with $x_1, x_2, \ldots, x_{2^k}$ as originators in each $T(i)$, $i = 1, 2, \ldots, 2^k$, respectively, and establish with them a Q_{2^k}-cube. The value of k that will give the optimal construction (depending on m), will be determined later.

We have only to describe the additional edges to be added between each x_i to vertices in $T(i)$. As was viewed before concerning BT_{2^t}, we can decompose each $BT_{2^{t-k}}$ into isomorphic copies of a $BT_{2^{m-1}}$ tree. There are exactly $\frac{2^{t-k}}{2^{m-1}}$ such copies. (We suppose that $m - 1 \le t - k$). Hence, each originator x_i is joined to one of the originators in each $BT_{2^{m-1}}$ tree, which is at distance at least 2 from x_i. So that we add (by using Lemma 2.1) $\frac{2^{t-k}}{2^{m-1}} - (t - k - m + 2)$ additional edges to obtain:

$$B_m(n) \le 2^k \left(\overbrace{2^{t-k} - 1}^{\#\text{ edges in each } 2^{t-k}\text{-tree}} + \overbrace{\frac{2^{t-k}}{2^{m-1}} - (t - k - m + 2)}^{\text{edges added to each } 2^{t-k}\text{-tree}} \right)$$

$$+ \overbrace{k2^{k-1}}^{\text{edges in } Q_{2^k}\text{-cube}}$$

$$= n\left(1 + \frac{1}{2^{m-1}}\right) - 2^k(t + 3 - m) + 3k2^{k-1}$$

$$= n\left(1 + \frac{1}{2^{m-1}}\right) - 2^{k-1}(2(t-m) + 6 - 3k), \qquad (1)$$

where the minimum is achieved by choosing:

$$k = \lceil \frac{2}{3}(t - m) \rceil,$$

satisfying $m - 1 \le t - k$. Now, defining $r \equiv (t - m) \pmod 3$ yields $k = \frac{2}{3}(t-m) + \frac{r}{3}$, so that the subtracted term in (1) is: $2^{\frac{r}{3}-1}(6 - r)\left(\frac{n}{2^m}\right)^{\frac{2}{3}}$, or $c_r\left(\frac{n}{2^m}\right)^{\frac{2}{3}}$ so that $c_0 = 3$, $c_1 = 5 \cdot 2^{-\frac{2}{3}}$, $c_2 = 4 \cdot 2^{-\frac{1}{3}}$. Hence, substituting the value of k in (1) yields (i).

To have (iii) we take G to be a $BT_{2^{m+1}}$. Then, one can check that within at most m time units each vertex of the tree can reach one of the originators. Then, that originator finishes broadcasting within at most $m+1$ additional time units. Since G was a tree we have $B_m(n) = n-1$, $n \leq 2^{m+1}$.

The broadcasting algorithm in G:

Case i: The originator is $v \notin \{x_i | 1 \leq i \leq 2^k\}$

Assume $v \in T(j)$. Then, v needs at most m time units to reach x_j. Then, another k time units are needed to x_j to broadcast in the Q_{2^k}-cube and at most another $t-k$ time units in any of the $T(i)$-trees to accomplish broadcasting, within at most $t+m$ time units.

Case ii: The originator is $v \in \{x_i | 1 \leq i \leq 2^k\}$

An originator x_i broadcast first to all other originators within k time units (using the Q_{2^k}-cube construction). Then it has $t-k$ time units to accomplish broadcasting in each tree $T(i)$, which makes all together at most t time units.

Case 2: $2^{t-1} < n < 2^t$

We start with the same construction as in Case 1 for $n = 2^t$ by taking 2^k copies of a 2^{t-k}-tree. Then, vertices are deleted, as follows: first we delete whole trees $T(i)$ (without the originators x_i) as needed, as long as the number $2^t - n$ is at least $|T(i)| - 1$. The remaining vertices to be deleted are from the same tree and we start with the leaves there. Observe that since $k = \lceil \frac{2}{3}(t - m) \rceil$ we never delete vertices from the Q_{2^k}-cube of originators. Further, since $\lceil \log_2 n \rceil = t$ the broadcasting scheme is the same as in the previous case. Hence from $A_{m,n}$, the number of edges obtained in the previous case, we delete,

$$\begin{aligned} C_{m,n} &= \lfloor \frac{2^t - n}{2^{t-k} - 1} \rfloor \left(2^{t-k} - 1 + \frac{2^{t-k}}{2^{m-1}} - (t - k - m + 2) \right) \\ &\quad + 2^t - n - (2^{t-k} - 1) \lfloor \frac{2^t - n}{2^{t-k} - 1} \rfloor \\ &= \lfloor \frac{2^t - n}{2^{t-k} - 1} \rfloor \left(\frac{2^{t-k}}{2^{m-1}} - (t - k - m + 2) \right) + 2^t - n. \end{aligned}$$

The first product in $C_{m,t}$ is exactly the total number of edges deleted with the deletion of "whole" trees $T(i)$. The rest of the amount deleted is due to the remaining vertices left to be deleted. Here we notice that actually the number of deleted edges is greater than the one subtracted, but it is difficult to determine that number using only parameters (namely, t, m). It is possible to do it for explicit values of t and m.

This completes the proof of the theorem. $\square$

As one can see Theorem 2.1 supplies a better general bound than Lemma 1.3. In the particular cases of $m = 1, 2$ we have the following theorems which produce better upper bounds than those in Lemmas 1.1 - 1.2. The calculation of the explicit values of $C_{1,n}$, $C_{2,n}$ are omitted since these are just substitutions $m = 1, 2$ and the relevant values of $t \equiv j \pmod 3$, $j = 0, 1, 2$.

Theorem 2.2.

(i)

$$B_1(n) \leq \begin{cases} 2n - c_0 n^{\frac{2}{3}}, & t \equiv 0 \pmod 3,\ n > 4 \\ 2n - c_1 n^{\frac{2}{3}}, & t \equiv 1 \pmod 3,\ n > 4 \\ 2n - c_2 n^{\frac{2}{3}}, & t \equiv 2 \pmod 3,\ n > 4, \end{cases}$$

where, $c_0 = 2$, $c_1 = 3 \cdot 2^{-\frac{2}{3}}$, $c_2 = \frac{5}{2} \cdot 2^{-\frac{2}{3}}$, *and* $n = 2^t$.

(ii) $B_1(n) \leq A_{1,n} - C_{1,n}$, *when* $2^{t-1} < n < 2^t$.

(iii) $B_1(n) = n - 1$, $n \leq 4$.

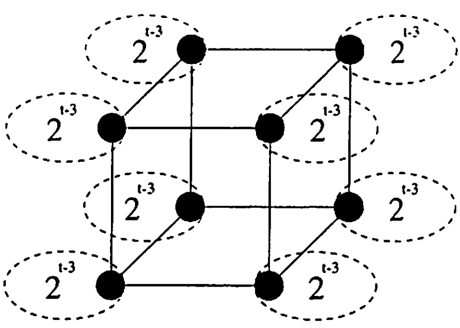


Figure 2: 1-RBG graph G for $k = 3$

Theorem 2.3.

(i)

$$B_2(n) \leq \begin{cases} \frac{3}{2}n - c_0 n^{\frac{2}{3}}, & t \equiv 0 \pmod 3,\ n > 8 \\ \frac{3}{2}n - c_1 n^{\frac{2}{3}}, & t \equiv 1 \pmod 3,\ n > 8 \\ \frac{3}{2}n - c_2 n^{\frac{2}{3}}, & t \equiv 2 \pmod 3,\ n > 8, \end{cases}$$

where, $c_0 = \frac{5}{4}$, $c_1 = 2^{\frac{1}{3}}$, $c_2 = \frac{3}{2} \cdot 2^{-\frac{1}{3}}$, *and* $n = 2^t$.

(ii) $B_2(n) \leq A_{2,n} - C_{2,n}$, *when* $2^{t-1} < n < 2^t$.

(iii) $B_2(n) = n - 1$, $n \leq 8$.

Remarks:

1. Although our bound in Theorem 2.1 is significantly better bound than that in Lemma 1.3, in the next theorem we give an ad-hoc construction which proves Conjecture 1.1 which is similar to a conjecture appeared in [16].

2. Substituting $m = t$ gives the obvious 2^t-binomial tree.

In the next theorem we prove Conjecture 1.1.

Theorem 2.4.

$$B_m(n) = n - 1, \text{ for all } n \text{ and } m \geq \lfloor \frac{1}{2} \log_2 n \rfloor + 1.$$

Proof:

Case 1: $n = 2^t$

We take the $BT_{2^{t+1}}$-tree, with originators x_1 and x_2, and delete vertices in each BT_{2^t}-subtree in a way that each vertex in the resulting tree is able to reach one of the originators within at most $\lfloor \frac{t}{2} \rfloor$ time units. We delete all vertices at distance greater than $\frac{t}{2}$, and (in case t is even) we delete half the vertices at distance exactly $\frac{t}{2}$.

By lemma 2.1, lemma 2.2 and the well-known combinatorial identity,

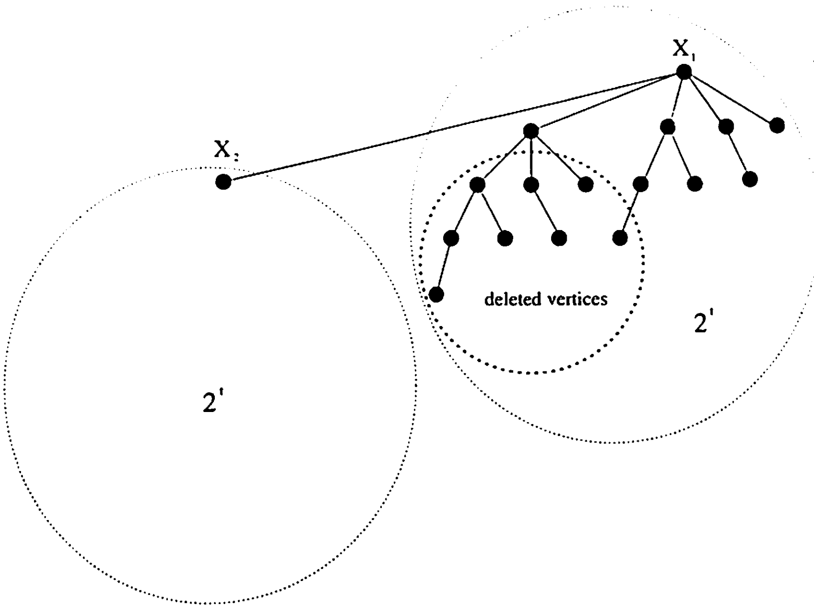
$$\sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \binom{t}{i} = s^{t-1} + \frac{1}{2} \binom{t}{\frac{t}{2}} \frac{1 + (-1^t)}{2},$$

the deletion stops at distance $\lfloor \frac{t}{2} \rfloor$ from one of the originators. Therefore, we obtain a subtree of order 2^{t-1} such that broadcasting from x_1 (or x_2) is accomplished within $t + 1$ time steps. Each other vertex can transmits a message to x_1 (or x_2) within at most $\lfloor \frac{t}{2} \rfloor$ time steps, so that the broadcasting procedure is finished within at most $t + m$ time steps.

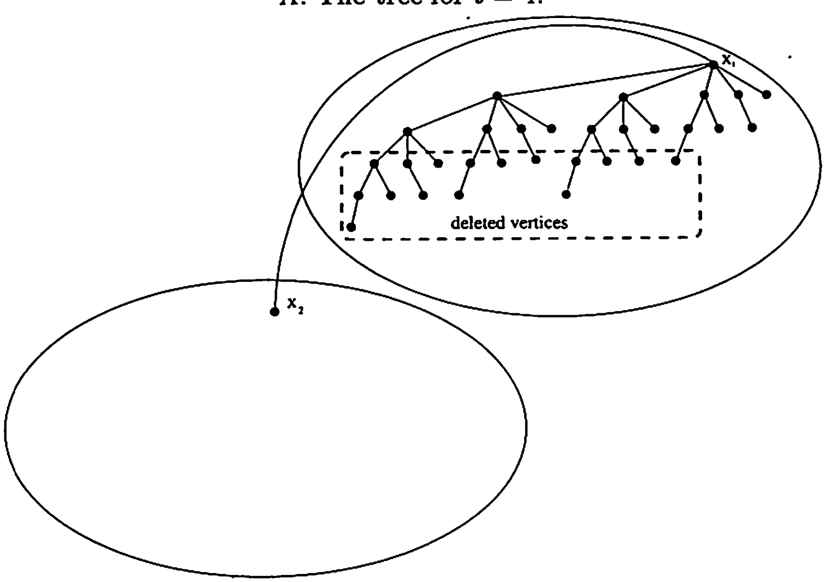
The result is obtained since the constructed graph is a tree (see Figure 3).

Case 2: $2^{t-1} < n < 2^t$

We take the tree that was constructed in the previous case and delete leaves as required. Since $\lceil \log_2 n \rceil = t$ we accomplish the broadcasting in that tree within the needed time. □



A: The tree for $t = 4$.



B: The tree for $t = 5$.

Figure 3: m-RMBG for $t = 4, 5$

3 Final Remarks

In [15] Shastri calculated values for $n \leq 65$, $m \leq 3$. For example: he conjectured that $B_1(15) = 18$ and $B_1(16) = 19$. The construction presented in Theorem 2.2 shows that in particular $B_1(15) \leq 18$ and $B_1(16) \leq 20$. Also it was obtained there that $B_2(16) = 16$, which is exactly the upper bound of Theorem 2.3.

Hence, we can conclude that for the values of m, $1 \leq m \leq \lceil \frac{1}{2} \log_2 n \rceil$, the obtained bounds here are almost the best possible.

Acknowledgment. We would like to thank the referee for his valuable remarks in shaping the presentation of the paper.

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