# Decompositions of the Complete Graph into Small 2-Regular Graphs

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**Abstract**

A $G$-decomposition of the complete graph $K_v$ is a set $S$ of subgraphs of $K_v$, each isomorphic to $G$, such that the edge set of $K_v$ is partitioned by the edge sets of the subgraphs in $S$. For all positive integers $v$ and every 2-regular graph $G$ with ten or fewer vertices, we prove necessary and sufficient conditions for the existence of a $G$-decomposition of $K_v$.

## 1 Introduction

A graph $H$ can be *decomposed* into subgraphs $G_1, G_2, \ldots, G_t$ if $H$ is the edge-disjoint union of $G_1, G_2, \ldots, G_t$. If, in addition, $G_i \cong G$ for each $i$, then the decomposition is called a *G-decomposition* of $H$. The problem of determining all values of $v$ for which there is a $G$-decomposition of the complete graph $K_v$ is called the *spectrum problem* for $G$.

The spectrum problem has been considered for many graphs (see [12] for a survey). If $G$ is a complete graph with $k$ vertices, then a $G$-decomposition of $K_v$ is a $(v, k, 1)$ balanced incomplete block design of order $v$ with blocks of size $k$ and index 1. After a great deal of previous work (see [16] for a survey), the spectrum problem for $m$-cycles has recently been solved in

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[4, 17]. The spectrum problem has been considered for all graphs with five or fewer vertices [5, 6], the $n$-cube [8, 14, 15], paths [11], stars [7], Platonic graphs [2], and the Petersen graph [1]. In this paper, we are interested in the spectrum problem for 2-regular graphs with at most ten vertices.

When $G$ is a 2-regular graph, the obvious necessary conditions for the existence of a $G$-decomposition of $K_v$ are given by the following lemma.

**Lemma 1.1** *If $G$ is a 2-regular graph with $m$ vertices and $G$-decomposition of $K_v$ exists then*

- $m \leq v$,
- *$v$ is odd, and*
- *$m$ divides $v(v-1)/2$.*

Let $C_m$ denote the cycle on $m$ vertices and let $C_{m_1} \cup C_{m_2} \cup \ldots \cup C_{m_k}$ denote that graph with components $C_{m_1}, C_{m_2}, \ldots, C_{m_k}$. In addition to the spectrum problem for $m$-cycles, another well-studied problem in the area of $G$-decompositions of $K_v$ for 2-regular graphs $G$ is the *Oberwolfach problem*. The Oberwolfach problem asks for a $G$-decomposition of $K_v$ when $G$ is 2-regular graph with $v$ vertices. There are two known cases when the necessary conditions of Lemma 1.1 are not also sufficient. In [3], it is shown there is no $(C_4 \cup C_5)$-decomposition of $K_9$, and no $(C_3 \cup C_3 \cup C_5)$-decomposition of $K_{11}$. It is widely believed that these are the only exceptions.

On the other hand, the necessary conditions in Lemma 1.1 are sufficient for the existence of a $C_m$-decomposition of $K_v$ [4, 17]. The problem considered in this paper encompasses both the problem of decomposing $K_v$ into $m$-cycles and the Oberwolfach problem. We will show that the only additional exception (other than the two exceptions covered by the Oberwolfach problem) for 2-regular graphs with at most ten vertices is that there is no $(C_3 \cup C_3)$-decomposition of $K_9$ (see Theorem 2.1).

# 2 Main Results

We begin with some notation. The complete multipartite graph with $r$ parts of $s$ vertices each is denoted by $K_{r(s)}$. If $G$ is a graph and $H$ is a subgraph of $G$, then $G - H$ will denote the graph obtained from $G$ by removing the edges of $H$. In particular, we will make use of the graph $K_v - K_u$, and we call the complete graph whose edges are removed the *hole*. For even $v$, the complete graph $K_v$ with $v/2$ independent edges (a 1-factor) removed will be denoted by $K_v - F$.

Let $K$ and $M$ be finite sets of positive integers and let $\lambda$ and $v$ be positive integers. A *group divisible design*, denoted by $(K, \lambda, M; v)$ GDD, is a collection of subsets of size $k \in K$ (called blocks), chosen from a $v$-set, where the $v$-set is partitioned into disjoint subsets (called groups) of size $m \in M$ such that each block contains at most one element from each group, and any two elements from distinct groups occur together in $\lambda$ blocks. If $M = \{m\}$ and $K = \{k\}$, we write $(k, \lambda, m; v)$ GDD. Also, a GDD with exactly one group of size $m_2$ and the remaining groups of size $m_1$ is denoted by $(K, \lambda, \{m_1, m_2^*\}; v)$ GDD. Similarly, a GDD with exactly one block of size $k_2$ and the remaining blocks of size $k_1$ is denoted by $(\{k_1, k_2^*\}, \lambda, M; v)$ GDD.

We will make extensive use of the following well-known construction given in [18].

**Lemma 2.1** *Let $s$ and $h$ be non-negative integers. Suppose there exists a $(K, 1, M, v)$ GDD with group set $S$. If, for some $g' \in S$, there exists*

*(1) a $G$-decomposition of $K_{r(s)}$ for each $r \in K$;*

*(2) a $G$-decomposition of $K_{s|g|+h} - K_h$ for each group $g \in S \setminus \{g'\}$; and*

*(3) a $G$-decomposition of $K_{s|g'|+h}$,*

*then there exists a $G$-decomposition of $K_{sv+h}$.*

The following lemmas will be useful in completing the proof of Theorem 2.1.

**Lemma 2.2** *Suppose $G$ is a 2-regular graph with 6 vertices. Then there exists a $G$-decomposition of $K_v$ if and only if $v \equiv 1, 9 \pmod{12}$ and $v \neq 9$ if $G = C_3 \cup C_3$.*

**Proof.** Let $G$ be a 2-regular graph with 6 vertices. By Lemma 1.1, $v \equiv 1, 9 \pmod{12}$ if a $G$-decomposition of $K_v$ exists, and the result has been proven in the case $G = C_6$. Hence, suppose that $v \equiv 1, 9 \pmod{12}$ and that $G = C_3 \cup C_3$. When $G = C_3 \cup C_3$, a $G$-decomposition also yields a $C_3$-decomposition. There is (up to isomorphism) only one $C_3$-decomposition of $K_9$ and it is straightforward to check that the copies of $C_3$ given in this decomposition cannot be partitioned into copies of $G$. Hence there is no $G$-decomposition of $K_9$ when $G = C_3 \cup C_3$. Existence for the remaining values of $v$ is given in [13].

□

**Lemma 2.3** *Suppose $G$ is a 2-regular graph with 7 vertices. Then there exists a $G$-decomposition of $K_v$ if and only if $v \equiv 1, 7 \pmod{14}$.*

**Proof.** Let $G$ be a 2-regular graph with 7 vertices. If a $G$-decomposition of $K_v$ exists, then, by Lemma 1.1, $v \equiv 1, 7 \pmod{14}$. The result has been proved for the case $G = C_7$ so assume that $v \equiv 1, 7 \pmod{14}$ and $G = C_3 \cup C_4$. Apply Lemma 2.1 with $s = 7$, $h = 1$ if $v \equiv 1 \pmod{14}$, $h = 0$ if $v \equiv 7 \pmod{14}$, and with $G$-decompositions and GDDs as indicated in Table 2. The $G$-decompositions in the table can be found in the Appendix and the GDDs are well-known to exist [10].

| $v$ | GDDs | $G$ − decompositions of |
|---|---|---|
| $v = 42x + 1, x \geq 1$ | $(3, 1, 2, 6x)$GDD | $K_{15}, K_{3(7)}$ |
| $v = 42x + 15, x \geq 1$ | $(3, 1, 2, 6x + 2)$GDD | $K_{15}, K_{3(7)}$ |
| $v = 42x + 29, x \geq 1$ | $(3, 1, \{2, 4^*\}, 6x + 4)$GDD | $K_{15}, K_{29}, K_{3(7)}$ |
| $v = 42x + 7, x \geq 1$ | $(3, 1, 1, 6x + 1)$GDD | $K_7, K_{3(7)}$ |
| $v = 42x + 21, x \geq 0$ | $(3, 1, 1, 6x + 3)$GDD | $K_7, K_{3(7)}$ |
| $v = 42x + 35, x \geq 0$ | $(\{3, 5^*\}, 1, 1, 6x + 5)$GDD | $K_7, K_{3(7)}, K_{5(7)}$ |

**Table 2**

□

**Lemma 2.4** *Suppose $G$ is a 2-regular graph with 8 vertices. Then there exists a $G$-decomposition of $K_v$ if and only if $v \equiv 1 \pmod{16}$.*

**Proof.** Let $G$ be a 2-regular graph with 8 vertices. By Lemma 1.1, $v \equiv 1 \pmod{16}$ if a $G$-decomposition of $K_v$ exists, and the result has been proved in the case $G = C_8$. So assume that $v \equiv 1 \pmod{16}$ and suppose $G = C_3 \cup C_5$ or $G = C_4 \cup C_4$. Apply Lemma 2.1 with $s = 8$, $h = 1$ and with $G$-decompositions and GDDs as indicated in Table 3. The $G$-decompositions in the table can be found in the Appendix and the GDDs are well-known to exist [10].

| $v$ | GDDs | $G$ − decompositions of |
|---|---|---|
| $v = 48x + 1, x \geq 1$ | $(3, 1, 2, 6x)$GDD | $K_{17}, K_{3(8)}$ |
| $v = 48x + 17, x \geq 1$ | $(3, 1, 2, 6x + 2)$GDD | $K_{17}, K_{3(8)}$ |
| $v = 48x + 33, x \geq 1$ | $(3, 1, \{2, 4^*\}, 6x + 4)$GDD | $K_{17}, K_{33}, K_{3(8)}$ |

**Table 3**

□

**Lemma 2.5** *Suppose $G$ is a 2-regular graph with 9 vertices. Then there exists a $G$-decomposition of $K_v$ if and only if $v \equiv 1, 9 \pmod{18}$.*

**Proof.** Let $G$ be a 2-regular graph with 9 vertices. From Lemma 1.1, if there exists a $G$-decomposition of $K_v$, then $v \equiv 1, 9 \pmod{18}$, and the problem has been settled in the case $G = C_9$. Hence we assume that

$v \equiv 1,9 \pmod{18}$ and that $G = C_3 \cup C_3 \cup C_3$, $G = C_3 \cup C_6$, or $G = C_4 \cup C_5$. Apply Lemma 2.1 with $s = 3$, $h = 1$ if $v \equiv 1 \pmod{18}$, $h = 0$ if $v \equiv 9 \pmod{18}$ and $G = C_3 \cup C_3 \cup C_3$ or $G = C_3 \cup C_6$, $h = 9$ if $v \equiv 9 \pmod{18}$ and $G = C_4 \cup C_5$, and with $G$-decompositions and GDDs as indicated in Table 4. The $G$-decompositions in the table can be found in the Appendix and the GDDS are well-known to exist [10]. This leaves the isolated cases $v = 37$ for $G = C_3 \cup C_3 \cup C_3, G = C_3 \cup C_6$ and $G = C_4 \cup C_5$, and $v = 45$ for $G = C_4 \cup C_5$. These decompositions are given in the Appendix.

| $v$ | GDDs | $G$ – decompositions of |
|---|---|---|
| $v = 18x + 1, x \geq 3$ | $(3, 1, 6, 6x)$GDD | $K_{19}, K_{3(3)}$ |
| $v = 18x + 9, x \geq 1$, $G = C_3 \cup C_6$ or $G = C_3 \cup C_3 \cup C_3$ | $(3, 1, 3, 6x + 3)$GDD | $K_9, K_{3(3)}$ |
| $v = 18x + 9, x \geq 3$, $G = C_4 \cup C_5$ | $(3, 1, 6, 6x)$GDD | $K_{27}, K_{27} - K_9, K_{3(3)}$ |

**Table 4**

□

**Lemma 2.6** *Suppose $G$ is a 2-regular graph with 10 vertices. Then there exists a $G$-decomposition of $K_{25} - K_5$.*

**Proof.** Let $G$ be a 2-regular graph with 10 vertices. The result is true if $G = C_{10}$ (see [9]) so assume that $G = C_3 \cup C_3 \cup C_4$, $C_3 \cup C_7$, $C_4 \cup C_6$ or $C_5 \cup C_5$. Let the vertex set of $K_{25} - K_5$ be $\mathbb{Z}_5 \times \mathbb{Z}_5$ and let the hole be on the vertices $(0,0), (1,0), (2,0), (3,0), (4,0)$. For each 2-regular graph $G$ with 10 vertices, a $G$-decomposition of $K_{10} - F$ is given in the Appendix. We place this decomposition on the vertex set $\{(x,y) : x \in \mathbb{Z}_5, y \in \{3,4\}\}$ ensuring that the edge set of $F$ is $\{\{(x,3),(x,4)\} : x \in \mathbb{Z}_5\}$. Note that there are 29 copies of $G$ in a $G$-decomposition of $K_{25} - K_5$ and 4 copies of $G$ in a $G$-decomposition of $K_{10} - F$. Thus we require 25 more copies of $G$ that form a $G$-decomposition of $(K_{25} - K_5) - (K_{10} - F)$ (where the $K_5$ and the $K_{10}$ are vertex disjoint) for each 2-regular graph $G$ with 10 vertices. The required $G$-decompositions, constructed cyclically modulo 5, are given in the Appendix. □

**Lemma 2.7** *Suppose $G$ is a 2-regular graph with 10 vertices. Then there exists a $G$-decomposition of $K_v$ if and only if $v \equiv 1,5 \pmod{20}$ and $v \neq 5$.*

**Proof.** Let $G$ be a 2-regular graph with 10 vertices. By Lemma 1.1, if a $G$-decomposition of $K_v$ exists then $v \equiv 1,5 \pmod{20}$ and $v > 5$, and the result has been proved for the case $G = C_{10}$. Hence assume that $v \equiv 1,5 \pmod{20}$, $v > 5$, and $G = C_3 \cup C_3 \cup C_4$, $G = C_3 \cup C_7$, $G = C_4 \cup C_6$, or $G = C_5 \cup C_5$. Apply Lemma 2.1 with $s = 10$, $h = 1$ if $v \equiv 1 \pmod{20}$, $h = 5$ if $v \equiv 5 \pmod{20}$, and with $G$-decompositions and GDDs as indicated in Table 5. The required $G$-decompositions of $K_{25} - K_5$ exist by Lemma 2.6 and the remaining $G$-decompositions in the table can be found in the Appendix. The GDDS are well-known to exist [10].

| $v$ | GDDs | $G$ – decompositions of |
|---|---|---|
| $v = 60x + 1, x \geq 1$ | $(3, 1, 2, 6x)$GDD | $K_{21}, K_{3(10)}$ |
| $v = 60x + 21, x \geq 1$ | $(3, 1, 2, 6x + 2)$GDD | $K_{21}, K_{3(10)}$ |
| $v = 60x + 41, x \geq 1$ | $(3, 1, \{2, 4^*\}, 6x + 4)$GDD | $K_{21}, K_{41}, K_{3(10)}$ |
| $v = 60x + 5, x \geq 1$ | $(3, 1, 2, 6x)$GDD | $K_{25}, K_{25} - K_5, K_{3(10)}$ |
| $v = 60x + 25, x \geq 1$ | $(3, 1, 2, 6x + 2)$GDD | $K_{25}, K_{25} - K_5, K_{3(10)}$ |
| $v = 60x + 45, x \geq 1$ | $(3, 1, \{2, 4^*\}, 6x + 4)$GDD | $K_{45}, K_{25} - K_5, K_{3(10)}$ |

**Table 5**

□

Combining the results of Lemmas 2.2, 2.3, 2.4, 2.5 and 2.7 (and the well-known existence results for $C_m$-decompositions of $K_v$ for $m \leq 10$), we have the following theorem.

**Theorem 2.1** *Let $G$ be a 2-regular graph with $m$ vertices where $m \leq 10$. Then there exists a $G$-decomposition of $K_v$ if and only if*

- *$m \leq v$;*
- *$v$ is odd;*
- *$m$ divides $v(v-1)/2$; and*
- *$v \neq 9$ when $G = C_3 \cup C_3$ or $G = C_4 \cup C_5$.*

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# 3 Appendix

$K_7$ Let the vertex set be $\mathbb{Z}_7$. Decomposition is:
$C_4 \cup C_3$: $((0,1,3,4),(2,5,6))$, $((0,2,3,5),(1,4,6))$, $((1,2,4,5),(0,3,6))$.

$K_{15}$ Let the vertex set be $\mathbb{Z}_{15}$. Develop the following mod 15:
$C_4 \cup C_3$: $((0,1,7,2),(3,6,10))$.

$K_{29}$ Let the vertex set be $\mathbb{Z}_{29}$. Develop the following mod 29:
$C_4 \cup C_3$: $((0,1,3,6),(2,11,15))$, $((0,5,19,7),(1,9,20))$.

$K_{3(7)}$ Let the vertex set be $\mathbb{Z}_{21}$ (with the parts obtained by cycling $\{0,3,6,9,12,15,18\}$ mod 21). Develop the following mod 21:
$C_4 \cup C_3$: $((0,1,3,8),(2,6,13))$.

$K_{5(7)}$ Let the vertex set be $\mathbb{Z}_{35}$ (with the parts obtained by cycling $\{0,5,10,15,20,25,30\}$ mod 35). Develop the following mod 35:
$C_4 \cup C_3$: $((0,1,3,6),(2,15,19))$, $((0,7,21,9),(1,12,20))$.

$K_{17}$ Let the vertex set be $\mathbb{Z}_{17}$. Develop the following mod 17:
$C_4 \cup C_4$: $((0,1,3,6),(2,7,14,10))$.
$C_5 \cup C_3$: $((0,1,3,10,4),(2,5,14))$.

$K_{33}$ Let the vertex set be $\mathbb{Z}_{33}$. Develop the following mod 33:
$C_4 \cup C_4$: $((0,1,3,6),(2,7,11,18))$, $((0,8,19,9),(1,13,28,14))$.
$C_5 \cup C_3$: $((0,1,3,6,10),(2,7,13))$, $((0,7,16,2,17),(1,9,21))$.

$K_{3(8)}$ Let the vertex set be $\mathbb{Z}_{24}$ (with the parts obtained by cycling $\{0,3,6,9,12,15,18,21\}$ mod 24). Develop the following mod 24:
$C_4 \cup C_4$: $((0,1,3,7),(2,10,5,15))$.
$C_5 \cup C_3$: $((0,1,3,7,14),(2,10,15))$.

$K_9$ Let the vertex set be $\mathbb{Z}_9$. Decompositions are:
$C_3 \cup C_3 \cup C_3$: $((0,1,2),(3,4,5),(6,7,8))$, $((0,3,6),(1,4,7),(2,5,8))$,
$((0,4,8),(1,5,6),(2,3,7))$, $((0,5,7),(1,3,8),(2,4,6))$.
$C_6 \cup C_3$: $((0,1,2,3,4,5),(6,7,8))$, $((0,2,4,6,1,7),(3,5,8))$,
$((1,3,7,2,6,5),(0,4,8))$, $((1,4,7,5,2,8),(0,3,6))$.

$K_{27}$ Let the vertex set be $\{i_j \mid 0 \leq i \leq 12, j = 1,2\} \cup \{\infty\}$. Develop the following mod $(13,-)$, with the vertex $\infty$ remaining fixed:
$C_3 \cup C_3 \cup C_3$: $((0_1,1_1,3_1),(2_1,6_1,0_2),(4_1,9_1,1_2))$,
$((0_1,6_1,1_2),(1_1,0_2,3_2),(2_2,4_2,9_2))$,
$((0_1,0_2,9_2),(1_1,4_2,5_2),(2_1,8_2,\infty))$.
$C_5 \cup C_4$: $((0_1,1_1,3_1,6_1,0_2),(2_1,7_1,11_1,1_2))$,
$((0_1,6_1,1_2,3_1,4_2),(1_1,3_2,10_1,6_2))$,
$((0_2,1_2,3_2,9_2,4_2),(0_1,10_2,7_2,\infty))$.
$C_6 \cup C_3$: $((0_1,1_1,3_1,6_1,2_1,7_1),(4_1,0_2,1_2))$,
$((0_1,0_2,1_1,2_2,4_1,6_2),(2_1,5_2,7_2))$,
$((0_1,4_2,0_2,5_2,2_2,8_2),(2_1,9_2,\infty))$.

$K_{19}$ Let the vertex set be $\mathbb{Z}_{19}$. Develop the following mod 19:

$C_3 \cup C_3 \cup C_3$: $((0,1,4),(2,7,13),(3,5,12))$.
$C_5 \cup C_4$: $((0,1,3,6,10),(2,7,15,8))$.
$C_6 \cup C_3$: $((0,1,3,6,2,9),(4,10,15))$.

$K_{37}$ Let the vertex set be $\mathbb{Z}_{37}$. Develop the following mod 37:

$C_3 \cup C_3 \cup C_3$: $((0,1,3),(2,6,11),(4,10,22))$, $((0,7,20),(1,9,23),(2,12,28))$.
$C_5 \cup C_4$: $((0,1,3,6,10),(2,7,13,20))$, $((0,8,17,4,20),(1,12,27,13))$.
$C_6 \cup C_3$: $((0,1,3,6,2,7),(4,10,18))$, $((0,9,24,5,25,12),(1,11,22))$.

$K_{27} - K_9$ Let the vertex set be $\mathbb{Z}_{27}$, with the hole on $\{0,1,\ldots,8\}$. Decomposition is:

$C_5 \cup C_4$: $((0,9,1,10,11),(2,12,3,13))$, $((0,10,2,9,12),(1,11,3,14))$,
$((0,13,1,12,14),(2,11,4,15))$, $((0,15,1,16,17),(2,14,4,18))$,
$((0,16,2,17,18),(1,19,3,20))$, $((0,19,2,20,21),(1,17,3,18))$,
$((0,20,4,9,22),(1,21,2,23))$, $((0,23,3,9,24),(1,22,2,25))$,
$((0,25,3,10,26),(4,12,5,13))$, $((1,24,3,15,26),(4,10,5,16))$,
$((2,24,4,17,26),(3,16,6,21))$, $((3,22,4,19,26),(5,9,6,11))$,
$((4,21,5,14,23),(6,10,7,12))$, $((4,25,5,18,26),(6,13,7,14))$,
$((5,15,6,17,19),(7,9,8,11))$, $((5,17,7,15,20),(6,18,8,19))$,
$((5,22,6,20,23),(7,16,8,21))$, $((5,24,6,23,26),(7,18,9,19))$,
$((6,25,7,20,26),(8,10,9,13))$, $((7,22,8,12,23),(9,11,13,14))$,
$((7,24,8,14,26),(9,15,10,16))$, $((8,15,11,12,17),(9,20,10,21))$,
$((8,20,11,14,25),(9,17,10,23))$, $((8,23,11,16,26),(10,12,13,18))$,
$((9,25,10,13,26),(11,17,14,18))$, $((10,14,15,12,19),(11,21,13,22))$,
$((10,22,12,16,24),(11,19,13,25))$, $((11,24,12,21,26),(13,15,16,20))$,
$((12,18,15,17,20),(13,16,14,24))$, $((12,25,15,22,26),(13,17,21,23))$,
$((14,19,15,21,22),(16,18,24,25))$, $((14,20,18,25,21),(19,22,23,24))$,
$((15,23,16,21,24),(18,19,25,22))$, $((17,22,24,20,25),(18,21,19,23))$,
$((17,23,25,26,24),(16,19,20,22))$.

$K_{45}$ Let the vertex set be $\{i_j \mid 0 \le i \le 10, j = 1,2,3,4\} \cup \{\infty\}$. Develop the following mod $(11,-)$, with the vertex $\infty$ remaining fixed:

$C_5 \cup C_4$: $((0_1,1_1,3_1,6_1,0_2),(0_3,1_3,1_4,\infty))$, $((0_1,4_1,9_1,0_2,1_2),(2_1,5_2,3_2,\infty))$,
$((0_1,4_2,5_1,0_2,7_2),(1_1,9_2,1_2,0_3))$, $((0_1,9_2,3_2,0_3,2_3),(1_1,1_3,3_1,4_3))$,
$((0_1,4_3,7_1,1_3,6_3),(1_1,8_3,1_2,0_4))$, $((0_1,0_4,2_1,3_4,2_4),(1_1,4_4,8_1,5_4))$,
$((0_1,5_4,0_2,0_3,6_4),(1_2,2_3,4_2,6_3))$, $((0_2,3_3,8_2,1_3,0_4),(1_2,2_4,4_2,7_4))$,
$((0_2,2_4,5_2,1_4,4_4),(0_3,3_3,7_3,3_4))$, $((0_3,1_4,3_3,0_4,2_4),(1_3,5_4,10_4,6_4))$.

$K_{10} - F$ Let the vertex set be $\mathbb{Z}_{10}$. Decompositions are:

$C_4 \cup C_3 \cup C_3$: $((0,2,1,3),(4,6,8),(5,7,9))$, $((0,4,1,5),(2,6,9),(3,7,8))$,
$((0,6,1,7),(2,5,8),(3,4,9))$, $((0,8,1,9),(2,4,7),(3,5,6))$.
$C_5 \cup C_5$: $((0,2,1,3,4),(5,6,8,7,9))$, $((0,3,7,4,8),(1,5,2,6,9))$,
$((0,5,8,1,7),(2,4,6,3,9))$, $((0,6,1,4,9),(2,7,5,3,8))$.
$C_6 \cup C_4$: $((0,2,1,3,4,6),(5,8,7,9))$, $((0,3,5,1,7,4),(2,8,6,9))$,
$((0,5,2,6,3,7),(1,8,4,9))$, $((1,4,2,7,5,6),(0,8,3,9))$.
$C_7 \cup C_3$: $((0,2,1,3,4,6,8),(5,7,9))$, $((0,3,5,1,4,8,7),(2,6,9))$,
$((0,4,7,1,6,3,9),(2,5,8))$, $((1,8,3,7,2,4,9),(0,5,6))$.

$\boxed{K_{3(3)}}$ Let the vertex set be $\{0,1,2\} \cup \{3,4,5\} \cup \{6,7,8\}$, with the obvious vertex partition. Decompositions are:

$C_3 \cup C_3 \cup C_3$: $((0,3,6),(1,4,7),(2,5,8))$, $((0,4,8),(1,5,6),(2,3,7))$, $((0,5,7),(1,3,8),(2,4,6))$.

$C_5 \cup C_4$: $((0,3,1,4,6),(2,7,5,8))$, $((0,4,2,6,5),(1,7,3,8))$, $((1,5,2,3,6),(0,7,4,8))$.

$C_6 \cup C_3$: $((0,3,6,1,4,7),(2,5,8))$, $((0,4,6,2,3,8),(1,5,7))$, $((1,3,7,2,4,8),(0,5,6))$.

$\boxed{(K_{25}-K_5)-(K_{10}-F)}$ Let the vertex set be $\{i_j \mid 0 \le i \le 4, 1 \le j \le 5\}$, with the $K_5$ on $\{0_1,1_1,2_1,3_1,4_1\}$, the $K_{10}-F$ on $\{i_j \mid 0 \le i \le 4,\ j=2,3\}$ and $F = \{(0_2,0_3),(1_2,1_3),(2_2,2_3),(3_2,3_3),(4_2,4_3)\}$. Develop the following mod $(5,-)$:

$C_4 \cup C_3 \cup C_3$: $((0_1,0_2,1_1,2_2),(2_1,0_3,0_4),(3_1,1_2,2_4))$,
$((0_1,0_3,1_1,2_3),(2_1,2_4,3_4),(3_1,0_4,0_5))$,
$((0_1,0_5,2_1,1_5),(0_2,0_3,2_4),(1_2,0_4,2_5))$,
$((0_2,0_4,2_2,0_5),(4_2,1_5,3_5),(0_3,1_4,2_5))$,
$((0_3,3_4,1_4,4_5),(2_3,2_5,3_5),(3_3,2_4,1_5))$.

$C_5 \cup C_5$: $((0_1,0_2,1_1,2_2,2_3),(2_1,4_2,0_4,3_1,1_3))$,
$((0_1,3_2,0_4,1_1,1_3),(2_1,2_4,4_1,0_5,3_4))$,
$((0_1,0_5,1_1,4_5,2_5),(0_2,0_4,1_2,4_4,3_5))$,
$((0_2,0_5,3_2,2_5,1_5),(0_3,0_4,1_3,2_4,3_4))$,
$((0_3,2_4,0_5,1_3,2_5),(3_3,1_5,0_4,3_4,3_5))$.

$C_6 \cup C_4$: $((0_1,0_2,1_1,2_2,4_1,0_3),(2_1,4_2,4_3,0_4))$,
$((0_1,2_3,3_1,1_3,0_4,1_4),(1_1,3_4,4_1,0_5))$,
$((0_1,0_4,0_2,1_4,2_2,0_5),(1_1,3_5,3_2,4_5))$,
$((0_2,2_4,4_2,1_5,0_3,4_5),(1_3,1_4,3_3,3_5))$,
$((0_3,2_4,0_4,0_5,1_4,3_5),(3_4,1_5,2_5,4_5))$.

$C_7 \cup C_3$: $((0_1,0_2,1_1,2_2,4_1,1_2,1_3),(2_1,0_3,0_4))$,
$((0_1,0_3,1_1,3_3,0_4,3_1,4_4),(2_1,2_4,0_5))$,
$((0_1,0_5,1_1,3_5,0_2,0_4,1_5),(1_2,2_4,3_4))$,
$((0_2,3_4,4_2,0_5,1_2,1_5,2_5),(0_3,1_4,4_4))$,
$((0_3,3_4,0_5,0_4,4_5,4_3,2_5),(2_3,1_5,3_5))$.

$\boxed{K_{25}}$ Let the vertex set be $\{i_j \mid 0 \le i \le 4, 1 \le j \le 5\}$. Develop the following mod $(5,-)$:

$C_4 \cup C_3 \cup C_3$:
$((0_1,1_1,3_1,0_2),(2_1,1_2,3_2),(4_1,2_2,0_3))$, $((0_1,0_3,1_1,3_3),(2_1,0_4,1_4),(3_1,3_4,0_5))$,
$((0_1,1_4,4_1,0_5),(3_1,1_5,2_5),(0_2,1_2,0_3))$, $((0_2,1_3,4_2,0_4),(1_2,3_4,1_5),(2_2,1_4,0_5))$,
$((0_2,3_4,0_3,1_5),(1_2,0_5,3_5),(1_3,2_3,4_5))$, $((0_3,2_3,3_4,4_5),(1_3,1_4,1_5),(3_3,0_4,2_4))$.

$C_5 \cup C_5$:
$((0_1,1_1,3_1,0_2,1_2),(2_1,2_2,4_1,3_2,0_3))$, $((0_1,0_3,1_1,2_3,0_4),(2_1,4_3,0_2,2_2,1_4))$,
$((0_1,1_4,3_1,0_4,0_5),(1_1,2_5,0_2,0_3,3_5))$, $((0_1,3_5,0_2,3_3,4_5),(1_2,2_3,0_3,1_3,1_4))$,
$((0_2,1_4,3_2,0_4,1_5),(4_2,3_5,1_3,2_4,4_5))$, $((0_3,2_4,3_3,2_5,0_5),(1_4,3_4,4_4,3_5,4_5))$.

$C_6 \cup C_4$:
$((0_1,1_1,3_1,0_2,2_1,1_2),(4_1,4_2,2_2,0_3))$, $((0_1,0_3,1_1,3_3,1_2,0_4),(2_1,1_4,3_1,4_4))$,
$((0_1,3_3,2_2,1_2,0_3,0_5),(1_1,2_5,4_1,3_5))$, $((0_2,0_3,1_3,3_3,0_4,1_4),(1_2,3_4,3_2,0_5))$,
$((0_2,3_4,0_3,0_4,1_3,0_5),(1_2,2_5,1_4,4_5))$, $((0_3,1_4,3_4,0_5,3_3,1_5),(4_4,3_5,2_5,4_5))$.

$C_7 \cup C_3$:
$((0_1,1_1,3_1,0_2,2_1,1_2,0_3),(4_1,4_2,1_3))$, $((0_1,1_2,0_2,2_2,0_3,1_1,0_4),(2_1,3_3,3_4))$,
$((0_1,3_3,2_2,2_3,0_3,1_3,2_4),(1_1,4_4,0_5))$, $((0_1,0_5,2_1,3_5,0_2,0_4,2_5),(1_2,2_4,3_4))$,
$((0_2,3_4,4_2,0_5,1_2,1_5,2_5),(0_3,2_4,4_4))$, $((0_3,3_4,1_5,1_3,3_5,4_4,4_5),(4_3,0_5,2_5))$.

**$K_{21}$** Let the vertex set be $\mathbb{Z}_{21}$. Develop the following mod 21:

$C_4 \cup C_3 \cup C_3$: $((0,1,3,8),(2,5,11),(4,14,18))$.
$C_5 \cup C_5$: $((0,1,3,9,4),(5,8,15,7,16))$.
$C_6 \cup C_4$: $((0,1,3,6,2,7),(4,14,8,16))$.
$C_7 \cup C_3$: $((0,1,3,6,10,2,9),(4,14,19))$.

**$K_{41}$** Let the vertex set be $\mathbb{Z}_{41}$. Develop the following mod 41:

$C_4 \cup C_3 \cup C_3$:
$((27,32,39,40),(10,26,36),(23,34,37))$, $((0,2,8,17),(1,5,23),(3,11,32))$.

$C_5 \cup C_5$:
$((0,7,15,1,18),(2,14,29,4,23))$, $((0,1,3,6,10),(2,7,13,4,15))$.

$C_6 \cup C_4$:
$((0,1,3,6,2,7),(4,10,18,27))$, $((0,10,21,1,13,26),(2,16,35,18))$.

$C_7 \cup C_3$:
$((0,1,3,6,2,7,13),(4,11,19))$, $((0,9,19,7,24,1,20),(2,13,27))$.

**$K_{45}$** Let the vertex set be $\{i_j \mid 0 \le i \le 8, 1 \le j \le 5\}$. Develop the following mod $(9,-)$:

$C_4 \cup C_3 \cup C_3$:
$((0_1,1_1,3_1,0_2),(2_1,5_1,1_2),(4_1,8_1,2_2))$,
$((0_1,4_3,6_1,0_4),(1_1,6_3,7_3),(2_1,1_4,3_4))$,
$((0_1,1_5,3_1,6_5),(1_1,3_5,5_5),(0_2,1_2,3_2))$,
$((0_2,3_3,5_2,0_4),(1_2,7_3,2_4),(2_2,1_4,4_4))$,
$((0_2,4_5,7_2,5_5),(0_3,0_4,5_4),(1_3,2_4,0_5))$,
$((0_3,2_4,3_3,0_5),(1_3,4_4,3_5),(5_3,3_4,6_5))$,
$((0_1,1_2,6_1,0_3),(1_1,3_2,2_3),(2_1,1_3,4_3))$,
$((0_1,2_4,4_1,0_5),(1_1,5_4,6_4),(2_1,8_4,1_5))$,
$((0_2,4_2,0_3,2_3),(1_2,1_3,5_3),(2_2,3_3,0_4))$,
$((0_2,3_4,6_2,0_5),(1_2,6_4,2_5),(2_2,1_5,4_5))$,
$((0_3,3_5,3_4,7_5),(1_3,5_5,6_5),(7_4,4_5,8_5))$.

$C_5 \cup C_5$:
$((0_1,1_2,5_1,7_2,0_3),(1_1,2_3,3_1,1_3,3_3))$,
$((0_1,1_4,4_1,0_4,3_4),(1_1,5_4,0_2,1_2,0_5))$,
$((0_1,5_5,0_2,4_2,6_5),(1_2,7_2,1_3,2_2,2_3))$,
$((0_2,2_4,3_2,0_4,4_4),(1_2,8_4,5_2,2_5,8_5))$,
$((0_3,4_4,7_3,3_4,2_5),(1_3,0_5,2_3,5_5,7_5))$,
$((0_1,1_1,3_1,6_1,0_2),(2_1,7_1,2_2,4_1,1_2))$,
$((0_1,3_3,6_1,1_3,5_3),(1_1,0_4,2_1,2_4,3_4))$,
$((0_1,0_5,2_1,3_5,2_5),(1_1,4_5,0_2,2_2,5_5))$,
$((0_2,4_3,6_2,2_3,0_4),(1_2,7_3,1_3,0_3,2_4))$,
$((0_2,0_5,0_3,0_4,1_5),(1_3,2_4,3_3,6_4,2_5))$,
$((0_3,4_5,0_4,7_4,5_5),(1_4,1_5,4_4,7_5,3_5))$.

$C_6 \cup C_4$:
$((0_1,3_2,4_1,8_2,0_2,0_3),(1_1,2_3,3_1,5_3))$,
$((0_1,3_4,5_1,2_4,0_2,0_5),(1_1,2_5,3_1,5_5))$,
$((0_2,4_2,0_3,2_2,3_3,2_3),(1_2,5_3,8_2,0_4))$,
$((0_2,3_5,0_3,2_3,6_3,8_5),(1_3,4_3,0_4,1_4))$,
$((0_3,3_4,0_4,2_4,6_4,1_5),(1_3,0_5,4_4,5_5))$,
$((0_1,1_1,3_1,6_1,2_1,0_2),(4_1,1_2,5_1,6_2))$,
$((0_1,3_3,5_1,1_3,4_1,0_4),(1_1,2_4,3_1,5_4))$,
$((0_1,3_5,5_1,1_5,0_2,6_5),(1_2,3_2,6_2,0_3))$,
$((0_2,0_4,2_2,5_4,1_2,6_4),(3_2,1_5,5_2,7_5))$,
$((0_3,1_4,2_3,0_4,3_3,0_5),(1_3,3_4,8_3,6_5))$,
$((0_4,0_5,1_4,3_5,1_5,6_5),(4_4,2_5,8_5,7_5))$.

$C_7 \cup C_3$:
$((6_1,5_4,6_5,8_1,2_2,1_3,5_5),(3_1,8_4,7_5))$,
$((6_2,2_5,6_3,6_4,5_3,1_4,7_5),(3_1,0_3,8_5))$,
$((3_2,8_3,6_2,4_5,7_3,1_4,7_4),(5_1,1_3,6_3))$,
$((0_1,1_1,3_1,6_1,2_1,0_2,1_2),(4_1,6_2,8_2))$,
$((0_1,0_4,3_1,5_4,4_2,4_3,2_5),(0_2,2_4,0_5))$,
$((7_1,4_5,2_5,8_3,5_2,8_1,8_5),(0_5,1_5,6_5))$,
$((4_1,8_4,3_2,4_3,5_5,5_3,7_5),(0_4,4_4,4_5))$,
$((0_2,0_4,2_3,5_2,7_5,8_2,6_4),(2_2,5_4,8_5))$,
$((5_1,4_2,8_2,8_1,6_4,4_4,0_3),(7_1,5_3,6_3))$,
$((0_1,5_2,2_2,0_3,6_1,6_3,3_4),(1_1,3_3,2_4))$,
$((0_2,4_3,6_3,0_3,2_4,1_4,3_5),(1_2,0_4,5_5))$.

$\boxed{K_{3(10)}}$ Let the vertex set be $\{i_j \mid 0 \le i \le 4, 1 \le j \le 6\}$, with the vertices with subscripts $2i-1$ and $2i$ in the $i$'th partition, $i = 1,2,3$. Develop the following mod 5:

$C_4 \cup C_3 \cup C_3$:
$((1_1,0_4,0_6,2_4),(1_2,0_3,2_6),(4_2,2_3,0_5))$,
$((2_1,2_5,2_3,3_5),(4_1,0_3,4_6),(2_2,1_4,0_5))$,
$((2_1,0_3,3_1,0_6),(0_2,2_3,4_5),(3_2,3_4,2_6))$,
$((0_1,0_3,1_1,3_4),(2_1,2_4,0_5),(0_2,1_3,2_6))$,
$((0_1,2_5,3_3,1_6),(0_2,0_3,0_6),(1_2,2_4,4_6))$,
$((0_1,4_5,3_4,4_6),(0_2,2_4,2_5),(1_2,4_4,1_5))$.

$C_5 \cup C_5$:
$((4_1,0_3,4_5,2_3,0_6),(2_2,1_6,1_3,3_2,4_6))$,
$((2_1,0_4,1_5,0_2,3_5),(2_2,2_3,3_6,4_4,4_5))$,
$((0_1,0_3,1_1,3_3,3_5),(2_1,1_4,4_1,2_3,1_6))$,
$((0_1,0_4,0_2,1_3,2_5),(1_1,2_4,1_2,3_3,1_5))$,
$((0_1,4_5,0_2,2_4,0_6),(4_1,1_6,1_2,0_4,2_6))$,
$((0_2,4_3,1_6,1_4,0_5),(1_2,4_4,1_5,3_4,4_6))$.

$C_6 \cup C_4$:
$((1_3,1_5,2_4,3_5,0_4,2_6),(4_2,4_3,4_6,4_4))$,
$((3_2,0_3,1_5,2_3,4_2,3_6),(0_2,1_4,2_6,4_4))$,
$((1_1,1_3,0_6,2_2,3_6,4_4),(0_2,3_4,1_2,1_5))$,
$((0_1,1_3,2_1,0_3,3_1,0_4),(1_1,0_5,4_1,1_5))$,
$((0_1,1_4,2_1,0_5,1_2,3_6),(1_1,0_6,4_1,1_6))$,
$((0_2,1_3,2_2,0_5,0_4,2_5),(3_3,1_5,4_3,1_6))$.

$C_7 \cup C_3$:
$((3_2,3_5,1_3,4_5,0_4,4_2,1_6),(1_1,4_4,0_5))$,
$((2_2,2_6,2_3,2_5,3_3,4_2,3_6),(1_1,1_4,1_6))$,
$((0_2,3_4,1_6,3_3,2_6,1_3,2_5),(1_2,1_4,4_5))$,
$((0_1,0_3,1_1,2_3,4_1,1_3,3_6),(2_1,3_4,4_6))$,
$((0_1,2_4,1_6,2_1,2_5,1_1,3_5),(4_1,3_4,0_6))$,
$((0_2,0_3,2_2,4_3,3_2,2_5,2_4),(4_2,3_4,0_5))$.