The Maximum Toughness of Sesqui-Cubic Graphs

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Abstract

We explore the maximum possible toughness among graphs with n vertices and m edges in the cases in which $\lceil \frac{3n}{2} \rceil \le m < 2n$. In these cases, it is shown that the maximum toughness lies in the interval $[\frac{4}{3}, \frac{3}{2}]$. Moreover, if $\lceil \frac{3n}{2} \rceil + 2 \le m < 2n$, then the value $\frac{3}{2}$ is achieved. However, if $m \in \{\lceil \frac{3n}{2} \rceil, \lceil \frac{3n}{2} \rceil + 1\}$, then the maximum toughness can be strictly less than $\frac{3}{2}$. This provides an infinite family of graphs for which the maximum toughness is not half of the maximum connectivity. The values of maximum toughness are computed for all $1 \le n \le 12$, and some open problems are presented.

1 Introduction

In this paper, we adopt and freely use the notation and terminology from [4]. A $K_{1,3}$ subgraph is an induced subgraph that is isomorphic to $K_{1,3}$. Its degree 3 vertex is called a $K_{1,3}$ center. The toughness of a non-complete graph G is

$$\tau(G) = \min\{\frac{|S|}{\omega(G-S)} \ : S \subseteq V \text{ and } \omega(G-S) > 1\},$$

while $\tau(K_n) = \frac{n-1}{2}$. Among all (n,m)-graphs, the maximum toughness is denoted by $T_n(m)$. An (n,m)-graph G is said to be maximally tough if $\tau(G) = T_n(m)$. A graph on n vertices is said to be sesqui-cubic if $n-1$ of the vertices have degree 3 and the remaining vertex has degree 3 or 4. That is, the graph is as close to cubic as possible given the parity of $3n$.

Chvátal [2] gives an important upper bound for toughness in terms of connectivity.

Theorem 1.1 ([2]). $\tau(G) \le \frac{\kappa(G)}{2}$.

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A connection between τ and κ that is stronger than Theorem 1.1 is given by Matthews and Sumner.

Theorem 1.2 ([7]). *If G is $K_{1,3}$-free, then $\tau(G) = \frac{\kappa(G)}{2}$.*

Theorem 1.2 will be used heavily to show that the graphs we construct are maximally tough.

2 Computing $T_n(m)$ for $\lceil \frac{3n}{2} \rceil \leq m < 2n$

The task of computing $T_n(m)$ for $\lceil \frac{3n}{2} \rceil \leq m < 2n$ was started and partially completed by Chvátal.

Theorem 2.1 ([2]). *For n even, $T_n(\frac{3n}{2}) = \frac{3}{2}$ if and only if either $n = 4$ or $n \equiv 0 \bmod 6$.*

Jackson and Katerinis further showed that the converse of Theorem 1.2 holds when $m = \frac{3n}{2}$.

Theorem 2.2 ([6]). *A cubic graph is $\frac{3}{2}$-tough if and only if it is 3-connected and $K_{1,3}$-free.*

2.1 Restrictions on maximally tough sesqui-cubic graphs

In light of Theorem 2.2, the most natural place to look for maximally tough (n, m)-graphs with $\lceil \frac{3n}{2} \rceil \leq m < 2n$ is among 3-connected $K_{1,3}$-free graphs. Such graphs only exist under certain conditions. Our proof of the following theorem is a generalization of Chvátal's proof of the necessity of $n \equiv 0 \bmod 6$ in Theorem 2.1.

Theorem 2.3. *If a sesqui-cubic graph G on n vertices is 3-connected and $K_{1,3}$-free, then $n = 4$ or $n \equiv 0$ or $5 \bmod 6$.*

Proof. Let w be the degree 4 vertex in the case that n is odd. Let a, b, c, and d be the neighbors of w. Since w is not a $K_{1,3}$ center, we may assume that $\{a, b\}$ is an edge. We claim that, if $n > 5$, then $\{c, d\}$ is the only other edge in the graph induced by the neighbors of w. This will then imply that the graph induced by $\{w, a, b, c, d\}$ must be as pictured in Figure 1.

Suppose toward a contradiction that $\{b, c\}$ is an edge. If $\{a, c\}$ were also an edge, then w would be a cut point, contradicting the fact that G is 3-connected. Since w is not a $K_{1,3}$ center, we may assume that $\{a, d\}$ is an edge. If $n = 5$, then the maximally tough graph pictured in Figure 2 is obtained. If $n > 5$, then c and d would form a cut set, contradicting the fact that G is 3-connected. We conclude that $\{b, c\}$ is not an edge. By symmetry, none of $\{a, c\}$, $\{a, d\}$, or $\{b, d\}$ can be edges either. Since w is

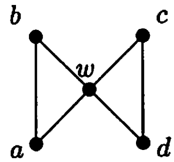

Figure 1: Around the degree 4 vertex

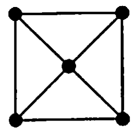

Figure 2: The maximally tough $(5,8)$-graph showing $T_5(8) = \frac{3}{2}$

not a $K_{1,3}$ center, $\{c,d\}$ must be the only other edge in $N(w)$ when $n > 5$ as claimed.

Since G is $K_{1,3}$-free, each degree 3 vertex must be adjacent to a triangle. We have already observed that w is adjacent to exactly two triangles, and they meet only at the vertex w. We claim that, besides those two triangles, all others are disjoint.

Certainly, any two triangles not containing w cannot meet only at a vertex, since that would yield a second degree 4 vertex. Consequently, two such meeting triangles must share an edge. Figure 3 pictures this situation, where the four vertices involved have been labeled u, v, x, and y. If $n = 4$,

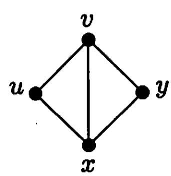

Figure 3: Triangles sharing an edge.

then the maximally tough graph $G = K_4$ is obtained. If $n > 4$, then u and y would form a cut set, contradicting the fact that G is 3-connected.

We conclude that, besided the two triangles in Figure 1, all other triangles are disjoint. If n is even, then w does not exist and the vertex set can be partitioned into a bunch of triangles. In that case, $n \equiv 0 \mod 6$. If n is odd, then the vertex set can be partitioned into a bunch of triangles plus the 5 vertices in Figure 1. In that case, $n \equiv 5 \mod 6$. □

Since $K_{1,3}$-free graphs are not always available, the next natural place to look for maximally tough (n,m)-graphs with $\lceil \frac{3n}{2} \rceil \le m < 2n$ is among graphs with as few $K_{1,3}$ centers as possible. In [5], Goddard noted that the proof of Theorem 1.2 given in [7] can be extended to show that a 3-connected cubic graph is $\frac{4}{3}$-tough if there is at most one $K_{1,3}$ center and $\frac{5}{4}$-tough if there are at most two $K_{1,3}$ centers. We further extend a version of this idea to sesqui-cubic graphs.

Theorem 2.4. *If G is sesqui-cubic, 3-connected, $K_{1,4}$-free, and has either*

(i) at most one $K_{1,3}$ center, or

(ii) two adjacent $K_{1,3}$ centers,

then G is $\frac{4}{3}$-tough.

Proof. Our proof closely follows the proof of Theorem 1.2 given in [7]. However, the necessary adaptations to allow an odd number of vertices warrant a complete proof being given here.

Suppose that S is a set of vertices for which $\tau(G) = \frac{|S|}{\omega(G-S)}$, and denote the components of $G - S$ by $C_1, \ldots, C_\omega$. Since G is 3-connected, for each $i \ne j$, $u \in C_i$, and $v \in C_j$, there are at least 3 internally disjoint paths from u to v. Since each such path must go through S, there must be at least 3 edges from each C_i to distinct vertices in S. Summing over all i shows that there are at least 3ω edges from $G - S$ to S, such that each C_i is adjacent to at least 3 vertices of S.

Each non-$K_{1,3}$ center is adjacent to at most 2 components of $G - S$. Since G is $K_{1,4}$-free, a $K_{1,3}$ center could be adjacent to at most 3 components of $G - S$. If there are two adjacent $K_{1,3}$ centers (and both are in S), then each can be adjacent to at most 2 components of $G - S$. Thus, if we count at most one edge from each C_i to any particular vertex in S, then there are at most $2(|S| - 1) + 3 = 2|S| + 1$ edges from $G - S$ to S.

We conclude that $3\omega \le 2|S| + 1$, and hence $\frac{3}{2} - \frac{1}{2\omega} \le \frac{|S|}{\omega}$. If $\omega = 2$, then $\frac{|S|}{2} = \tau(G) \le \frac{\kappa(G)}{2} = \frac{3}{2}$. Since $\kappa(G) = 3$, this implies that $|S| = 3$ and hence $\tau(G) = \frac{3}{2} > \frac{4}{3}$. If $\omega \ge 3$, then $\tau(G) = \frac{|S|}{\omega} \ge \frac{3}{2} - \frac{1}{2\omega} \ge \frac{3}{2} - \frac{1}{6} = \frac{4}{3}$. □

The restrictions on n given in Theorem 2.1 can be reduced to allow the possibilities of sesqui-cubic graphs on an odd number of vertices.

Theorem 2.5. *If $T_n(\lceil \frac{3n}{2} \rceil) = \frac{3}{2}$, then*

(i) $n = 4$, or

(ii) $n \equiv 0$ or $5 \bmod 6$.

Our proof of Theorem 2.5 is an extended version of the proof of Theorem 2.1 given by Chvátal. Chvátal's proof uses Brooks' theorem [1] to guarantee a 3-coloring of a cubic graph. Consequently, we need the following result of Dirac on graph colorings of non-regular graphs and its corollary.

Theorem 2.6 ([3]). *Let G be a graph on n vertices such that $n-1$ of the vertices have degree 3 and one vertex has degree $d \geq 3$. If G is 2-connected and 4-colorable but not 3-colorable, then $n \leq 3d-5$.*

Corollary 2.7. *All 3-connected sesqui-cubic graphs (except K_4) are 3-colorable.*

Proof. Let G be a sesqui-cubic graph on n vertices. It suffices to consider the case in which G has one vertex of degree $d = 4$, and hence Theorem 2.6 tells us that $n \leq 7$. All graphs on 7 or fewer vertices are pictured in [8] where they are grouped according to their degree sequence. There are only four sesqui-cubic graphs on 7 vertices and only the one pictured in Figure 4 is not 3-colorable. However, that graph is not 3-connected. The two cubic

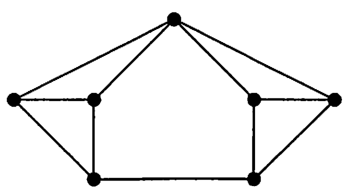

Figure 4: The unique non-complete 4-chromatic sesqui-cubic graph

graphs on 6 vertices and the unique sesqui-cubic graph on 5 vertices are easily seen to be 3-colorable. Of course, K_4 is not 3-colorable. □

The proof of Theorem 2.5. Certainly, $T_4(6) = \tau(K_4) = \frac{3}{2}$. Hence, it suffices to assume that $n > 4$ and to let G be a $\frac{3}{2}$-tough $(n, \lceil\frac{3n}{2}\rceil)$-graph. By Theorem 1.1, G must be 3-connected. So G has $n-1$ vertices of degree 3 and one vertex of degree 3 or 4. Let w denote the degree 4 vertex if it exists.

Note that G cannot be 2-colored, since that would imply that G is bipartite and $\tau(G) \leq 1$. By Corollary 2.7, G is 3-colorable. Let A, B, and C be the color classes of a 3-coloring such that $|A| \leq |B| \leq |C|$ and $|A|$ is chosen as small as possible. Further, if $|A| = |B|$, then we choose that $w \notin A$.

Observe that each vertex $a \in A$ is adjacent to some $b \in B$. Otherwise, a 3-coloring could be chosen with color classes $A' = A - \{a\}$, $B' = B \cup \{a\}$, and $C' = C$. The fact that $|A'| < |A|$ would contradict our choices of A, B, and C. Similarly, each $a \in A$ is adjacent to some $c \in C$.

Define subsets A_B and A_C of A by $a \in A_B$ if a is adjacent to exactly one vertex in B and $a \in A_C$ if a is adjacent to exactly one vertex in C. This accounts for all of the vertices of A except possibly w. In any case, note that the subgraph induced by $B \cup A_B$ has exactly $|B|$ components and the subgraph induced by $C \cup A_C$ has exactly $|C|$ components. Also, note that $|C| \geq |B| \geq 2$. Otherwise $|A \cup B| \leq 2$ would contradict the fact that each vertex of C is adjacent to at least 3 vertices of $A \cup B$.

Case 1: $A_B \cup A_C \neq A$.

This implies that $w \in A$, and hence $|A| < |B|$ and n is odd. Note that

$$\omega(G - (C \cup A_C \cup \{w\})) = |B| \text{ and } \omega(G - (B \cup A_B \cup \{w\})) = |C|.$$

Since G is $\frac{3}{2}$-tough, $|C \cup A_C \cup \{w\}| \geq \frac{3}{2}|B|$ and $|B \cup A_B \cup \{w\}| \geq \frac{3}{2}|C|$. Adding these inequalities together gives $|A| + |B| + |C| + 1 \geq \frac{3}{2}(|B| + |C|)$. Hence, $|A| + 1 \geq \frac{1}{2}(|B| + |C|) \geq |B| > |A|$. This forces $|B| = |A| + 1$. The equality $|A| + 1 = \frac{1}{2}(|A| + 1 + |C|)$ then gives that $|C| = |A| + 1$. Hence, $n = |A| + |B| + |C| = 3|A| + 2 \equiv 2 \mod 3$. Since n is odd, $n \equiv 5 \mod 6$.

Case 2: $A_B \cup A_C = A$.

In this case,

$$\omega(G - (C \cup A_C)) = |B| \text{ and } \omega(G - (B \cup A_B)) = |C|.$$

Since G is $\frac{3}{2}$-tough, $|C \cup A_C| \geq \frac{3}{2}|B|$ and $|B \cup A_B| \geq \frac{3}{2}|C|$. Adding these inequalities together gives $|A| + |B| + |C| \geq \frac{3}{2}(|B| + |C|)$. Hence, $|A| \geq \frac{1}{2}(|B| + |C|)$. Since $|A| \leq |B| \leq |C|$, it follows that $|A| = |B| = |C|$. Therefore $n \equiv 0 \mod 3$. Moreover, we now have

$$|C \cup A_C| = \frac{3}{2}|B| = \frac{3}{2}|C| = |B \cup A_B|$$

and therefore $|A_C| = |A_B|$. Hence, n is even and $n \equiv 0 \mod 6$. □

2.2 Constructing maximally tough sesqui-cubic graphs

For each $n > 4$, we construct a sesqui-cubic graph $SC(n)$ on n vertices. In some cases, $SC(n)$ is $\frac{3}{2}$-tough and hence maximally tough. In other cases, one or two edges are added to form new graphs which are $\frac{3}{2}$-tough. The definition of $SC(n)$ depends on the congruence class of n modulo 6. However, the majority of the construction is common for all n and is presented first.

Let the vertex set of $SC(n)$ be given by $V = \{0, 1, \ldots, n-1\}$. The edge set E is most easily described as a disjoint union $E = E_1 \cup E_2 \cup E_3$. First, let $E_1 = \{ \{i, i+1\} : 0 \leq i \leq n-1\}$. We are taking addition modulo n.

So $SC(n)$ contains the n-cycle C_n. Let

$$\begin{aligned} E_2 &= \{\ \{3i+2, 3i+4\}\ : 0 \le i \le \lfloor \frac{n-10}{6} \rfloor \} \cup \\ &\quad \{\ \{n-3i-2, n-3i-4\}\ : 0 \le i \le \lfloor \frac{n-10}{6} \rfloor \} \cup \\ &\quad \{\ \{n-1, 1\}\ \} \cup \{\ \{3i+3, n-3i-3\}\ : 0 \le i \le \lfloor \frac{n-10}{6} \rfloor \}. \end{aligned}$$

The set E_3 depends on the value of n modulo 6. If $n \equiv 0 \bmod 6$, then

$$E_3 = \{\ \{0, \frac{n}{2}\}, \{\frac{n}{2}-1, \frac{n}{2}+1\}\ \}.$$

If $n \equiv 1 \bmod 6$, then

$$E_3 = \{\ \{0, \lfloor \frac{n}{2} \rfloor\}, \{0, \lceil \frac{n}{2} \rceil\}, \{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1\}\ \}.$$

If $n \equiv 2 \bmod 6$, then

$$E_3 = \{\ \{0, \frac{n}{2}\}, \{\frac{n}{2}-1, \frac{n}{2}+1\}, \{\frac{n}{2}-2, \frac{n}{2}+2\}\ \}.$$

If $n \equiv 3 \bmod 6$, then

$$E_3 = \{\ \{0, \lfloor \frac{n}{2} \rfloor - 1\}, \{0, \lceil \frac{n}{2} \rceil + 1\}, \{\lfloor \frac{n}{2} \rfloor - 2, \lfloor \frac{n}{2} \rfloor\}, \{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 2\}\ \}.$$

If $n \equiv 4 \bmod 6$, then

$$E_3 = \{\ \{0, \frac{n}{2}\}\ \}.$$

If $n \equiv 5 \bmod 6$, then

$$E_3 = \{\ \{0, \lfloor \frac{n}{2} \rfloor\}, \{0, \lceil \frac{n}{2} \rceil\}\ \}.$$

The graphs $SC(n)$ for $5 \le n \le 22$ are pictured in Figures 5 through 10, where vertex 0 is always the topmost vertex.

One of the important properties of the graphs $SC(n)$ is that they are 3-connected. This is proven in Appendix A by using a characterization of 3-connected graphs due to Tutte [9]. Here, the 3-connectivity of $SC(n)$ is used in the proof of the following theorem.

Theorem 2.8. *Let $n > 4$.*

(a) If $n \equiv 0$ or $5 \bmod 6$, then $T_n(\lceil \frac{3n}{2} \rceil) = \tau(SC(n)) = \frac{3}{2}$.

(b) If $n \equiv 1, 2, 3,$ or $4 \bmod 6$, then $T_n(\lceil \frac{3n}{2} \rceil) \ge \tau(SC(n)) = \frac{4}{3}$.

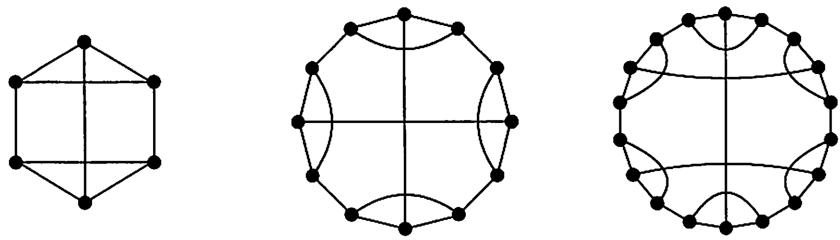

Figure 5: $SC(n)$ for $n \equiv 0 \mod 6$

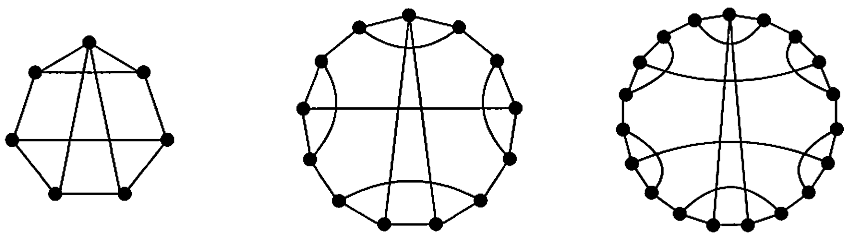

Figure 6: $SC(n)$ for $n \equiv 1 \mod 6$

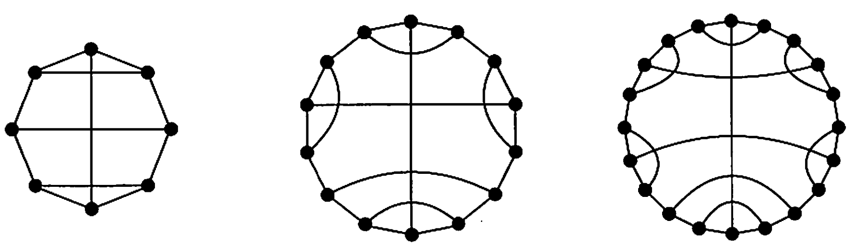

Figure 7: $SC(n)$ for $n \equiv 2 \mod 6$

Remark 2.9. *Theorems 2.5 and 2.8 together give that, if* $n > 4$ *and* $n \equiv 1, 2, 3, or\, 4 \mod 6$, *then* $\frac{4}{3} \leq T_n(\lceil \frac{3n}{2} \rceil) < \frac{3}{2}$.

Remark 2.10. *In the case that* $n \equiv 0 \mod 6$, *our proof of Theorem 2.8 provides an alternative proof to the one given by Chvátal [2] for the corresponding result in Theorem 2.1. Chvátal's proof uses the notion of graph inflations.*

The proof of Theorem 2.8. Since Theorem A.3 tells us that the sesqui-cubic graph $SC(n)$ is 3-connected, our proof is a simple application of Theorems

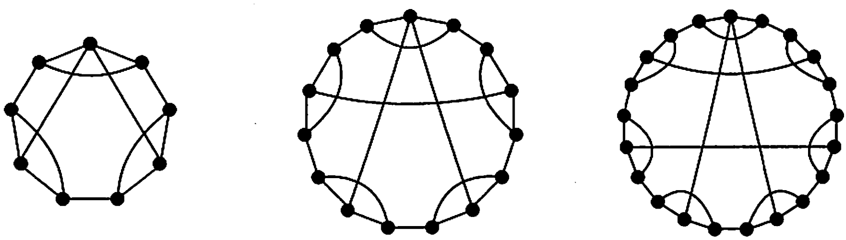

Figure 8: $SC(n)$ for $n \equiv 3 \mod 6$

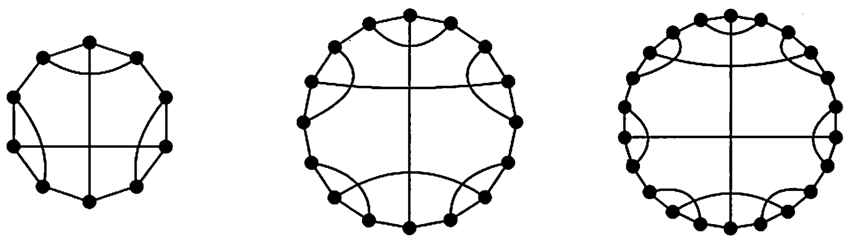

Figure 9: $SC(n)$ for $n \equiv 4 \mod 6$

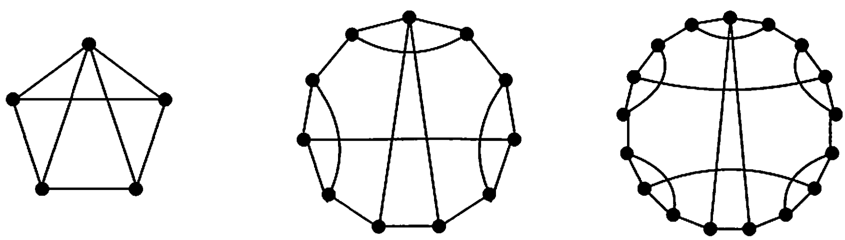

Figure 10: $SC(n)$ for $n \equiv 5 \mod 6$

1.2 and 2.4. It is straightforward to check that $SC(n)$ is $K_{1,4}$-free and to count the number of $K_{1,3}$ centers for each congruence class of n mod 6. If $n \equiv 0$ or 5 mod 6, then $SC(n)$ is $K_{1,3}$-free and hence $\frac{3}{2}$-tough. For $n \equiv 1, 2, 3,$ or 4 mod 6, we also give a disconnecting set S which demonstrates that $\tau(SC(n)) \leq \frac{|S|}{\omega(SC(n)-S)} = \frac{4}{3}$. If $n \equiv 3$ mod 6, then $SC(n)$ has one $K_{1,3}$ center at $v = 0$, and we choose $S = \{0, \lfloor \frac{n}{2} \rfloor - 3, \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil + 2\}$. If $n \equiv 4$ mod 6, then $SC(n)$ has one $K_{1,3}$ center at $v = \frac{n}{2}$, and we choose $S = \{\frac{n}{2} - 4, \frac{n}{2} - 2, \frac{n}{2}, \frac{n}{2} + 3\}$. If $n \equiv 1$ mod 6, then the two $K_{1,3}$ centers $v' = \lfloor \frac{n}{2} \rfloor - 1$ and $v'' = \lceil \frac{n}{2} \rceil + 1$ are adjacent, and we choose $S = \{0, \lfloor \frac{n}{2} \rfloor - 2, \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil + 1\}$.

If $n \equiv 2 \bmod 6$, then the two $K_{1,3}$ centers $v' = \frac{n}{2} - 2$ and $v'' = \frac{n}{2} + 2$ are adjacent, and we then choose $S = \{0, \frac{n}{2} - 2, \frac{n}{2} + 1, \frac{n}{2} + 3\}$. □

Theorem 2.8 tells us that not all of the graphs $SC(n)$ are $\frac{3}{2}$-tough. Hence, for $n \equiv 1, 2, 3, \text{or } 4 \bmod 6$, we aim to increase the toughness of $SC(n)$ by adding an edge e' and thereby defining a new graph $SC'(n)$. If $n \equiv 1 \bmod 6$, then define $e' = \{\lfloor \frac{n}{2} \rfloor - 2, \lfloor \frac{n}{2} \rfloor\}$. If $n \equiv 2 \bmod 6$, then $e' = \{\frac{n}{2} - 2, \frac{n}{2} + 1\}$. If $n \equiv 3 \bmod 6$, then $e' = \{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1\}$. Finally, if $n \equiv 4 \bmod 6$, then $e' = \{\frac{n}{2} - 1, \frac{n}{2} + 1\}$.

Theorem 2.11. *Let* $n > 4$.

(a) If $n \equiv 3$ *or* $4 \bmod 6$, *then* $T_n(\lceil \frac{3n}{2} \rceil + 1) = \tau(SC'(n)) = \frac{3}{2}$.

(b) If $n \equiv 1$ *or* $2 \bmod 6$, *then* $T_n(\lceil \frac{3n}{2} \rceil + 1) \geq \tau(SC'(n)) = \frac{4}{3}$.

Proof. This proof follows that same basic argument as that of Theorem 2.8. Clearly, $SC'(n)$ is 3-connected. If $n \equiv 3$ or $4 \bmod 6$, then $SC'(n)$ is $K_{1,3}$-free, and Theorem 1.2 applies. If $n \equiv 1$ or $2 \bmod 6$, then the same disconnecting set S used in the proof of Theorem 2.8 demonstrates that $\tau(SC'(n)) \leq \frac{4}{3}$. Of course, $\tau(SC'(n)) \geq \tau(SC(n)) = \frac{4}{3}$. □

Remark 2.12. *If* $n \equiv 1$ *or* $2 \bmod 6$, *then* $SC'(n)$ *has one* $K_{1,3}$ *center (at* $v = \lceil \frac{n}{2} \rceil + 1$ *if* $n \equiv 1 \bmod 6$ *and at* $v = \frac{n}{2} - 2$ *if* $n \equiv 1 \bmod 6$*).*

Conjecture 2.13. *The graphs* $SC(n)$ *and* $SC'(n)$ *are maximally tough.*

Theorem 2.11 tells us that even the graphs $SC'(n)$ are not all $\frac{3}{2}$-tough. Hence, for $n \equiv 1$ or $2 \bmod 6$, we add an edge e'' to $SC'(n)$ to define a new graph $SC''(n)$. If $n \equiv 1 \bmod 6$, then $e'' = \{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 2\}$. If $n \equiv 2 \bmod 6$, then $e'' = \{\frac{n}{2} - 1, \frac{n}{2} + 2\}$.

Theorem 2.14. *If* $n \equiv 1$ *or* $2 \bmod 6$, *then* $T_n(\lceil \frac{3n}{2} \rceil + 2) = \tau(SC''(n)) = \frac{3}{2}$.

Proof. Since $SC''(n)$ is 3-connected and $K_{1,3}$-free, Theorem 1.2 applies. □

3 Computing $T_n(m)$ for small n

The results in [4] give all of the values of $T_n(m)$ for $n \leq 6$ and most of the values for $7 \leq n \leq 12$. The values of $T_n(m)$ for $\lceil \frac{3n}{2} \rceil \leq m < 2n$ were left open in [4] and are handled here.

Theorem 3.1. $T_7(11) = \frac{4}{3}$.

Proof. Remark 2.9 tells us that $\frac{4}{3} \leq T_7(11) < \frac{3}{2}$. Since there are no possible fractions $\frac{|S|}{\omega(G-S)}$ strictly between $\frac{4}{3}$ and $\frac{3}{2}$ when there are only 7 vertices, it must be that $T_7(11) = \frac{4}{3}$. □

Remark 3.2. *A much less satisfying proof of Theorem 3.1 was given in [4].*

Theorem 3.3. $T_8(12) = T_8(13) = \frac{4}{3}$.

Proof. By Theorem 2.8, $T_8(12) \geq \frac{4}{3}$. Hence, it suffices to show that $T_8(13) \leq \frac{4}{3}$. That is, any $(8, 13)$-graph G has $\tau(G) \leq \frac{4}{3}$.

Case 1: G has 7 vertices of degree 3 and one vertex of degree 5.
Let v be the vertex of degree 5. So, $H = G - \{v\}$ is a $(7, 8)$-graph with degree sequence $2, 2, 2, 2, 2, 3, 3$. There are 7 such H as pictured in [8]. In each case, it is easy to see that v is an element of a disconnecting set S for G such that $\tau(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{4}{3}$.

Case 2: G has 6 vertices of degree 3 and 2 vertices of degree 4.
Let v be one of the degree 4 vertices. So, $H = G - \{v\}$ is a $(7, 9)$-graph, and H either has degree sequence $2, 2, 2, 2, 3, 3, 4$ or $2, 2, 2, 3, 3, 3, 3$ (depending on the adjacency of v with the other degree 4 vertex of G). There are 24 such H as pictured in [8]. In each case, it is easy to see that v is an element of a disconnecting set S for G such that $\tau(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{4}{3}$. □

Remark 3.4. *We can also see that* $T_8(12) = \frac{4}{3} < \frac{3}{2}$ *by using Theorem 2.1. First,* $SC(8)$ *is a* $\frac{4}{3}$*-tough* $(8, 12)$*-graph. Second, there are no possible fractions* $\frac{|S|}{\omega(G-S)}$ *strictly between* $\frac{4}{3}$ *and* $\frac{3}{2}$ *when there are only* 8 *vertices.*

Theorem 3.5. $T_9(14) = \frac{4}{3}$.

Proof. By Theorem 2.8, $T_9(14) \geq \frac{4}{3}$. It suffices to show that, if G is any $(9, 14)$-graph, then $\tau(G) \leq \frac{4}{3}$.

The graph G must have 8 vertices of degree 3 and one vertex, say v, of degree 4. The graph $H = G - \{v\}$ has degree sequence $2, 2, 2, 2, 3, 3, 3, 3$. Let w be one of the degree 3 vertices in H. So, $K = H - \{w\}$ is a $(7, 7)$-graph.

Case 1: $\kappa(K) = 0$.
In this case, $\omega(G - \{v, w\}) \geq 2$. Hence, $\tau(G) \leq \frac{|\{v,w\}|}{\omega(G-\{v,w\})} \leq 1$.

Case 2: $\kappa(K) = 1$.
There are 32 such graphs K as pictured in [8]. In each case, it is easy to see that the set $\{v, w\}$ is a subset of a disconnecting set S for G such that $\tau(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{4}{3}$.

Case 3: $\kappa(K) \geq 2$.
The only possibility is that $K = C_7$, the 7-cycle. Moreover, the graph G must then be as pictured in Figure 11. It is now easy to see that the vertices labelled x and y give a disconnecting set such that

$$\tau(G) \leq \frac{|\{x, y\}|}{\omega(G - \{x, y\})} = 1.$$

□

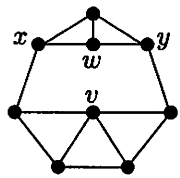

Figure 11: The unique G for which $\kappa(K) \geq 2$.

Theorem 3.6. $T_{10}(15) = \frac{4}{3}$.

Proof. Remark 2.9 tells us that $\frac{4}{3} \leq T_{10}(15) < \frac{3}{2}$. There are no possible fractions $\frac{|S|}{\omega(G-S)}$ strictly between $\frac{4}{3}$ and $\frac{3}{2}$ when there are only 10 vertices. □

Remark 3.7. *The Petersen graph is an example of a* $(10, 15)$*-graph that is* $\frac{4}{3}$*-tough [2] and hence maximally tough.*

The values of $T_n(m)$ for $7 \leq n \leq 12$ and $\lceil \frac{3n}{2} \rceil \leq m < 2n$ are listed in Tables 1, 2, and 3. Note that Tables 1 and 2 display values of maximum toughness which are strictly less than half the maximum connectivity.

m	$n = 7$	Thm		m	$n = 8$	Thm
11	$\frac{4}{3}$	3.1		$12 - 13$	$\frac{4}{3}$	3.3
$12 - 13$	$\frac{3}{2}$	2.11		$14 - 15$	$\frac{3}{2}$	2.14

Table 1: Maximum Toughness Values for $7 \leq n \leq 8$

m	$n = 9$	Thm		m	$n = 10$	Thm
14	$\frac{4}{3}$	3.5		15	$\frac{4}{3}$	3.6
$15 - 17$	$\frac{3}{2}$	2.11		$16 - 19$	$\frac{3}{2}$	2.11

Table 2: Maximum Toughness Values for $9 \leq n \leq 10$

m	$n = 11$	Thm		m	$n = 12$	Thm
$17 - 21$	$\frac{3}{2}$	2.8		$18 - 23$	$\frac{3}{2}$	2.1

Table 3: Maximum Toughness Values for $11 \leq n \leq 12$

A 3-connectivity

In this appendix, the graphs $SC(n)$ defined in Subsection 2.2 are shown to be 3-connected. The main tool used in our proof is the characterization of 3-connected graphs given by Tutte [9]. For our purposes, Tutte's result is given in the following form.

Theorem A.1 ([9]). *If a graph G is 3-connected and a new graph G' is obtained from G in one of the following two ways:*

(a) a new edge is added, or

(b) a vertex v of G with $deg_G(v) \geq 4$ is replaced by two new adjacent vertices v' and v'', and each neighbor of v in G is joined by an edge to exactly one of v' or v'' in G' so that $deg_{G'}(v') \geq 3$ and $deg_{G'}(v'') \geq 3$,

then G' is 3-connected.

The second type of operation in Theorem A.1 is referred to as splitting a vertex. The actual workhorse used in the proof of our result is a corollary of Theorem A.1. Before stating that result, we need some notation.

Given a graph $G = (V, E)$ and two edges $e_1 = \{u_1, v_1\}$ and $e_2 = \{u_2, v_2\}$ in E, we define a new graph $G(e_1, e_2) = (V', E')$. Let $V' = V \cup \{w_1, w_2\}$, where w_1 and w_2 are two new vertices that are not in V. Let

$$E' = (E - \{e_1, e_2\}) \cup \{\ \{u_1, w_1\}, \{v_1, w_1\}, \{u_2, w_2\}, \{v_2, w_2\}, \{w_1, w_2\}\ \}.$$

It is possible that e_1 and e_2 share vertex. However, a picture of this construction in the case that e_1 and e_2 do not meet is given in Figure 12.

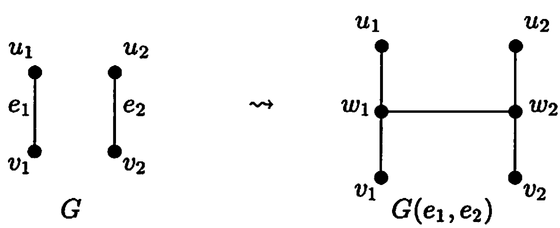

Figure 12: Constructing $G(e_1, e_2)$ from G

Corollary A.2. *Let G be a graph, and let e_1 and e_2 be two distinct edges in G. If G is 3-connected, then $G(e_1, e_2)$ is also 3-connected.*

Proof. Say $e_1 = \{u_1, v_1\}$ and $e_2 = \{u_2, v_2\}$. We may assume that $v_1 \neq v_2$. The key to our proof is the fact that $G(e_1, e_2)$ can be constructed from G in three stages, each of which is covered by Theorem A.1. The first step is to form a graph G' from G by adding the edge $\{v_1, v_2\}$. Since G is 3-connected, we must have $deg_G(v_1) \geq 3$ and $deg_G(v_2) \geq 3$. Hence, $deg_{G'}(v_1) \geq 4$ and $deg_{G'}(v_2) \geq 4$. In the second step, G'' is formed from G' by splitting the vertex v_1. Specifically, the neighbors u_1 and v_2 of v_1 are joined to v_1' by edges, and all other neighbors of v_1 are joined to v_1'' by edges. Note that $deg_{G''}(v_1') = 3$ and $deg_{G''}(v_2) \geq 4$. Since $deg_{G'}(v_1) \geq 4$, it follows that $deg_{G''}(v_1'') \geq 3$. A picture of the construction of G'' from G' is given in Figure 13. In the third and final step, $G(e_1, e_2)$ is obtained from G'' by

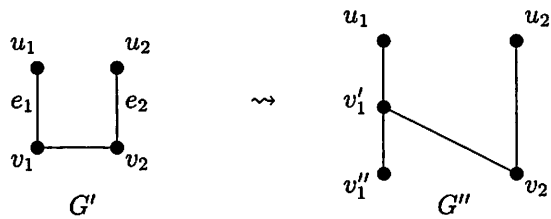

Figure 13: Splitting v_1 in G' to form G''.

splitting v_2. Specifically, the neighbors u_2 and v_1' of v_2 are joined to v_2' by edges, and all other neighbors of v_2 are joined to v_2'' by edges. It is clear that this last splitting forms $G(e_1, e_2)$. □

The main result of this appendix can now be proven.

Theorem A.3. *For each $n > 4$, the graph $SC(n)$ is 3-connected.*

Proof. Our proof is by induction on n. It is easy to verify (by computer) that the graphs $SC(n)$ for $5 \leq n \leq 10$ are 3-connected. Suppose that $n > 4$, $SC(n)$ is 3-connected, and its vertex set is given by $V = \{v_0, v_1, \ldots, v_{n-1}\}$. We construct $SC(n+6)$ in three stages, each of which is covered by Corollary A.2. Note that our construction is independent of the congruence class of $n \mod 6$.

Let $e_1 = \{v_{n-2}, v_{n-1}\}$ and $e_2 = \{v_1, v_2\}$ be edges in $G = SC(n)$. Define $G' = G(e_1, e_2)$, and denote the new vertex between v_{n-2} and v_{n-1} by w_{n+3} and the new vertex between v_1 and v_2 by w_3.

Let $e_1' = \{v_1, w_3\}$ and $e_2' = \{w_3, v_2\}$ be edges in G'. Now we define $G'' = G'(e_1', e_2')$, and denote the new vertex between v_1 and w_3 by w_2 and the new vertex between w_3 and v_2 by w_4. The graph G'' is pictured in Figure 14.

Let $e_1'' = \{w_{n+3}, v_{n-1}\}$ and $e_2'' = \{v_{n-2}, w_{n+3}\}$ be edges in G''. Define $G''' = G''(e_1'', e_2'')$, and denote the new vertex between w_{n+3} and v_{n-1} by

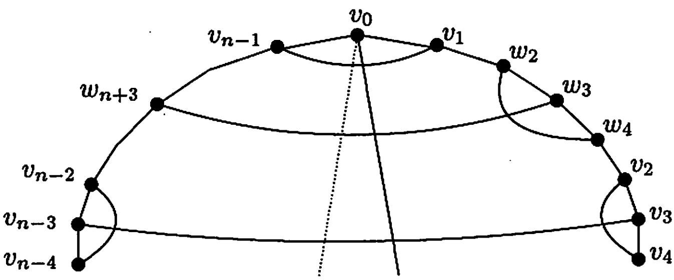

Figure 14: Building $SC(n+6)$ from $SC(n)$

w_{n+4} and the new vertex between v_{n-2} and w_{n+3} by w_{n+2}. Finally, rename the vertices of G''' by $w_0 = v_0, w_1 = v_1, w_{n+5} = v_{n-1}$, and, for $2 \le i \le n-2$, $w_{i+3} = v_i$. On the vertex set $W = \{w_0, w_1, \ldots, w_{n+5}\}$, it is now easy to see that $SC(n+6) = G'''$ is also 3-connected. □

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