On the Asymptotics of Colouring Plane Multigraphs

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Abstract

For loopless plane multigraphs G, the edge-face chromatic number and the entire chromatic number are asymptotically their fractional counterparts (LP relaxations) as these latter invariants tend to infinity. Proofs of these results are based on analogous theorems for the chromatic index and the total chromatic number, due, respectively, to Kahn [3] and to the first author [6]. Our two results fill in the missing pieces of a complete answer to the natural question: which of the seven invariants associated with colouring the nonempty subsets of $\{V, E, F\}$ exhibit "asymptotically good" behaviour?

This paper is concerned with loopless plane multigraphs G and the asymptotic behaviour of various colouring invariants associated with such graphs. We use V, E and F, respectively, to denote the vertex, edge and face sets of G. Each nonempty subset of $\{V, E, F\}$ corresponds to a colouring invariant of G; for example, $\{E\}$ corresponds to the chromatic index χ_e and $\{V, E\}$ to the total chromatic number χ_{ve}.

The seven resulting colouring invariants are optimal solutions to integer programming problems, the linear relaxations of which yield the fractional versions of these parameters. In [5], the first author essentially asked the question: which of the seven integral invariants are asymptotic to their fractional counterparts, as the latter invariants tend to infinity? The primary

1991 *AMS Subject Classification*: Primary 05C15; Secondary 90C05, 90C10.

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†This paper is based on the author's M.A. thesis [10].

motivation for this question was a result of Kahn [3] giving an affirmative answer for χ_e; see Theorem 3 below. As observed in [6] (Theorem 4 below), Kahn [4] also proved a result implying an affirmative answer for χ_{ve}. These two positive results lead one naturally to wonder if this "asymptotically good" behaviour is enjoyed by any of the other five colouring invariants presently under consideration.

Heawood's five-colour theorem (or its famous improvement) quickly classifies χ_v, χ_f and χ_{vf} as uninteresting for our question: each is bounded by a (small) constant; so too is its fractional version. This leaves the *edge-face* and the *entire* chromatic numbers, respectively χ_{ef}, χ_{vef}, as the only interesting, as yet unaddressed, cases. This paper fills in these missing pieces to complete the answer to our question with two more positive results.

A word on definitions

We aim to be brief, and thus point to the references for any omitted terminology: [1] for general graph theory; [2] for colouring invariants and related history; [9] for LP/IP background; [8] for fractional concepts. All our graphs are loopless; in particular, when we write *multigraph*, we mean loopless multigraph. After seeing the definition of χ_{vef}, the reader will have no trouble formulating definitions for the other chromatic numbers introduced above. An *entire colouring* of a plane multigraph G is a map $\sigma : V \cup E \cup F \to S$ — where S is a set of "colours" — such that $\sigma(X) \neq \sigma(Y)$ whenever X, Y are incident or adjacent elements, i.e. a pair of adjacent vertices, a vertex-edge pair with the edge incident on the vertex, an edge-face pair with the edge on the boundary of the face, etc.; a face touching either another face or an edge only in a vertex is not considered an adjacency. The *entire chromatic number*, χ_{vef}, is the least size of an S admitting such a colouring. For example, if G consists of two copies of K_3 joined at a single common vertex, then $\chi_{vef}(G) = 6$.

As noted above, each of the seven colouring invariants under consideration has a fractional analogue, its linear relaxation. We use asterisks to denote fractional parameters, so, e.g., χ_e^* denotes the fractional chromatic index. In the proof of our first main result (Theorem 1), we need a few concepts underlying a detailed definition of χ_e^*. Writing $\mathfrak{M}$ for the set of matchings in G, we call an $f : \mathfrak{M} \to [0,1]$ satisfying

$$\sum_{A \in M \in \mathfrak{M}} f(M) = 1 \quad \text{for each } A \in E$$

a *fractional edge colouring* of G. Note that an ordinary (integral) edge colouring arises if we restrict the range of f to $\{0,1\}$. Now

$$\chi_e^*(G) = \min\left\{ \sum_{M\in\mathfrak{M}} f(M) : f \text{ is a fractional edge colouring of } G \right\} \quad (1)$$

makes the LP defining χ_e^* explicit. Likewise we may define χ_{ef}^*, but the analogue of $\mathfrak{M}$ is more complicated. For $\Phi \subseteq F$, let $\partial_e(\Phi)$ denote the set of edges of G on the boundary of some face in Φ. An *edge-face stable set* of G is a subset of $E \cup F$ of the form $M \cup \Phi$, where $M \subseteq E$ is a matching, $\Phi \subseteq F$ is a collection of faces, no two sharing a common boundary edge, and $M \cap \partial_e(\Phi) = \varnothing$. We write $\mathfrak{S}$ for the family of edge-face stable sets of G. An $f : \mathfrak{S} \to [0,1]$ satisfying

$$\sum_{A\in S\in\mathfrak{S}} f(S) = 1 \quad \text{for each } A \in E \cup F$$

is a *fractional edge-face colouring* of G. We then have

$$\chi_{ef}^*(G) = \min\left\{ \sum_{S\in\mathfrak{S}} f(S) : f \text{ is a fractional edge-face colouring of } G \right\}. \quad (2)$$

We often abbreviate the objective functions in (1) and (2) to $f(G)$.

Results

We are ready to state precisely first our results then two precursors that we need as lemmas.

Theorem 1 *For plane multigraphs,*

$$\chi_{ef} \sim \chi_{ef}^* \quad \textit{as} \quad \chi_{ef}^* \to \infty. \quad (3)$$

That is, for each $\varepsilon > 0$ there exists $D = D(\varepsilon)$ such that every plane multigraph G with $\chi_{ef}^*(G) > D$ satisfies

$$(1+\varepsilon)^{-1} < \frac{\chi_{ef}(G)}{\chi_{ef}^*(G)} < 1+\varepsilon. \quad (4)$$

Theorem 2 *For plane multigraphs,*

$$\chi_{vef} \sim \chi_{vef}^* \quad \textit{as} \quad \chi_{vef}^* \to \infty.$$

Thus, both of χ_{ef}, χ_{vef} are asymptotically good invariants.

The analogous statement for χ_e — true for general (not just plane) multigraphs — was proved by Kahn [3]:

Theorem 3 *For multigraphs,*

$$\chi_e \sim \chi_e^* \quad \textit{as} \quad \chi_e^* \to \infty.$$

The convergence here is in the same sense as that in (3), but we again spell out the quantifiers for reference in the proof of Theorem 1: for each $\gamma > 0$ there exists $B = B(\gamma)$ such that every multigraph G with $\chi_e^*(G) > B$ satisfies $\chi_e(G) < (1+\gamma)\chi_e^*(G)$.

That χ_{ve} is also asymptotically good — again for general multigraphs — was observed by the first author in [6]:

Theorem 4 *For multigraphs,*

$$\chi_{ve} \sim \chi_{ve}^* \quad \textit{as} \quad \chi_{ve}^* \to \infty.$$

Proof of Theorem 1

In addition to Theorem 3, we need the following elementary inequalities connecting the edge-face chromatic numbers with the chromatic indices (in (6) and (7), C is a small constant, say 4 or 5):

$$\chi_{ef}^* \le \chi_{ef}; \tag{5}$$

$$\chi_{ef}^* \le \chi_e^* + C; \tag{6}$$

$$\chi_{ef} \le \chi_e + C; \tag{7}$$

$$\chi_e^* \le \chi_{ef}^*. \tag{8}$$

Proof of (5). The left side is the optimal value of the linear relaxation of the IP defining the right. ∎

Proof of (6). We may obtain a fractional colouring h of $E \cup F$ by fractionally χ_e^*-colouring E and (integrally) colouring F with a set $\mathcal{C}$ of additional colours. Depending on how hard we wish to hit, we may take $C := |\mathcal{C}| = 5$ (using Heawood's five-colour theorem) or $C = 4$ (using Appel and Haken's four-colour theorem). Since $h(G) = \chi_e^* + C$, (6) now follows. ∎

Proof of (7). An optimal edge colouring can be expanded to an edge-face colouring using at most $C \in \{4, 5\}$ additional colours for the faces. ∎

Proof of (8). From an optimal fractional edge-face colouring $f : \mathfrak{S} \to [0, 1]$, we may obtain a fractional edge colouring $h : \mathfrak{M} \to [0, 1]$ by shifting the weight $f(S)$ from each edge-face stable set $S = M \cup \Phi$ to the matching M in the natural way. This yields an h with $h(G) = f(G) = \chi^*_{ef}(G)$, and (8) follows since $\chi^*_e(G) \leq h(G)$. ∎

We are now equipped to complete the proof of Theorem 1. Since we already have (5), it remains only to establish the right-hand inequality in (4) for arbitrary $\varepsilon > 0$ and sufficiently large χ^*_{ef}. Given $\varepsilon > 0$, let $\gamma = \varepsilon/2$, and choose B so large (according to Theorem 3) that

$$\chi^*_e > B \quad \text{implies} \quad \chi_e < (1 + \gamma)\chi^*_e. \tag{9}$$

Let C be as in (6), (7). If $\chi^*_{ef} > D := \max\{B + C, 2C\varepsilon^{-1} + C\}$, then, since $\chi^*_e \geq \chi^*_{ef} - C$ (by (6)), we see that χ^*_e exceeds both B and $2C/\varepsilon = C/\gamma$. Thus, as long as $\chi^*_{ef} > D$, we have

$$\chi_{ef} \leq \chi_e + C < (1 + \gamma)\chi^*_e + \gamma\chi^*_e = (1 + \varepsilon)\chi^*_e \leq (1 + \varepsilon)\chi^*_{ef}$$

(justifying the inequalities, respectively, by: (7); the preceding sentence and (9); and (8)), as desired. ∎

Proof of Theorem 2

Since the proof mirrors that of Theorem 1, we simply sketch it. The following inequalities are analogous to (5)–(8) and may be proved similarly:

$$\chi^*_{vef} \leq \chi_{vef}; \tag{10}$$
$$\chi^*_{vef} \leq \chi^*_{ve} + C; \tag{11}$$
$$\chi_{vef} \leq \chi_{ve} + C; \tag{12}$$
$$\chi^*_{ve} \leq \chi^*_{vef}. \tag{13}$$

To prove Theorem 2, one may now use the proof of Theorem 1 with the following replacements: $(\chi_{ef}, \chi_e, \chi^*_{ef}, \chi^*_e, \text{Theorem 3}, (5)\text{–}(8)) \mapsto (\chi_{vef}, \chi_{ve}, \chi^*_{vef}, \chi^*_{ve}, \text{Theorem 4}, (10)\text{–}(13))$. ∎

Remarks

On plane duality

Let G^* denote the dual of a plane multigraph G. Of course,

$$\chi_f(G^*) = \chi_v(G), \tag{14}$$

and vice versa, perhaps leading one to guess that analogous relationships exist between χ_{ef} and χ_{ve}. If so, then one could obtain alternately Theorem 1 from Theorem 4 via duality; however, simple examples reveal this guess to be incorrect, therefore suggesting that such a proof strategy may be fruitless.

Consider the cycle C_n with n edges. Since its dual C_n^* contains a vertex of degree n, evidently $\chi_{ef}(C_n^*) \geq n$. On the other hand, one easily checks that $\chi_{ve}(C_n) = 4$ if $n \geq 4$. This example shows emphatically that in general, $\chi_{ef}(G^*) \neq \chi_{ve}(G)$, hence dashing any hope that the analogue of (14), with edges included, might hold. It illustrates, moreover, that Theorem 1 can be relevant for G^* (here $\chi_{ef}(C_n^*) \to \infty$ as $n \to \infty$) without implying the relevance of Theorem 4 for G (since $\chi_{ve}(C_n)$ is constant). The intuition that duality might yield a quick route from one theorem to the other proves specious at best.

On list-colouring

A natural question is whether the list-colouring analogues of Theorems 1–4 also hold. For example, can χ_{ef} in Theorem 1 be replaced by the edge-face choice number? For list-colouring edges, Kahn [4] gave an affirmative answer in 1995, though the article appeared in published form only recently. Positive answers for total, edge-face, and entire list-colouring were established by the first author in [7], a follow-up to the present paper.

Acknowledgement

The authors thank Evan Wantland for insightful conversations at the white board.

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