DOUBLE YOUDEN RECTANGLES FOR THE FOUR BIPLANES WITH $k = 9$

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Abstract

A cyclic or bicyclic 9×37 double Youden rectangle (DYR) is provided for each of the four biplanes with $k = 9$. These DYRs were obtained by computer search.

Keywords: Bicyclic; Biplane; Cyclic; Difference set; Double Youden rectangle; Hussain biplane; Initial blocks; Lehigh biplane; Symmetric balanced incomplete block design.

1 Introduction

Let k and v be integers with $0 < k < v$, let n be the integer part of v/k, and let m be the corresponding remainder, *i.e.* $m = v - nk$. Let $\mathcal{X}$ be a set of v elements, and $\mathcal{Y}$ a set of k elements.

A *double Youden rectangle* (DYR) of size $k \times v$ is [4, p. 40] a $k \times v$ arrangement of the kv distinct ordered pairs x, y ($x \in \mathcal{X}$, $y \in \mathcal{Y}$) that satisfies the following conditions:

(i) each element of $\mathcal{X}$ appears exactly once per row,

(ii) each element of $\mathcal{Y}$ appears exactly once per column,

(iii) the sets of members of $\mathcal{X}$ in the columns are the blocks of a symmetric balanced incomplete block design (SBIBD, or symmetric 2-design), and

(iv) each element of $\mathcal{Y}$ appears either n or $n + 1$ times in each row, and either (a) $m = 1$ or (b) the sets of elements of $\mathcal{Y}$ in the rows become the blocks of size m of an SBIBD when n occurrences of each element from $\mathcal{Y}$ are removed from each set.

An example of a 7×15 DYR (for which we must have $n = 2$ and $m = 1$) is the following:

B2	C3	A1	I6	G4	H5	L4	J5	K6	O2	M3	N1	F0	D0	E0
M5	B4	F6	O3	H2	C0	A3	E6	G0	I1	K5	J2	L5	N4	D1
D4	N6	C5	A0	M1	I3	H0	B1	F4	K3	G2	L6	E2	J6	O5
A6	E5	O4	G1	B0	N2	D5	I0	C2	J4	L1	H3	M6	F3	K4
J0	H1	B3	D2	F5	M4	G6	L2	E1	A5	I4	O0	N3	K2	C6
C1	K0	I2	N5	E3	D6	F2	H4	J3	M0	B6	G5	A4	O1	L3
G3	A2	L0	E4	O6	F1	K1	D3	I5	H6	N0	C4	J1	B5	M2

Here $\mathcal{X} = \{A, B, \ldots, O\}$ and $\mathcal{Y} = \{0, 1, \ldots, 6\}$. As indicated by the horizontal and vertical lines, this DYR is 3-cyclic in the sense that columns 1, 4, 7, 10 and 13 are *initial columns* from which succeeding columns can be obtained by successive use of the cyclic permutation

$$(ABC)(DEF)(GHI)(JKL)(MNO)$$

of the elements of $\mathcal{X}$, simultaneously with the cylic permutation

$$(1\ 2\ 3)(4\ 5\ 6)$$

of the elements of $\mathcal{Y}$ and of the row numbers, these too being taken as $0, 1, \ldots, 6$. Thus, within each 3×3 subsection of the DYR, the cycling is down the main diagonal and along broken diagonals parallel to the main diagonal. The element 0 of $\mathcal{Y}$ is fixed under the permutations, just as row 0 is a fixed row.

DYRs with $k = v - 1$ are known to exist for all $k > 3$ (see [18]). Otherwise, few DYRs are known. Vowden [25, 26] gave two infinite series of DYRs, but other constructions for particular sizes of DYR have lacked generality [17, 18, 19, 20, 21]. Also, little is known about the scope for constructing DYRs for a parameter set (v, k, λ), $k < v-1$, for which several or many non-isomorphic SBIBDs exist. For $(v, k, \lambda) = (15, 7, 3)$ there are five non-isomorphic SBIBDs, labelled C1, C2, ..., C5 in [5]. DYRs based on C1, C2, C3 and C5 were given in [20], and the above 7×15 DYR is based on C4, so a DYR is now known for each of the five non-isomorphic SBIBDs for $(v, k, \lambda) = (15, 7, 3)$; but this happy state of affairs is paralleled only for $(v, k, \lambda) = (37, 9, 2)$, as described in the present paper.

If we write $\lambda = k(k-1)/(v-1)$, a DYR with $\lambda = 2$ has the property that, in the terminology of [7], the SBIBD in condition (iii) is a *biplane of order* $k - 2$. For biplanes with k even, condition (iv) can be met only if $k = 4$, and yes, 4×7 DYRs do indeed exist [9, 16]. For k odd, biplanes are known to exist for $k = 3, 5, 9, 11$ and 13, and any corresponding DYRs must have $m = 1$ in condition (iv). The value $k = 3$ is too small for the existence of a DYR [8], whereas Preece [18] obtained DYRs for $k = 5$. This leaves us to consider the values $k = 9, 11$ and 13. The present paper reports the existence of DYRs for each of the four non-isomorphic biplanes for $k = 9$. We have not found any DYR for any of the biplanes with $k = 11, 13$.

2 The four biplanes and the new 9×37 DYRs

The properties of the four biplanes with $k = 9$ can be summarised as follows (see [2, pp. 210–211], [22]):

$B_d(9)$: The *difference-set biplane* [6, p. 393], constructed using the quartic residues modulo 37. It is self-dual, the order of its automorphism group is 333, and this group has $\mathbb{C}_9$ as a subgroup.

$B_H(9)$: The *Hussain biplane* given in 1945 by Hussain [13, Section 3]. This appeared also as designs (A) and $(E)'$ in [3] and as design A in [12, p. 15]. The order of its automorphism group is 1512, and this group has $\mathbb{C}_9$ as a subgroup.

$B_H'(9)$: The dual of the Hussain biplane.

$B_L(9)$: The *Lehigh biplane*, announced in [2, p. 211] and [22]. It is self-dual, the order of its automorphism group is 54, and this group contains $\mathbb{C}_3 \times \mathbb{C}_3$, but not $\mathbb{C}_9$, as a subgroup.

Each of $B_d(9)$, $B_H(9)$ and $B_H'(9)$ can be represented by five initial blocks, four of which are developed using cycles of length 9 to give $4 \times 9 = 36$ blocks; the remaining initial block is fixed under the cyclic development, thus giving us the required $36 + 1 = 37$ blocks in total. This cyclic property was the basis of our successful computer search for a 9×37 DYR for each of these three biplanes. Only two searches were needed, as a DYR for $B_H'(9)$ can be obtained by duality from a DYR for $B_H(9)$ (see [18, pp. 310–312]). The biplane $B_L(9)$ can also be represented by five initial blocks, but these are developed bicyclically with cycles of length 3. This was the basis of our success in obtaining a DYR for $B_L(9)$.

Table 1 gives the 9×37 DYR obtained for each of the biplanes $B_d(9)$, $B_H(9)$ and $B_L(9)$. In each of these DYRs, the elements of $\mathcal{X}$ are denoted A,B,...,Z,a,b,...,k; those of $\mathcal{Y}$ are denoted 1,2,...,9.

The DYRs for $B_d(9)$ and $B_H(9)$ are 9-cyclic, and each of them has $\mathbb{C}_9$ as automorphism group. In each, the columns corresponding to the initial blocks of the cyclically generated SBIBDS are the 1^{st} (fixed), 2^{nd}, 11^{th}, 20^{th} and 29^{th}. Thus, after the first column, each of these two DYRs consists of four 9×9 sections, starting respectively in columns 2, 11, 20 and 29. Within each such section, each successive column is obtained from the previous one by using the cyclic permutation

(BCDEFGHIJ)(KLMNOPQRS)(TUVWXYZab)(cdefghijk)

of the elements of $\mathcal{X}$, simultaneously with the cyclic permutation

(1 2 3 4 5 6 7 8 9)

of the elements of $\mathcal{Y}$ and of the row numbers, the rows now being numbered 1, 2, ..., 9. The element A of $\mathcal{X}$ is fixed, just as the first column is a fixed column. Thus, under the cyclic development, the entry A1 in row 1 and column 2 of the DYR for $B_d(9)$ becomes A2 in row 2 and column 3, whereas the entry K4 in row 2 and column 2 becomes L5 in row 3 and column 3, and so on.

The DYR for $B_L(9)$ is bicyclic, with 3-cycles, and has $\mathbb{C}_3 \times \mathbb{C}_3$ as automorphism group. Again, column 1 and the element A from $\mathcal{X}$ are fixed. Now, however, the 9×9 sections starting in columns 2, 11, 20 and 29 are each composed of nine 3×3 subsections. Within the 3 columns of such a subsection, the permutation

(BCD)(EFG)(HIJ)(KLM)(NOP)(QRS)(TUV)(WXY)(Zab)(cde)(fgh)(ijk)

of the elements of $\mathcal{X}$ is used along with the permutation

(1 2 3)(4 5 6)(7 8 9)

of the elements of $\mathcal{Y}$ and of the row numbers. Each 9×9 section is completed by use of

(BEH)(CFI)(DGJ)(KNQ)(LOR)(MPS)(TWZ)(UXa)(VYb)(cfi)(dgj)(ehk)

along with

(1 4 7)(2 5 8)(3 6 9) .

3 Comments

The cyclic methodology for generating the 9×37 DYRs given in this paper is exactly analogous to that used in many of the earlier DYR papers listed as references below. However, this methodology cannot be adapted readily, if at all, to finding DYRs for the biplanes with $k = 11$, 13. The orders of the automorphism groups of the five known biplanes $\mathcal{B}_1$, $\mathcal{B}_2$, ... , $\mathcal{B}_5$ with $k = 11$ (see [11, 15, 23, 2, 10, 14]) are 80640, 288, 144, 64 and 24 respectively, none of which is a multiple of k; indeed, Denniston [10, p.177] showed that "no biplane with 56 points can have a period-11 automorphism." The order of the automorphism groups of each of the two known biplanes with $k = 13$ (see [1, 24]) is 110, which also is not a multiple of k. We cannot suggest any promising starting point in the search for DYRs of sizes 11×56 and 13×79, even for the 7-cyclic biplane $\mathcal{B}_1$.

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TABLE 1

9-cyclic double Youden rectangle based on $B_d(9)$

G1	A1	D6	i5	P2	g1	b7	H4	a7	S3	B2	c1	O9	C4	L8	U2	K9	N6	T4	R9	Y2	h5	E9	k6	j3	f5	F3	M3	X7	e6	I8	d4	Z8	V7	Q5	W1	J8
H2	K4	A2	E7	j6	Q3	h2	T8	I5	b8	U5	C3	d2	P1	D5	M9	V3	L1	O7	N4	S1	Z3	i6	F1	c7	k4	g6	G4	B9	Y8	f7	J9	e5	a9	W8	R6	X2
I3	T9	L5	A3	F8	k7	R4	i3	U9	J6	P8	V6	D4	e3	Q2	E6	N1	W4	M2	H5	O5	K2	a4	j7	G2	d8	c5	h7	Y3	C1	Z9	g8	B1	f6	b1	X9	S7
J4	B7	U1	M6	A4	G9	c8	S5	j4	V1	N3	Q9	W7	E5	f4	R3	F7	O2	X5	i8	I6	P6	L3	b5	k8	H3	e9	d6	K8	Z4	D2	a1	h9	C2	g7	T2	Y1
B5	W2	C8	V2	N7	A5	H1	d9	K6	k5	Y6	O4	R1	X8	F6	g5	S4	G8	P3	e7	j9	J7	Q7	M4	T6	c9	I4	f1	Z2	L9	a5	E3	b2	i1	D3	h8	U3
C6	c6	X3	D9	W3	O8	A6	I2	e1	L7	Q4	Z7	P5	S2	Y9	G7	h6	K5	H9	g2	f8	k1	B8	R8	N5	U7	d1	J5	V4	a3	M1	b6	F4	T3	j2	E4	i9
D7	M8	d7	Y4	E1	X4	P9	A7	J3	f2	I1	R5	a8	Q6	K3	Z1	H8	i7	L6	B6	h3	g9	c2	C9	S9	O6	V8	e2	j1	W5	b4	N2	T7	G5	U4	k3	F5
E8	g3	N9	e8	Z5	F2	Y5	Q1	A8	B4	M7	J2	S6	b9	R7	L4	a2	I9	j8	f3	C7	i4	h1	d3	D1	K1	P7	W9	G6	k2	X6	T5	O3	U8	H6	V5	c4
F9	C5	h4	O1	f9	a6	G3	Z6	R2	A9	k9	N8	B3	K7	T1	S8	M5	b3	J1	X1	g4	D8	j5	i2	e4	E2	L2	Q8	d5	H7	c3	Y7	U6	P4	V9	I7	W6

9-cyclic double Youden rectangle based on $B_H(9)$

B1	A1	F7	W7	d1	C6	j9	L6	Z6	O2	U3	Y2	I6	g8	T3	X3	h8	R7	E5	M3	e1	K4	D2	G9	f9	k4	i4	b9	c1	P8	a5	S5	Q8	N2	J5	V7	H4
C2	P3	A2	G8	X8	e2	D7	k1	M7	a7	F6	V4	Z3	J7	h9	U4	Y4	i9	S8	T1	N4	f2	L5	E3	H1	g1	c5	j5	I5	d2	Q9	b6	K6	R9	O3	B6	W8
D3	b8	Q4	A3	H9	Y9	f3	E8	c2	N8	K9	G7	W5	a4	B8	i1	V5	Z5	j1	k6	U2	O5	g3	M6	F4	I2	h2	d6	X9	J6	e3	R1	T7	L7	S1	P4	C7
E4	O9	T9	R5	A4	I1	Z1	g4	F9	d3	k2	L1	H8	X6	b5	C9	j2	W6	a6	e7	c7	V3	P6	h4	N7	G5	J3	i3	D8	Y1	B7	f4	S2	U8	M8	K2	Q5
F5	e4	P1	U1	S6	A5	J2	a2	h5	G1	b7	c3	M2	I9	Y7	T6	D1	k3	X7	j4	f8	d8	W4	Q7	i5	O8	H6	B4	R6	E9	Z2	C8	g5	K3	V9	N9	L3
G6	H2	f5	Q2	V2	K7	A6	B3	b3	i6	Y8	T8	d4	N3	J1	Z8	U7	E2	c4	C5	k5	g9	e9	X5	R8	j6	P9	I7	M4	S7	F1	a3	D9	h6	L4	W1	O1
H7	j7	I3	g6	R3	W3	L8	A7	C4	T4	d5	Z9	U9	e5	O4	B2	a9	V8	F3	J8	D6	c6	h1	f1	Y6	S9	k7	Q1	P2	N5	K8	G2	b4	E1	i7	M5	X2
I8	U5	k8	J4	h7	S4	X4	M9	A8	D5	G4	e6	a1	V1	f6	P5	C3	b1	W9	R2	B9	E7	d7	i2	g2	Z7	K1	c8	Y3	Q3	O6	L9	H3	T5	F2	j8	N6
J9	E6	V6	c9	B5	i8	K5	Y5	N1	A9	X1	H5	f7	b2	W2	g7	Q6	D4	T2	d9	S3	C1	F8	e8	j3	h3	a8	L2	O7	Z4	R4	P7	M1	I4	U6	G3	k9

TABLE 1 (continued)

(3,3)-cyclic double Youden rectangle based on $B_L(9)$

B1	A1	X8	f1	U9	Q8	F1	P2	J4	h8	D4	j7	M5	T6	Z2	Y6	g5	G3	a7	k5	R2	d6	W3	C4	L8	i5	H9	e6	c2	b7	N4	V9	O3	K3	S1	E7	I9
C2	g2	A2	Y9	G2	V7	R9	f9	N3	H5	K6	B5	k8	W4	U4	a3	b8	h6	E1	e4	i6	S3	M9	X1	D5	c4	j6	I7	O5	d3	Z8	L1	T7	P1	J7	Q2	F8
D3	W7	h3	A3	S7	E3	T8	I6	g7	O1	i9	L4	C6	b1	X5	V5	F2	Z9	f4	Q1	c5	j4	B6	K7	Y2	J8	d5	k4	a9	P6	e1	N2	M2	U8	G9	H8	R3
E4	S5	D7	k2	A4	a2	i4	X3	K2	I4	j8	J6	U1	G7	d1	P8	W9	T5	b9	c8	B3	h9	e8	L5	g9	Z6	F7	O2	M4	H1	C3	f5	V1	Q7	Y3	R6	N6
F5	i3	Q6	B8	j5	A5	b3	J5	Y1	L3	V2	k9	H4	N9	E8	e2	Z7	X7	U6	f7	d9	C1	h7	c9	M6	P3	a4	G8	D1	K5	I2	R8	g6	T2	O4	W1	S4
G6	C9	j1	R4	Z1	k6	A6	M1	H6	W2	I5	T3	i7	c3	O7	F9	V4	a8	Y8	D2	g8	e7	K4	f8	d7	E9	N1	b5	J3	B2	L6	U3	S9	h4	Q5	P5	X2
H7	a6	N5	C7	M8	G1	e5	A7	U5	c7	Z3	W8	V3	d2	D9	X4	J1	g4	S2	T9	I1	R5	f2	E6	k3	h2	O8	j3	b6	L9	Q9	P7	B4	F6	i8	Y4	K1
I8	D8	b4	O6	c6	K9	E2	d8	A8	V6	T1	a1	X9	Y5	e3	B7	Q3	H2	h5	S6	U7	J2	i1	g3	F4	k1	f3	P9	R7	Z4	M7	G4	N8	C5	L2	j9	W5
J9	P4	B9	Z5	F3	d4	L7	T4	e9	A9	Y7	U2	b2	C8	W6	c1	f6	R1	I3	H3	Q4	V8	G5	j2	h1	N7	i2	g1	K8	S8	a5	D6	E5	O9	X6	M3	k7