Trajectories in the $3x + 1$ Problem

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Abstract

This paper presents a new approach in the quest for a solution to the $3x+1$ problem. The method relies on the convergence of the trajectories of the odd positive integers by exploiting the role of the positive integers of the form $1+4n$, where n is a non-negative integer.

AMS 2000 Classification: 11A25, 11Y55, 11Y70

1. Introduction

Research on the $3x + 1$ problem has produced a wealth of papers and the references at [4,5,6] present a comprehensive bibliography of the literature resulting from such research. Yet, hitherto, no formal proof of the conjecture associated with the problem has been achieved. This paper uses a selection of the results appearing in the literature to lay the foundation for an alternative approach in the quest for a solution.

To fix our ideas we include the following basic definitions, descriptions and terminology which will be used. Our discussion is confined to non-negative integers.

Let $\mathbf{N} = \{1,2,3,....\}$ denote the set of positive integers and let $\mathbf{O}$ be the subset of odd integers in $\mathbf{N}$. Let $\mathbf{N_0} = \mathbf{N} \cup \{0\} = \{0,1,2,....\}$.

For any $x \in \mathbf{N}$, let the function $f : \mathbf{N} \to \mathbf{N}$ be given by,

$$f(x) = \begin{cases} 1+3x, & \text{if } x \text{ is odd,} \\ \frac{x}{2}, & \text{if } x \text{ is even.} \end{cases} \tag{1.1}$$

The $3x + 1$ conjecture asserts that in the sequence of iterates $x,\ f(x),\ f^2(x),\ f^3(x),...$ for any $x \in \mathbf{N}$, $\exists\, k \in \mathbf{N_0}$, such that $f^k(x) = 1$.

By convention $f^o(x) = x$, $\forall\, x \in \mathbf{N}$, and the least value of k satisfying $f^k(x) = 1$ is what is being sought.

In [3] an alternative formulation of the conjecture was provided by means of the function $h : \mathbf{O} \to \mathbf{O}$ given by:

$$h(x) = \frac{1+3x}{2^{m(x)}},\ x \in \mathbf{O}, \tag{1.2}$$

where $2^{m(x)}$ is the maximum power of 2 dividing $1 + 3x$. The conjecture then reduces to showing that for each odd integer x there is an integer k such that $h^k(x) = 1$.

Definition 1.1. The trajectory, under f, of $x \in \mathbf{N}$ is the set $L(x) = \{x,\ f(x),\ f^2(x),\ f^3(x), ...\}$.

Definition 1.2. The trajectory, under h, of $x \in \mathbf{O}$ is the set $T(x) = \{x,\ h(x),\ h^2(x),\ h^3(x), ...\}$.

The trajectory $T(x)$ of any $x \in \mathbf{O}$ is obtained from $L(x)$ by selecting the odd integers in $L(x)$.

Definition 1.3. Two trajectories are said to coalesce if they have a common element.

2. Previous Results

For the establishment of formal results we use the function f. Since even integers in $\mathbf{N}$ are reduced to odd integers by direct application of f we concentrate our attention on $\mathbf{O}$. First we determine a partition of $\mathbf{O}$.

Now,

$$\begin{aligned}\mathbf{O} &= \{1+2n : n \in \mathbf{N_0}\}\\ &= \{1+2(2n') : n' \in \mathbf{N_0}\} \cup \{1+2(1+2n') : n' \in \mathbf{N_0}\}\\ &= \{1+4n' : n' \in \mathbf{N_0}\} \cup \{3+4n' : n' \in \mathbf{N_0}\}\\ &= R_1 \cup \overline{R}_1, \text{ with } R_1 \cap \overline{R}_1 = \emptyset,\end{aligned}$$

where $R_1 = \{1+4n' : n' \in \mathbf{N_0}\}, \overline{R}_1 = \{3+4n' : n' \in \mathbf{N_0}\}$; that is, R_1 and $\overline{R}_1$ form a partition of $\mathbf{O}$.

Also,

$$\begin{aligned}\overline{R}_1 &= \{3+4n' : n' \in \mathbf{N_0}\}\\ &= \{3+4(2n'') : n'' \in \mathbf{N_0}\} \cup \{3+4(1+2n'') : n'' \in \mathbf{N_0}\}\\ &= \{3+8n'' : n'' \in \mathbf{N_0}\} \cup \{7+8n'' : n'' \in \mathbf{N_0}\}\\ &= R_2 \cup \overline{R_1 \cup R_2}, \text{ with } R_2 \cap \overline{R_1 \cup R_2} = \emptyset,\end{aligned}$$

where $R_2 = \{3+8n'' : n'' \in \mathbf{N_0}\}, \overline{R_1 \cup R_2} = \{7+8n'' : n'' \in \mathbf{N_0}\}$, so that R_1, R_2 and $\overline{R_1 \cup R_2}$ form a more refined partition of $\mathbf{O}$.

Repeated application of this procedure gives

$$\mathbf{O} = R_1 \cup R_2 \cup R_3 \cup \ldots, \tag{2.1}$$

with $R_i \cap R_j = \emptyset$ for $i, j \in \mathbf{N}$, $i \neq j$, leading to the partition of $\mathbf{O}$ represented in a grid, a section of which is displayed in Table 1 below.

The subsets R_i in (2.1) can also be characterised as follows:

$$R_i = \{x \in \mathbf{O} : x \equiv 2^i - 1 (\text{mod } 2^{i+1})\} \tag{2.2}$$

with the $R_i, i \in \mathbf{N}$, forming the rows in Table 1. The columns are $C_j, j \in \mathbf{N}_0$.

Table 1

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	.
R_1	1	5	9	13	17	21	25	29	33	37	41	45	.
R_2	3	11	19	27	35	43	51	59	67	75	83	91	.
R_3	7	23	39	55	71	87	103	119	135	151	167	183	.
R_4	15	47	79	111	143	175	207	239	271	303	335	367	.
R_5	31	95	159	223	287	351	415	479	543	607	671	735	.
R_6	63	191	319	447	575	703	831	959	1087	1215	1343	1471	.
R_7	127	383	639	895	1151	1407	1663	1919	2175	2431	2687	2943	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.

The element $x \in \mathbf{O}$ in row i and column j, that is, in position R_i, C_j of Table 1, is $2^{i+1}j + 2^i - 1 = 2^i(2j+1) - 1$.

Given a positive integer $x \in \mathbf{O}$, the first task would therefore be to determine its position in the grid in Table 1. In [2] the procedure for locating the appropriate "cell" for $x \in \mathbf{O}$ in Table 1 was given and can be obtained by means of the following algorithm.

Algorithm	Initialisation	:	set $i = 0$
	Input	:	$x \in \mathbf{O}$
	Step 1	:	set $z = x$
	Step 2	:	set $q = \frac{z-1}{2}, i = i + 1$
	Step 3	:	if $q \equiv 0 \text{ mod } 2$, set $j = q/2$, print x, i, j end
	Step 4	:	else set $z = q$ repeat step 2.

*/comment - x occurs in R_i, C_j.

Example. The row and column can easily be read from the binary representation of any number in Table 1. Consider the rightmost zero of the binary representation. (Add on a zero at the left if the representation consists of all ones, in which case the chosen number in Table 1 is in C_0.) The number of ones to the right of this zero gives the row, while the binary number to the left of this zero gives the column. For example, the binary representation of 479 is 111011111, with 5 ones to the right of the zero and the binary number 111, or decimal 7, to the left of the zero; hence 479 is in row 5 and column 7 of Table 1.

Theorem 2.1. (Cadogan [1]). For any $i \geq 2, j \geq 0, x \in R_i \Rightarrow f^2(x) \in R_{i-1}$.

Proof: In Table 1, $x \in R_i, C_j \Rightarrow x = 2^i - 1 + j.2^{i+1}$
$\Rightarrow f(x) = 1 + 3.2^i - 3 + 3j.2^{i+1} = 3.2^i - 2 + 3j.2^{i+1}$
$\Rightarrow f^2(x) = 3.2^{i-1} - 1 + 3j.2^i = 2^{i-1} - 1 + (3j+1)2^i$,
so that , $f^2(x)) \in R_{i-1}$. ■

Corollary 2.2. For any $i \geq 2, j \geq 0, x \in R_i, C_j \Rightarrow f^2(x)$ is the element in position R_{i-1}, C_{3j+1} of Table 1. ■

Corollary 2.3. For any $i \geq 2, j \geq 0, x \in R_i, C_j \Rightarrow f^{2(i-1)}(x) = 2.3^{i-1}(2j+1) - 1$ and is in position R_1, C_n of Table 1, where, $n = 3^{i-1}.j + \sum_{r=0}^{i-2} 3^r = \frac{3^{i-1}(2j+1)-1}{2}$. ■

Remark: Corollary 2.3 provides a means of determining, at least, the initial elements in the trajectory $T(x)$ of any $x \in \mathbf{O}$. For example, $x = 23$ is in R_3, C_1 of Table 1, hence $h(23)$ is in R_2, C_4 and $h^2(23)$ is in R_1, C_{13} so $T(23)$ starts with the numbers 23, 35, 53. ■

For each $m = \sum_{r=0}^{n-1} 2^{2r}$ in R_1, $h(m) = 1$, and $m \in \{1, 5, 21, 85, 341, \cdots\} \subset R_1$.

In Table 1, let $x_i \in R_i$, $x_{i+1} \in R_{i+1}$ with x_i, $x_{i+1} \in C_j$, $i > 0$, $j \geq 0$.

Then, we have,

Lemma 2.4. $x_{i+1} = 1 + 2x_i$, $i \in \mathbf{N}$. ■

Remark: The result of Lemma 2.4. greatly simplifies the construction, from R_1, of the subsets $R_i \subset \mathbf{O}$, $i \geq 2$. ■

Lemma 2.5. Let $n \in \mathbf{N}$. Then,
$f^3(1+4n) = 1 + 3n$, for all $n \in \mathbf{N}$,
$f^3(1+4n)) = f(n)$, for all $n \in \mathbf{O}$. ■

A consequence of the result of Lemma 2.5 is the following theorem.

Theorem 2.6. Let x_1, x_2, x_3, ... be a sequence of odd integers such that $x_i = 1 + 4x_{i-1}, i \in \mathbf{N}, i \geq 2$. Then,

(i) $f(x_n) = 4^{n-1} f(x_1)$,
(ii) $f^{2n-1}(x_n) = f(x_1)$,
(iii) $f^k(x_1) = 1$, $k \in \mathbf{N}_0 \Rightarrow f^{2n+k-2}(x_n) = 1$. ■

3. Main Results

Corollary 2.3 provides the basis for the results which follow. It declares that the trajectories $L(x_i)$ and $T(x_i)$, of each $x_i \in \overline{R}_1 = \cup_{i\geq 2} R_i$ contain elements of R_1 and signifies that R_1 operates as a filter for each $L(x_i)$, $T(x_i)$. In order, therefore, to complete the telescoping process towards 1, it is essential to show that for each $x_1 \in R_1, L(x_1)$, or $T(x_1)$, ends at 1, in the sense expressed by the main conjecture.

We refer to the tableaux in Appendix 1 which contain values of $n \in \mathbf{N}_0$ for $0 \leq n \leq 150$, the corresponding values of each $x \in R_1$, where $x = 1+4n$, and the values of $f^3(x) = 1 + 3n = x - n$. Our principal approach then entails showing that for each $x \in R_1, \exists\, x' \in R_1$ such that $x' < x$, and $L(x), L(x')$ or $T(x), T(x')$ coalesce.

It follows that if $L(x)$, $L(x')$ coalesce, then so do $T(x)$, $T(x')$.

Definition 3.1. Let $\sim$ be the relation defined on $\mathbf{N}$ as follows: for $x, y \in \mathbf{N}$ $x \sim y$ iff the trajectories $L(x)$ and $L(y)$ coalesce.

Henceforth, we shall write $x \sim y$ to mean that the trajectories of $x, y \in \mathbf{N}$ coalesce.

The following result is an immediate consequence of Definition 3.1.

Lemma 3.1. $\sim$ is an equivalence relation on $\mathbf{N}$ and partitions $\mathbf{N}$ into $\sim$-classes. ■

Corollary 3.2. Let $n \in \mathbf{N}$. Then,

(i) $1 + 4n \sim 1 + 3n$,
(ii) $1 + 4n \sim 1 + 3n \sim n$, if $n \in \mathbf{O}$.

Proof: Results follow from Lemma 2.5 . ■

We now commence the process of determining specific patterns of coalescence in the trajectories based on the elements of R_1.

First, we use the elements of R_1 as **markers** to partition the values of n in the tableaux in Appendix 1 into intervals, each interval containing three elements between markers. For example, between the two markers $n = 1$ and $n = 5$ are the three values $n = 2, 3, 4$, and between markers $n = 5$ and $n = 9$ are three values $n = 6, 7, 8$; and so on. We now consider the total collection of markers noting that in each interval the integers n can be characterised as (i) $n = 1 + 4k$ (the markers), (ii) $n = 4k$, (iii) $n = 2 + 4k$ and (iv) $n = 3 + 4k, k \in \mathbf{N_0}$, with the corresponding values of $x = 1 + 4n$ given respectively by $5 + 16k, 1 + 16k, 9 + 16k$ and $13 + 16k$. Thus, we have the following results which are a consequence of Corollary 3.2. for all $k \in \mathbf{N_0}$.

Lemma 3.3. $5 + 16k \sim 1 + 4k$ ■

Lemma 3.4. $1 + 16k \sim 1 + 12k$ ■

Lemma 3.5. $9 + 16k \sim 7 + 12k$ ■

Lemma 3.6. $13 + 16k \sim 3 + 4k$ ■

Remark: In Lemmas 3.3 and 3.4, each $x \sim n$ with $n < x$ and $x, n \in R_1$. Also in Lemma 3.5, $7 + 12k = 3 + 4(1 + 3k), k \in \mathbf{N_0}$, so that for each x in Lemma 3.5, $\exists\ x'$ in Lemma 3.6 such that $x \sim x'$. ■

Corollary 3.7. By Lemma 2.5,

$$f^4(13+16k) = f^2(3+4k) = 5+6k = \begin{cases} 1 + 4\left(1 + 3\frac{k}{2}\right) \in R_1, & k \text{ even}, \\ 3 + 4\left(\frac{1+3k}{2}\right) \in \overline{R}_1, & k \text{ odd}. \end{cases}$$ ■

Lemma 3.8. Let $x_i \in R_i, x_{i+1} \in R_{i+1}, i \in \mathbf{N}$, with $x_i, x_{i+1} \in C_j, j \in \mathbf{N_0}$. Then,

$$f^{2i+1}(x_i) \in \mathbf{O} \Rightarrow i + j \in \mathbf{O}.$$

Proof: From Corollary 2.3, $f^{2i+1}(x_i) = 2.3^{i-1}(2j+1) - 1 = 1 + 4n$ is in R_1, C_n with $n = \frac{3^{i-1}(2j+1)-1}{2}$, so that, by Lemma 2.5, $f^{2i+1}(x_i) = \frac{3^i(2j+1)-1}{2}$.

Thus, $f^{2i+1}(x_i) \in \mathbf{O} \Rightarrow n \equiv 0 (\text{mod } 2)$
$\Rightarrow \frac{3^{i-1}(2j+1)-1}{2} \equiv 0(\text{mod } 2)$,
$\Rightarrow 3^{i-1}(2j+1) \equiv 1(\text{mod } 4)$.

Now, $i - 1$ is odd $\Rightarrow 3(2j + 1) \equiv 1(\text{mod } 4) \Rightarrow j \equiv 1(\text{mod } 2)$,
and $i - 1$ is even $\Rightarrow 2j + 1 \equiv 1(\text{mod } 4) \Rightarrow j \equiv 0(\text{mod } 2)$,
hence, $i + j \in \mathbf{O}$ in each case. ■

We turn our attention now to the **major result** of this Section.

Main Theorem 3.9. Let $x_i \in R_i, x_{i+1} \in R_{i+1}, i \in \mathbf{N}$, with $x_i, x_{i+1} \in C_j, j \in \mathbf{N_0}$, of Table 1. If $f^{2i+1}(x_i) \in \mathbf{O}$, then, $x_i \sim x_{i+1}$.

Proof: From Corollary 2.3., $x_i = 2^i y - 1, x_{i+1} = 2^{i+1} y - 1, y = (2j+1) \in \mathbf{O}$, so that,

$$x_i = 2^iy - 1 \sim 2^{i-1}(3y) - 1 \sim 2^{i-2}(3^2y) - 1 \sim \cdots \sim 3^iy - 1,$$

$$x_{i+1} = 2^{i+1}y - 1 \sim 2^i3y - 1 \sim 2^{i-1}3^2y - 1 \sim \cdots \sim 3^{i+1}y - 1,$$

where, $a \sim b$ is derived by applying f^2 to each term a to produce b. Also,

$$3^iy - 1 = f^{2i}(x_i),\ 3^{i+1}y - 1 = f^{2i+2}(x_{i+1}),$$

and since $3^{i+1}y - 1$ and $3^iy - 1$ are both even for $i \in \mathbf{N}$,

$$f(3^{i+1}y - 1) = \tfrac{3^{i+1}y-1}{2} \in \mathbf{N}, f(3^iy - 1) = \tfrac{3^iy-1}{2} \in \mathbf{N}.$$

Further, $1 + 2(3^k \cdot 2^{i-k}y - 1) = 3^k \cdot 2^{i-k+1}y - 1,\ \forall\ 0 \le k \le i$, so that, $3^i \cdot 2y - 1 = 1 + 2(3^iy - 1) = 1 + 4\left(\frac{3^iy-1}{2}\right)$, that is, $f^3(3^i \cdot 2y - 1) = 1+3\left(\frac{3^iy-1}{2}\right) = \frac{3^{i+1}y-1}{2}$. Hence, $\frac{3^iy-1}{2} \in \mathbf{O} \Rightarrow f\left(\frac{3^iy-1}{2}\right) = 1+3\left(\frac{3^iy-1}{2}\right) = \frac{3^{i+1}y-1}{2}$, that is, $f^{2i+2}(x_i) = f^{2i+3}(x_{i+1})$, so that $x_i \sim x_{i+1}$. ■

Example. In Table 1, $27, 55 \in C_3$ with $27 \in R_2$ and $55 \in R_3$, $55 = 1+2\cdot 27$. $L(55)$ and $L(27)$ coalesce at the number 94, with $f^7(27) = f^8(55) = 94$. Table 1 provides further examples.

Corollary 3.10. If $f^{2i+1}(x_i)$ is even, then $x_{i-1} \sim x_i$, when $i > 1$, and $x_1 \sim j$.

Proof: $f^{2i+1}(x_i)$ is even $\Rightarrow \frac{3^iy-1}{2}$ is even $\Rightarrow \frac{3^{i-1}y-1}{2} \in \mathbf{O}$ with $x_{i-1} = j$ when $i = 1$, hence by Theorem 3.9 $x_{i-1} \sim x_i$, if $i > 1$, and $x_1 \sim j$ by Corollary 2.3 and Lemma 2.5. ■

From Theorem 3.9 and Corollary 3.10 we obtain,

$$\frac{3^{i+1}y-1}{2} = 1+3\left(\frac{3^iy-1}{2}\right) = 1+3\left(1+3\left(\frac{3^{i-1}y-1}{2}\right)\right).$$

Now, let $\frac{3^{i-1}y-1}{2} = D \in \mathbf{O}$ in Corollary 3.10. Then, by Theorem 2.6 and Corollary 3.2,

(i) $x_{i-1} \sim D$,
(ii) $x_i \sim 1 + 4D \sim 1 + 3D \sim D$,
(iii) $x_{i+1} \sim 1 + 4(1 + 3D) \sim 1 + 3(1 + 3D)$.

From (i) and (ii) above, $x_{i-1} \sim x_i$ since $D \in \mathbf{O} \Rightarrow f(D) = 1 + 3D$. But $D \in \mathbf{O} \Rightarrow 1 + 3D$ is even, so that $x_i \sim x_{i+1}$ cannot be derived by a straightforward application of f. Nevertheless, it has been verified that $L(1+3D)$ and $L(1+3(1+3D))$ coalesce for $D \in \mathbf{O}$, $D \leq 10^6 - 1$ and this leads to the following conjecture.

Conjecture 3.11. $D \in \mathbf{O} \Rightarrow 1+3(1+3D) \sim 1+4(1+3D) \sim 1+3D$. ■

4. Conclusion

It has been shown in the literature [1, 4, 5, 6] that the analysis of the trajectories of the integers in R_1 of Table 1 is critical to the solution of the original conjecture. The results in Section 3 from Lemma 3.1 to Corollary 3.10 have laid the foundation for further work on the problem.

The establishment of Conjecture 3.11 would extend the results obtained in Corollary 3.2, and would provide a means of proving that each element in the first row of Table 1 'hits' another element in that row closer to the integer 1. In this way, the result of the original conjecture would be formally established.

The research is continuing.

Acknowledgements

The author wishes to thank all those who contributed to the success of this research, in particular, Professors Larry Cummings and William Gilbert of the Department of Pure Mathematics at the University of Waterloo in Canada.

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APPENDIX 1

Tables for $n \in \mathbf{N}$, $x \in R_1$ and $f^3(x) = x - n$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
x	1	5	9	13	17	21	25	29	33	37	41	45	49	53
$f^3(x)$	1	4	7	10	13	16	19	22	25	28	31	34	37	40

n	14	15	16	17	18	19	20	21	22	23	24	25	26
x	57	61	65	69	73	77	81	85	89	93	97	101	105
$f^3(x)$	43	46	49	52	55	58	61	64	67	70	73	76	79

n	27	28	29	30	31	32	33	34	35	36	37
x	109	113	117	121	125	129	133	137	141	145	149
$f^3(x)$	82	85	88	91	94	97	100	103	106	109	112

n	38	39	40	41	42	43	44	45	46	47	48
x	153	157	161	165	169	173	177	181	185	189	193
$f^3(x)$	115	118	121	124	127	130	133	136	139	142	145

n	49	50	51	52	53	54	55	56	57	58	59
x	197	201	205	209	213	217	221	255	229	233	237
$f^3(x)$	148	151	154	157	160	163	166	169	172	175	178

n	60	61	62	63	64	65	66	67	68	69	70
x	241	245	249	253	257	261	265	269	273	277	281
$f^3(x)$	181	184	187	190	193	196	199	202	205	208	211

n	71	72	73	74	75	76	77	78	79	80	81
x	285	289	293	297	301	305	309	313	317	321	325
$f^3(x)$	214	217	220	223	226	229	232	235	238	241	244

n	82	83	84	85	86	87	88	89	90	91	92
x	329	333	337	341	345	349	353	357	361	365	369
$f^3(x)$	247	250	253	256	259	262	265	268	271	274	277

n	93	94	95	96	97	98	99	100	101	102	103
x	373	377	381	385	389	393	397	401	405	409	413
$f^3(x)$	280	283	286	289	292	295	298	301	304	307	310

n	104	105	106	107	108	109	110	111	112	113	114
x	417	421	425	429	433	437	441	445	449	453	457
$f^3(x)$	313	316	319	322	325	328	331	334	337	340	343

n	115	116	117	118	119	120	121	122	123	124	125
x	461	465	469	473	477	481	485	489	493	497	501
$f^3(x)$	346	349	352	355	358	361	364	367	370	373	376

n	126	127	128	129	130	131	132	133	134	135	136
x	505	509	513	517	521	525	529	533	537	541	545
$f^3(x)$	379	382	385	388	391	394	397	400	403	406	409

n	137	138	139	140	141	142	143	144	145	146	147
x	549	553	557	561	565	569	573	577	581	585	589
$f^3(x)$	412	415	418	421	424	427	430	433	436	439	442

n	148	149	150
x	593	597	601
$f^3(x)$	445	448	451