Hall's Multicoloring Condition and Common Partial Systems of Distinct Representatives

by

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Abstract

If L is a list assignment function and κ is a multiplicity function on the vertices of a graph G, a certain condition on (G, L, κ), known as Hall's multicoloring condition, is obviously necessary for the existence of a multicoloring of the vertices of G. A graph G is said to be in the class MHC if it has a multicoloring for any functions L and κ such that (G, L, κ) satisfies Hall's multicoloring condition. It is known that if G is in MHC then each block of G is a clique and each cutpoint lies in precisely two blocks. We conjecture that the converse is true as well. It is also known that if G is a graph consisting of two cliques joined at a point then G is in MHC. We present a new proof of this result which uses common partial systems of distinct representatives, the relationship between matching number and vertex covering number for 3-partite hypergraphs, and Menger's Theorem.

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1. Introduction

A vertex list assignment to a finite simple graph G is a function L from $V(G)$ to the power set $\mathcal{P}(C)$ of C where C is a finite set of colors. Let κ be a function from $V(G)$ to the natural numbers. An (L, κ)-multicoloring of G is a function $\varphi : V(G) \to P(C)$ such that

(i) $\varphi(v) \subseteq L(v)$ for each $v \in V(G)$.

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(ii) $|\varphi(v)| = \kappa(v)$ for each $v \in V(G)$.
(iii) if vertices u and v are adjacent in G then $\varphi(u) \cap \varphi(v) = \phi$.

If $\kappa(v) = 1$ for all $v \in V(G)$ then the function φ is called an L-coloring of G.

If H is an induced subgraph of G with list assignment L and σ is a color in C, we let $\alpha_{H,L}(\sigma)$ denote the maximum size of an independent set of vertices in H each of which has σ in its list. Since $\alpha_{H,L}(\sigma)$ is the maximum number of occurrences of the color σ in any (L, κ)-multicoloring of H, the following condition, known as Hall's multicoloring condition, is obviously necessary for the existence of an (L, κ)-multicoloring of G.

Hall's multicoloring condition

$$\sum_{\sigma \in C} \alpha_{H,L}(\sigma) \geq \sum_{v \in V(H)} \kappa(v) \tag{1}$$

for each induced subgraph H of a graph G with list assignment L and multiplicity function κ.

In the special case when $\kappa(v) = 1$ for all $v \in V(G)$, (1) reduces to what is known as Hall's condition, first defined in [5], which is obviously necessary for the existence of an L-coloring of G:

Hall's condition

$$\sum_{\sigma \in C} \alpha_{H,L}(\sigma) \geq |V(H)|$$

for each induced subgraph H of a graph G with list assignment L.

Let HC be the set of all graphs such that Hall's condition is sufficient for the existence of an L-coloring of G for each list assignment L, and let MHC be the set of all graphs such that Hall's multicoloring condition is sufficient for an (L, κ)-multicoloring of G for each list assignment L and multiplicity function κ. Trivially MHC is a subset of HC. That complete graphs are in HC is just a restatement of Hall's well-known theorem for the existence of a system of distinct representatives (SDR) for a set system (for example, see [2]), and that they are in MHC is a slight generalization of Hall's Theorem. The set HC is just a bit larger:

Theorem 1. (Hilton and Johnson [5]) G is in HC if and only if each block of G is a clique.

So the graph $K_{1,3}$ (a central vertex adjacent to three vertices of degree one) is in HC. But as discovered by Cropper, it is not in MHC, as can be seen by considering the following list assignment L and multiplicity function κ. Let v be the central vertex and x, y, z be the pendant vertices; let $L(v) = \{1,2,3\}$, $L(x) = \{1,2\}$, $L(y) = \{1,3\}$, $L(z) = \{2,3\}$; and let $\kappa(v) = 2$ and $\kappa(x) = \kappa(y) = \kappa(z) = 1$. It is easy to check that G with L and κ satisfies Hall's multicoloring condition, but that no (L, κ)-multicoloring exists. Any graph containing $K_{1,3}$ as an induced subgraph is not in MHC (use the same L and κ on the $K_{1,3}$ and long lists at all other vertices). So if G is in MHC then each block of G is a clique and each cutpoint is in precisely two blocks (As pointed out by Doug West this is precisely the class of graphs which are line graphs of a forest). In [3] the authors "suspect, perhaps wishfully" that the converse is also true. We state that as a conjecture.

Conjecture. G is in MHC if and only if each block of G is a clique and each cutpoint lies in precisely two blocks.

All of the above ideas are also discussed in [3], where it is proved that paths, two cliques joined at a point, and two specific graphs of order 5 are in MHC. The proof for two cliques joined at a point is surprizingly long and has many technical details. The purpose of this paper is to present another proof of this result, which we state as Theorem 2, using common partial systems of distinct representatives, matching and vertex covering number for 3-uniform hypergraphs, and Menger's Theorem. The relationships among these concepts illuminated by the proof may well yield a proof of the above conjecture and turn out to be useful for other list-coloring problems as well.

Theorem 2. (Cropper, et al. [3])
If G is any graph consisting of two cliques joined at a point then G is in MHC.

2. Simplifying assumptions

Let G be a graph with list assignment function L and multiplicity function κ. We form a new graph G' by replacing each $v \in V(G)$ with a clique of size $\kappa(v)$. Vertices x and y in $V(G')$ in cliques replacing u and v in G are adjacent in G' if and only if u and v are adjacent in G. We define a list assignment L' on G' by $L'(x) = L(u)$ for each vertex x in the clique replacing u. We call the pair (G', L') the *explosion* of the triple (G, L, κ). Clearly, (G', L') satisfies Hall's condition if and only if (G, L, κ) satisfies Hall's multicoloring condition and G' has an L'-coloring if and only if G has an (L, κ)-multicoloring.

We remark that it is possible that G is in MHC, G has an (L, κ)-multicoloring, yet G' is not in HC. By Theorem 1, if G' has a block which is not a clique, then it is not in HC. For example, if G is the path with 3 vertices then G is in MHC (it is a path and it is also two cliques joined at a point). If $\kappa = 2$ at the joining vertex and $\kappa = 1$ at the other two vertices then for any list assignment L to G the explosion (G', L') of (G, L, κ) will be a cycle $wxyz$ plus the chord $[x, z]$ with $L'(x) = L'(z)$. If (G', L') satisfies Hall's condition then G' will have an L'-coloring. But by Theorem 1, G' is not in HC. If $L''(w) = \{1,2\}$, $L''(x) = \{1\}$, $L''(y) = \{1,3\}$, $L''(z) = \{2,3\}$, then (G', L'') satisfies Hall's condition, but G' has no L''-coloring (of course $L''(x) \neq L''(z)$).

A graph $G(A, K, B)$ is called a "three linked cliques graph" if $V(G)$ can be partitioned into three non-empty sets A, K, B such that any two vertices of G are adjacent except if one is in A and the other is in B. If H is a graph consisting of two cliques joined at the vertex v, with list assignment L and multiplicity function κ then the explosion of (H, L, κ) is a three linked cliques graph $G(A, K, B)$ with $|K| = \kappa(v)$ and list assignment L' such that $L'(x) = L(v)$ for each x in K.

Let $G(A, K, B)$ be a three linked cliques graph with list assignment L. If some color σ in C appears in the list of some vertex in A, the list of some vertex in B, but not in the common central list, replace all occurrences of σ in lists in B by the new color σ'. If some color τ appears in the list of some vertex in A, but not in the list of any vertex in B or K, add a new vertex to B with list $\{\tau\}$ and add τ to the common central list. Add a vertex to A in a similar way if τ appears in the list of some vertex in B, but not in the list of any vertex in A or K. Repeat these operations until a (uniquely determined) graph G' with list assignment L' is obtained where the common central list is the set of all colors. It is not hard to see that (G', L') satisfies Hall's condition if and only if (G, L) does, and G' has an L'-coloring if and only if G has an L-coloring. So Theorem 2 is equivalent to the following.

Theorem 2′. Let $G(A, K, B)$ be a three linked cliques graph with list assignment L from a set of colors C such that $L(v) = C$ for each v in K. If (G, L) satisfies Hall's condition then G has an L-coloring.

3. Common partial systems of distinct representatives

Let $A_1, A_2, ..., A_a$ and $B_1, B_2, ...B_b$ be subsets of a set $\mathcal{C}$. A common partial system of distinct representatives (CPSDR) for the set systems $\mathcal{A} = \{A_1, ..., A_a\}$ and $\mathcal{B} = \{B_1, B_2, ...B_b\}$ is a subset T of $\mathcal{C}$ such that T is a partial system of distinct representatives (PSDR) for both $\mathcal{A}$ and $\mathcal{B}$, i.e., T is an SDR for some subset of $\mathcal{A}$ and some subset of $\mathcal{B}$. We will need the following elementary result about completing PSDR's.

Lemma 1. Let $\mathcal{A}$ be a set system of subsets of a set S with t the maximum size of a PSDR. If P is any PSDR for $\mathcal{A}$ then there exists a PSDR Q for $\mathcal{A}$ of size t such that $P \subseteq Q$.

Proof. Construct the related bipartite graph G with $V(G) = S \cup \mathcal{A}$ and $E(G) = \{[x, A] \,|\, x \in S,\, A \in \mathcal{A}, \text{ and } x \in A\}$. There is an obvious correspondence between maximum size PSDR's of $\mathcal{A}$ and maximum matchings of G. At each step in the well-known alternating chain algorithm for finding a maximum matching in G (for example, see [2]), the size of the matching increases by one, and one vertex is added to the set of vertices in S and the set of vertices in $\mathcal{A}$ which meet the matching. □

If $G(A, K, B)$ is a three linked cliques graph with list assignment L, then we associate with G the pair of set systems $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A}$ consists of the lists of vertices in A and $\mathcal{B}$ consists of the lists of vertices in B.

Lemma 2. Let $G(A, K, B)$ be a three linked clique graph of order n with list assignment L from a set $\mathcal{C}$ of m colors such that $L(v) = \mathcal{C}$ for each $v \in K$. Let $\mathcal{A}$ and $\mathcal{B}$ be the associated set systems, let t be the maximum size of a CPSDR for $\mathcal{A}$ and $\mathcal{B}$, and assume both $\mathcal{A}$ and $\mathcal{B}$ have SDR's. Then G has an L-coloring if and only if $m + t \geq n$.

Proof. If φ is an L-coloring of G, let p be the number of colors which are used twice, and q be the number of colors which are used once. Since $p + q \leq m$ and $p \leq t$ we have

$$n = 2p + q = (p + q) + p \leq m + t.$$

Conversely, suppose $m + t \geq n$, let Q be a CPSDR of size t for $\mathcal{A}$ and $\mathcal{B}$, and let the cardinalities of A, B, and K be a, b, and k respectively. By

Lemma 1, Q can be extended to get SDR's for $\mathcal{A}$ and $\mathcal{B}$. The associated coloring of A and B uses $a+b-t$ colors. This leaves

$$m-a-b+t=m-(n-k)+t=k+m+t-n\geq k.$$

colors to use on vertices in K. □

Lemma 2 shows that the maximum size t of a CPSDR for $\mathcal{A}$ and $\mathcal{B}$ is the determining factor as to whether or not G has an $\mathcal{L}$-coloring. But whether or not Hall's condition is satisfied depends on the sum of the independence numbers for the colors, and this depends on the number r of colors which occur in both $\mathcal{A}$ and $\mathcal{B}$ (and in both subsets of $\mathcal{A}$ and subsets of $\mathcal{B}$). Trivially $t \leq r$, but clearly equality need not hold in general. We will define a 3-uniform hypergraph to reveal more about the relationship between t and r.

4. Hypergraphs and the König Property

Recall that a hypergraph H is a set of vertices (or points) $V(H)$ together with a collection $E(H)$ of subsets of $V(H)$ called edges (or lines). A hypergraph is r-uniform if each line has size r and is r-partite if there is a partition $V_1, V_2, ..., V_r$ of $V(H)$ such that each edge of H is of the form $(x_1, x_2, ..., x_r)$ where $x_i \in V_i$ for each i. A set of disjoint lines in $E(H)$ is called a matching. The matching number $m(H)$ of H is the maximum number of lines in a matching. A subset of $V(H)$ is called a vertex cover if it meets every line.

The minimum size $\tau(H)$ of a vertex cover is called the vertex covering number of H. Obviously,

$$m(H) \leq \tau(H). \tag{2}$$

If H is a bipartite graph, then König's Theorem (for example, see [2]) says that equality holds in (2), but of course equality need not hold for a general hypergraph.

Example 1. The 3-partite hypergraph H with $V(H) = \{1,2,3,4,5,6\}$ and $E(H) = \{(1,3,5), (1,3,6), (2,3,5), (1,4,5)\}$ has matching number equal to 1 and vertex covering number equal to 2.

No complete solution of the hypergraph matching problem is known, and the problem is NP-complete even for 3-uniform hypergraphs [4].

Following Lovasz and Plummer [6], we will say that a hypergraph H has the König Property if equality holds in (2). Berge and Las Vergnas [1] found a class of hypergraphs, called balanced hypergraphs, a generalization of bipartite graphs in that they do not contain a certain type of generalized odd circuit, which do have the König Property.

Theorem 3. Let $\mathcal{A} = \{A_1, ...A_a\}$ and $\mathcal{B} = \{B_1, ...B_b\}$ be set systems with ground set $\mathcal{C}$. Then the associated 3-partite hypergraph $H = H(\mathcal{A}, \mathcal{B}, \mathcal{C})$ with $V(H) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and $E(H) = \{(A_i, B_j, \sigma) : A_i \in \mathcal{A},\ B_j \in \mathcal{B}$, and $\sigma \in (A_i \cap B_j)$ has the König Property.

Proof. For each σ in $\mathcal{C}$ we define the projection of H onto σ to be the bipartite graph B_σ with $V(B_\sigma) = \{A_i \in \mathcal{A} : \sigma \in A_i\} \cup \{B_j \in \mathcal{B} : \sigma \in B_j\}$ and $E(B_\sigma) = \{[A_i, B_j] : \sigma \in (A_i \cap B_j)\}$. The graph B_σ is clearly complete bipartite for each $\sigma \in \mathcal{C}$. Now we define a tripartite graph $T = T(H)$ with $V(T) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and $E(T) = \{[A_i, \sigma] : \sigma \in A_i \in \mathcal{A}\} \cup \{[B_j, \sigma] : \sigma \in B_j \in \mathcal{B}\}$. Finally, we form the graph $F(H)$ from $T(H)$ by adding vertices x and y, x adjacent to all vertices in $\mathcal{A}$ and y adjacent to all vertices in $\mathcal{B}$. For each edge (A_i, B_j, σ) in H there is a path x, A_i, σ, B_j, y in $F(H)$. Because B_σ is complete bipartite, for each path x, A_i, σ, B_j, y in $F(H)$ there is an edge (A_i, B_j, σ) in H. If P is any path from x to y in $F(H)$ (possibly with more than five vertices) then there must be a segment A_i, σ, B_j in P (a vertex $\sigma \in \mathcal{C}$ adjacent to a vertex in $\mathcal{A}$ and a vertex in $\mathcal{B}$). We call the path $P' = x, A_i, \sigma, B_j, y$ a shortening of P. If we have any set of m vertex disjoint paths from x to y, then by taking a shortening of each we obtain a set of m vertex disjoint paths from x to y with five vertices in each. Hence the matching number for H is equal to the maximum number of vertex disjoint paths from x to y in $F(H)$. By Merger's Theorem (for example see [7]) this is equal to the minimum size of an (x, y)-cutset for $F(H)$, i.e., the smallest subset of $V(T)$ whose removal disconnects x from y. But every (x, y)-cutset for $F(H)$ is a vertex cover for H. □

Note that for the hypergraph H in Example 1 the projection of H onto the color 5 has edge set $\{[1,3], [1,4], [2,3]\}$, which is not complete bipartite. That is why there can exist a path $x, 2, 5, 4, y$ in $T(H)$, but no edge $(2, 5, 4)$ in H, and hence why H does not have the König Property.

5. Proof of the main result

Lemma 3. Let $\mathcal{A} = \{A_1, A_2, ..., A_\sigma\}$ and $\mathcal{B} = \{B_1, B_2, ..., B_b\}$ be set systems with ground set $\mathcal{C}$ and let t be the maximum size of a CPSDR for $\mathcal{A}$ and $\mathcal{B}$. Suppose that for each $A_i \in \mathcal{A}$ and $B_j \in \mathcal{B}$ the pairs of set systems $(\mathcal{A} - A_i), \mathcal{B}$ and $\mathcal{A}, (\mathcal{B} - B_j)$ each have a CPSDR of size t. Then there are precisely t elements of C which appear in both some set in $\mathcal{A}$ and some set in $\mathcal{B}$.

Proof. By Theorem 3 the associated 3-partite hypergraph $H = H(\mathcal{A},\mathcal{B},C)$ has the König Property. Since the matching number of H is equal to the maximum size of a CPSDR we have $m(H) = \tau(H) = t$. Let P be a vertex cover for H of size t. P cannot contain an element of $\mathcal{A}$ or of $\mathcal{B}$, because if it did then deleting that element leaves a set system with covering number $t-1$, a contradiction. Hence P is a subset of C and every edge (A_i, B_j, σ) of H must have σ in P. That means precisely t elements occur in both some set in $\mathcal{A}$ and some set in $\mathcal{B}$. □

We remark that the idea of Lemma 3 can be used to construct a minimum size vertex cover for the hypergraph $H(\mathcal{A}, \mathcal{B}, C)$. If t is the maximum size of a CPSDR for the set systems $\mathcal{A}$ and $\mathcal{B}$ and if some element of $\mathcal{A}$ or $\mathcal{B}$ can be removed resulting in a pair of set systems with a smaller maximum size of a CPSDR (it would be $t-1$), then remove that element, and repeat the process until removal of any element of $\mathcal{A}$ or $\mathcal{B}$ leaves the maximum size of a CPSDR unchanged. The removed elements of $\mathcal{A}$ and $\mathcal{B}$, along with the elements of C occurring in both remaining set systems, is a minimum size vertex cover.

We are now ready to prove Theorem 2′ (and hence Theorem 2).

Proof of Theorem 2′. Let $\mathcal{A}$ and $\mathcal{B}$ be the associated set systems and t be the maximum size of a CPSDR for $\mathcal{A}$ and $\mathcal{B}$. Since (G, L) satisfies Hall's condition and cliques are in HC, both $\mathcal{A}$ and $\mathcal{B}$ have SDR's. By Lemma 2, G has an L-coloring if and only if $m + t \geq n$ where $|C| = m$ and n is the order of G. It thus suffices to show that if (G, L) satisfies Hall's condition then $m + t \geq n$.

Assume G is minimal order such that there exists a set of colors C and list assignment L where $n > m + t$. If deletion of some element of $\mathcal{A}$ or $\mathcal{B}$ decreased the maximum size of a CPSDR then G would not be minimal. Hence, by Lemma 3, precisely t colors occur in both $\mathcal{A}$ and $\mathcal{B}$. Thus

$$\sum_{\sigma \in C} \alpha_{G,L}(\sigma) = m + t < n = |V(G)|,$$

a violation of Hall's condition. □

The conjecture that G is in MHC if and only if each block of G is a clique and each cutpoint lies in precisely two blocks has recently been proved by Cropper, Gyárfás, and Lehel.

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