On d-antimagic labelings of type (1,1,1) for prisms

Martin Bača,
Department of Applied Mathematics, Technical University,
Letná 9, 042 00 Košice, Slovak Republic
hollbaca@ccsun.tuke.sk
Mirka Miller,
Department of Computer Science and Software Engineering,
University of Newcastle, NSW 2308, Australia
mirka@cs.newcastle.edu.au

Abstract

We deal with the problem of labeling the vertices, edges and faces of a plane graph in such a way that the label of a face and the labels of the vertices and edges surrounding that face add up to a weight of that face and the weights of all the faces constitute an arithmetical progression of difference d.

1 Introduction and definitions

G is a finite connected graph without loops and multiple edges, $V(G)$ is its vertex-set and $E(G)$ is its edge-set. A general reference for graph theoretic notions is [15].

A graph G is said to be *plane* if it is drawn on the Euclidean plane in such a way that edges do not cross each other except at vertices of the graph. We make the convention that all plane graphs considered in this paper possess no vertices of degree one. For a plane graph G, it makes sense to determine its faces, including the unique face of infinite area. Let $F(G)$ be the face-set of G and $|V(G)| = v$, $|E(G)| = e$ and $|F(G)| = f$ are the number of vertices, edges and faces of G.

A labeling of type $(1, 1, 1)$ assigns labels from the set $\{1, 2, \ldots, v+e+f\}$ to the vertices, edges and faces of plane graph G in such a way that each vertex, edge and face receives exactly one label and each number is used exactly once as a label. If we label only vertices or only edges or only faces, we call such

JCMCC 44 (2003), pp. 199-207

a labeling a vertex labeling, an edge labeling or a face labeling, respectively. Alternatively we refer to these labelings as labelings of type $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$ respectively.

A labeling of type $(1,1,0)$ is a bijection from the set $\{1,2,\ldots,v+e\}$ onto the vertices and edges of $G(V,E,F)$.

The *weight* of a face under a labeling is the sum of the labels (if present) carried by that face and the edges and vertices surrounding it.

A labeling is said to be *face-magic* if for every number s all s-sided faces have the same weight. We allow different weights for different s.

The notion of *face-magic* labeling of plane graphs was defined by Ko-Wei Lih in [11]. Ko-Wei Lih called such labeling *magic* but this notion of being *magic* is entirely different from those defined in [6, 8, 9,10, 12, 14]. Face-magic labelings of type $(1,1,0)$ for wheels, friendship graphs and prisms are given in [11] and face-magic labelings of type $(1,1,1)$ for special classes of plane graphs are described in [1] and [2].

A connected plane graph $G=(V,E,F)$ is said to be *(a,d)-face antimagic* if there exist positive integers a, d and a bijection from the set $\{1,2,\ldots,e\}$ onto the edges of G such that the induced mapping $\varphi: F(G) \to W$ is also a bijection, where $W=\{a,a+d,\ldots,a+(f-1)d\}$ is the set of weights of all faces of G.

The (a,d)-face antimagic graph was defined in [3] and further results can be found in [4]. Other types of antimagic labelings were considered by Hartsfield and Ringel [7], Bodendiek and Walther [5] and Simanjuntak, Miller and Bertault [13].

Now, let us define the concept of d-antimagic labeling of plane graph in the following way.

A bijection $g: V(G) \cup E(G) \to \{1,2,\ldots,v+e\}$ (respectively a bijection $h: V(G)\cup E(G)\cup F(G) \to \{1,2,\ldots,v+e+f\}$) is called a *$d$-antimagic labeling of type* $(1,1,0)$ (respectively *of type* $(1,1,1)$) of plane graph G if for every number s the set of s-sided face weights is $W_s=\{a_s,a_s+d,a_s+2d,\ldots,a_s+(f_s-1)d\}$ for some integers a_s and d, where f_s is the number of s-sided faces. We allow different sets W_s for different s.

In this paper we describe various d-antimagic labelings of type $(1,1,1)$ for prisms.

2 Face-magic and d-antimagic labeling

In this section we present a relationship between face-magic labeling of type $(1,1,0)$ and d-antimagic labeling of type $(1,1,1)$ of plane graph.

Let us consider a plane graph G with s_i-sided faces, $i = 1,2,\dots,n$. Let f_{s_i} be the number of s_i-sided faces of G. Suppose G has the face-magic labeling of type $(1,1,0)$, i.e., there exists a bijection $g : V(G) \cup E(G) \to \{1,2,\dots,v+e\}$ such that all s_i-sided faces ($i = 1,2,\dots,n$) have the same weight.

Define a face labeling h with values in the set $\{v+e+1,\dots,v+e+f_{s_1}\}\cup\{v+e+f_{s_1}+1,\dots,v+e+f_{s_1}+f_{s_2}\}\cup\dots\cup\{v+e+f_{s_1}+\dots+f_{s_{n-1}}+1,\dots,v+e+f_{s_1}+\dots+f_{s_n}\}$ such that the s_1-sided faces are labeled with $v+e+1,\dots,v+e+f_{s_1}$, the s_2-sided faces with $v+e+f_{s_1}+1,\dots,v+e+f_{s_1}+f_{s_2}$, and so on.

By combining g and h we obtain the 1-antimagic labeling of type $(1,1,1)$, thus obtaining the following theorem.

Theorem 1 *If G has a face-magic labeling of type $(1,1,0)$ then G has a 1-antimagic labeling of type $(1,1,1)$.*

3 The results

The *prism* D_n, $n \geq 3$, is a trivalent graph which can be defined as the cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices. The prism D_n, $n \geq 3$, is the plane graph of a convex polytope.

Let $I = \{1,2,\dots,n\}$ be an index set. Prism D_n, $n \geq 3$, consists of an outer n-cycle $y_1\ y_2\ \dots\ y_n$, an inner n-cycle $x_1\ x_2\ \dots\ x_n$, and a set of n spokes x_iy_i, $i \in I$. $|V(D_n)| = 2n$, $|E(D_n)| = 3n$ and $|F(D_n)| = n+2$. (See Figure 1).

Ko-Wei Lih [11] proved that if $n \geq 3$, then the prism D_n has a face-magic labeling of type $(1,1,0)$. The common weight for all 4-sided faces under the magic labeling of type $(1,1,0)$ is $18n+4$ (if n is even) and $17n+4$ (if n is odd). The two n-sided faces have the same weight $\frac{4n^2+25n+8}{2}$ (if n is even) and $3n^2+9n$ (if n is odd).

If h_1 is face labeling with values in the set $\{v+e+1, v+e+2\} \cup \{v+e+3, v+e+4,\dots,v+e+f\}$ then from Theorem 1 it follows that

Corollary 1 *For $n \geq 3$, the prism D_n has the 1-antimagic labeling of type* $(1,1,1)$.

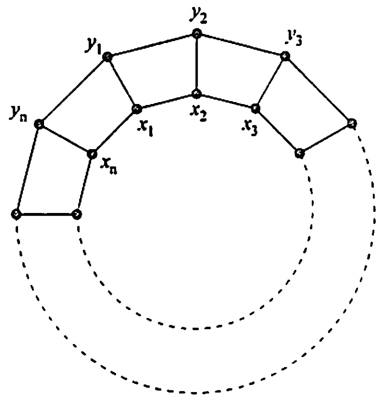

Figure 1: The prism D_n

If $n \equiv 3 \pmod 4$, $n \geq 3$, we construct a vertex labeling g_1 and an edge labeling g_2 of a prism D_n as given below, where

$$\delta(x) = \begin{cases} 0 & \text{if} \quad x \equiv 0 \pmod 2 \\ 1 & \text{if} \quad x \equiv 1 \pmod 2 \end{cases} \tag{1}$$

$$\rho(x,y,z) = \begin{cases} 1 & \text{if} \quad x \leq y \leq z \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and $i \in I$ with indices taken modulo n.

$$\begin{aligned}
g_1(x_i) &= \left[(2n+1-i)\rho(1,i,\frac{n-1}{2}) + \frac{i+1}{2}\rho(\frac{n+3}{2},i,n)\right]\delta(i) + \\
&\quad \left[\frac{n+1+i}{2}\rho(2,i,\frac{n+1}{2}) + (2n+1-i)\rho(\frac{n+5}{2},i,n-1)\right]\delta(i+1), \\
g_1(y_i) &= \left[\frac{i+1}{2}\rho(1,i,\frac{n-1}{2}) + (2n+1-i)\rho(\frac{n+3}{2},i,n)\right]\delta(i) + \\
&\quad \left[(2n+1-i)\rho(2,i,\frac{n+1}{2}) + \frac{n+1+i}{2}\rho(\frac{n+5}{2},i,n-1)\right]\delta(i+1), \\
g_2(x_iy_i) &= (2n+1)\rho(1,i,1) + (3n+2-i)\rho(2,i,n), \\
g_2(x_ix_{i+1}) &= 4n\rho(1,i,1) + \left[\frac{9n+3-2i}{2}\rho(3,i,\frac{n-1}{2}) + \frac{11n+3-2i}{2}\rho(\frac{n+3}{2},i,n)\right]\delta(i) + \\
&\quad \left[(3n-2+2i)\rho(2,i,\frac{n+1}{2}) + (2n+2i-2)\rho(\frac{n+5}{2},i,n-1)\right]\delta(i+1), \\
g_2(y_iy_{i+1}) &= \left[(3n-2+2i)\rho(3,i,\frac{n-1}{2}) + (2n+2i-2)\rho(\frac{n+3}{2},i,n)\right]\delta(i) +
\end{aligned}$$

$$\frac{9n+1}{2}\rho(1,i,1) +$$
$$\left[\frac{9n+3-2i}{2}\rho(2,i,\frac{n+1}{2}) + \frac{11n+3-2i}{2}\rho(\frac{n+5}{2},i,n-1)\right]\delta(i+1).$$

Theorem 2 *For $n \geq 3$, $n \equiv 3 \pmod 4$, the prism D_n has a 3-antimagic labeling of type* $(1,1,1)$.

Proof. Label the vertices and the edges of D_n by g_1 and g_2, respectively. We obtain a labeling of type $(1,1,0)$ with labels from the set $\{1,2,\dots,v+e\}$. The weights of 4-sided faces successively assume the values $16n+5$, $16n+7,\dots,18n+3$ and the weight of the internal (external) n-sided face is $5n^2+n+1$ ($5n^2+n-1$).

If h_1 is face labeling with values in the set $\{v+e+1, v+e+2\} \cup \{v+e+3, v+e+4,\dots,v+e+f\}$ then the resulting labeling of type $(1,1,1)$ can be

(i) 1-antimagic with the weights of 4-sided faces in the set $\{22n+7,\ 22n+8,\dots,23n+6\}$ and the weights of n-sided faces $5n^2+6n+2$ and $5n^2+6n+1$ or

(ii) 3-antimagic, where the weights of 4-sided faces constitute a set $\{21n+8,\ 21n+11,\ 21n+14,\dots,24n+5\}$ and the weights of n-sided faces are $5n^2+6n$ and $5n^2+6n+3$. This proves that labeling of type $(1,1,1)$ of D_n is 3-antimagic. □

If $i \in I$, we define the vertex labeling g_3 of D_n, where $n \geq 3$, $n \equiv 3 \pmod 4$, and the function $\delta(x)$ is defined in (1).

$$g_3(x_i) = i\delta(i) + (2n-i+1)\delta(i+1),$$

$$g_3(y_i) = (2n+1-i)\delta(i) + i\delta(i+1).$$

Theorem 3 *For $n \geq 3$, $n \equiv 3 \pmod 4$, the prism D_n has a 2-antimagic labeling of type* $(1,1,1)$.

Proof. Label the vertices and the edges of D_n by vertex labeling g_3 and edge labeling g_2, respectively. The labelings g_3 and g_2 combine to a labeling of type $(1,1,0)$. The weights of 4-sided faces successively attain values $\frac{33n+9}{2}, \frac{33n+11}{2},\dots,\frac{35n+7}{2}$ and the weights of both n-sided faces are $5n^2+\frac{n+1}{2}$ and $5n^2+\frac{3n-1}{2}$.

Let h_2 be a face labeling with values in the set $\{v+e+1, v+e+f\} \cup \{v+e+2, v+e+3, \ldots, v+e+f-1\}$. If the 4-sided faces successively assume the values of $\{v+e+2, v+e+3, \ldots, v+e+f-1\}$, the internal n-sided face receives the value $v+e+f$ and the external n-sided face receives the value $v+e+1$ then we obtain 2-antimagic labeling of type $(1,1,1)$. □

Theorem 4 *For $n \geq 3$, $n \equiv 3 \pmod 4$, the prism D_n has a 4-antimagic labeling and 6-antimagic labeling of type $(1,1,1)$.*

Proof. We assume that $n \equiv 3 \pmod 4$, $n \geq 3$. In this case we construct the vertex labelings g_4, g_5 and the edge labelings g_6, g_7 in the following way, where the functions $\delta(x)$ and $\rho(x,y,z)$ are as defined in (1) and (2).

$$
\begin{aligned}
g_4(x_i) &= \begin{cases} i & \text{if} \quad i \equiv 2 \pmod 4 \\ i-1 & \text{if} \quad i \equiv 0 \pmod 4 \\ 2n+1-i & \text{if} \quad i \equiv 1 \pmod 4 \\ 2n-i & \text{if} \quad i \equiv 3 \pmod 4 \end{cases} && \text{for } i \in I - \{n\}, \\
g_4(x_i) &= n && \text{for } i = n. \\
g_4(y_i) &= \begin{cases} i & \text{if} \quad i \equiv 1 \pmod 4 \\ i+1 & \text{if} \quad i \equiv 3 \pmod 4 \\ 2n+1-i & \text{if} \quad i \equiv 2 \pmod 4 \\ 2n+2-i & \text{if} \quad i \equiv 0 \pmod 4 \end{cases} && \text{for } i \in I - \{n\}, \\
g_4(y_i) &= n+1 && \text{for } i = n. \\
g_5(x_i) &= g_4(x_i) && \text{for } i \in I - \{n\}, \\
g_5(x_i) &= n+1 && \text{for } i = n. \\
g_5(y_i) &= g_4(y_i) && \text{for } i \in I - \{n\}, \\
g_5(y_i) &= n && \text{for } i = n.
\end{aligned}
$$

$$
\begin{aligned}
g_6(x_i x_{i+1}) &= 2n\rho(1,i,1) + [(4i-4)\rho(2,i,\frac{n+1}{2}) \\
&+ (4i-2n-4)\rho(\frac{n+5}{2},i,n-1)]\delta(i+1) \\
&+ [(3n+3-2i)\rho(3,i,\frac{n-1}{2}) + (5n+3-2i)\rho(\frac{n+3}{2},i,n)]\delta(i) \\
g_6(y_i y_{i+1}) &= (3n+1)\rho(1,i,1) + [(3n+3-2i)\rho(2,i,\frac{n+1}{2}) \\
&+ (5n+3-2i)\rho(\frac{n+5}{2},i,n-1)]\delta(i+1) \\
&+ [(4i-4)\rho(3,i,\frac{n-1}{2}) + (4i-2n-4)\rho(\frac{n+3}{2},i,n)]\delta(i) \\
g_6(x_i y_i) &= \rho(1,i,1) + (2n+3-2i)\rho(2,i,n) \qquad \text{for } i \in I. \\
g_7(x_i x_{i+1}) &= g_6(x_i x_{i+1}),
\end{aligned}
$$

$$
\begin{aligned}
g_7(y_i y_{i+1}) &= g_6(y_i y_{i+1}),\\
g_7(x_i y_i) &= (2n-1)\rho(1,i,1) + (i-2)\rho(3,i,n)\delta(i) \;+\; (n+i-2)\delta(i+1)\\
&\quad \text{for } i \in I.
\end{aligned}
$$

Label the vertices and the edges of D_n by g_4 and $g_6 + |V(D_n)|$, respectively. In the resulting labeling the weights of 4-sided faces constitute an arithmetical progression of difference 2 $(17n+5, 17n+7, \ldots, 19n+3)$ and the weights of n-sided faces constitute an arithmetical progression of difference 3 $(\frac{10n^2+3n-3}{2}, \frac{10n^2+3n+3}{2})$.

Let h_4 be face labeling with values in the set $\{4n+1, 4n+3, 4n+5, \ldots, 6n-1\} \cup \{6n+1, 6n+2\}$.

It is not difficult to check that combining the labelings g_4, $g_6 + |V(D_n)|$ and h_4 we can obtain a labeling of type $(1,1,1)$ such that the weights of 4-sided faces constitute the arithmetical progression $21n+2+4i$, $i \in I$, and the weights of n-sided faces are $\frac{10n^2+15n-1}{2}, \frac{10n^2+15n+7}{2}$. It means that the labeling of type $(1,1,1)$ is 4-antimagic.

Now, we label the vertices and the edges of D_n by g_5 and $g_7 + |V(D_n)|$, respectively.

We get a labeling in which the weights of 4-sided faces constitute an arithmetical progression of difference 4 $(16n+6, 16n+10, \ldots, 20n+2)$ and the weight of the internal (respectively external) n-sided face is $\frac{10n^2+3n+5}{2}$ (respectively $\frac{10n^2+3n-5}{2}$).

If h_5 is face labeling with values in the set $\{4n-1+2i, i \in I\} \cup \{6n+1, 6n+2\}$ then by combining the labelings g_5, $g_7 + |V(D_n)|$ and h_5 we obtain a 6-antimagic labeling of type $(1,1,1)$. □

4 Conclusion

In this paper we proved that for $n \geq 3$, $n \equiv 3 \pmod 4$ and $d = 1, 2, 3, 4, 6$, there exist d-antimagic labelings of type $(1,1,1)$ for graphs of prisms. We have not yet found a construction that will produce 5-antimagic labeling (for $n \equiv 3 \pmod 4$) and d-antimagic labeling of the graph D_n for $n \not\equiv 3 \pmod 4$ and $d = 2, 3, 4, 6$. However, we suggest the following

Conjecture 1 *There is a d-antimagic labeling of type $(1,1,1)$ for the plane graph D_n for $2 \leq d \leq 6$ and for all $n \geq 3$.*

We conclude with the following open problem.

Problem 1 *Find other possible values of the parameter d and the corresponding d-antimagic labeling of type* $(1, 1, 1)$ *for prisms* D_n.

References

[1] Bača, M., On magic and consecutive labelings for the special classes of plane graphs. *Utilitas Math.* **32** (1987), 59–65.

[2] Bača, M., On magic labelings of honeycomb. *Discrete Math.* **105** (1992),s 305–311.

[3] Bača, M. Face antimagic labelings of convex polytopes. *Utilitas Math.* **55** (1999), 221–226.

[4] Bača, M. and Miller, M., Antimagic face labeling of convex polytopes based on biprisms. *J. Combin. Math. Combin. Comput.* to appear.

[5] Bodendiek, R. and Walther, G., On number theoretical methods in graph labelings. *Res. Exp. Math.* **21** (1995), 3–25.

[6] Doob, M., Characterizations of regular magic graphs. *J. Combin. Theory, Ser. B* **25** (1978), 94–104.

[7] Hartsfield, N. and Ringel, G., *Pearls in Graph Theory.* (Academic Press, Boston - San Diego - New York - London, 1990).

[8] Jeurissen, R. H., Magic graphs, a characterization. *Europ. J. Combinatorics* **9** (1988), 363–368.

[9] Jezný, S. and Trenkler, M., Characterization of magic graphs. *Czechoslovak Math. J.* **33** (1983), 435–438.

[10] Kotzig, A. and Rosa, A. Magic valuations of finite graphs. *Canad. Math. Bull.* **13** (1970), 451–461.

[11] Ko-Wei Lih, On magic and consecutive labelings of plane graphs. *Utilitas Math.* **24** (1983), 165–197.

[12] Sedláček, J., Problem 27, in: *Theory of graphs and its applications, Proc. Symposium Smolenice* (June 1963), 163–167.

[13] Simanjuntak, R., Miller, M. and Bertault, F., Two new (a, d)-antimagic graph labelings. Preprint.

[14] Stewart, B. M. Magic graphs. *Can. J. Math.* **18** (1966), 1031–1059.

[15] West, D. B. *An Introduction to Graph Theory* (Prentice-Hall, 1996).