The simultaneous metamorphosis of small-wheel systems

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Abstract

Metamorphoses of small k-wheel systems for $k = 3$, 4 and 6 are obtained. In particular, we obtain simultaneous metamorphoses of: 3-wheel systems into Steiner triple systems and into $K_{1,3}$-designs; 4-wheel systems into 4-cycle systems, $K_{1,4}$-designs and bowtie systems; 6-wheel systems into 6-cycle systems, $K_{1,6}$-designs and 3-windmill designs or near-3-windmill designs.

1 Introduction and preliminaries

We start with some definitions. The *k-cycle* ($k \geq 3$) on the vertices $a_1, a_2, \ldots, a_k$ with edges $\{a_i, a_{i+1}\}$, $1 \leq i \leq k-1$, $\{a_1, a_k\}$, will be denoted by $(a_1, a_2, \ldots, a_k)$ or $(a_1, a_k, a_{k-1}, \ldots, a_2)$, or by any cyclic shift of these.

The bipartite graph $K_{1,k}$ (or *k-star*) on vertices $a_0, a_1, \ldots, a_k$ and with edges $\{a_0, a_i\}$, $1 \leq i \leq k$, will be denoted by $[a_0 : a_1, a_2, \ldots, a_k]$, where of

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course the order of the symbols $a_1, a_2, \ldots, a_k$ is arbitrary.

The *k-wheel* W_k is a simple graph with $k+1$ vertices $a_0, a_1, a_2, \ldots, a_k$ and $2k$ edges consisting of those in the k-star $[a_0 : a_1, a_2, \ldots, a_k]$ and the k-cycle $(a_1, a_2, \ldots, a_k)$. This k-wheel will be denoted by a_0–$(a_1, a_2, \ldots, a_k)$. The vertex a_0 is the *centre* of the wheel, and the edges $\{a_0, a_i\}$ for $1 \le i \le k$ are the *spokes* of the k-wheel. The k-cycle $(a_1, a_2, \ldots, a_k)$ will sometimes be referred to as the *rim* of the k-wheel.

Let G and H be simple graphs. A *G-design* or *G-system* of H is a pair (X, B) where X is the vertex set of the graph H and B is a set of isomorphic copies of the graph G whose edges partition the edge-set of H. If the graph H is a complete graph of order n then we also refer to this as a G-design or G-system of *order n*.

A *λ-fold* G-design or G-system of order n is, similarly, a pair (X, B) where X is the vertex set of a complete graph K_n and B is a collection of copies of the graph G whose edges exactly cover λ copies of K_n.

A *bowtie* is a simple graph on 5 vertices consisting of two triangles sharing one common vertex. We use the notation $\{a, b, c; a, d, e\}$ to denote the bowtie with triangles (a, b, c) and (a, d, e).

A *3-windmill* is a simple graph on 7 vertices consisting of three triangles all sharing one common vertex. We use the notation $\{a, b, c; a, d, e; a, f, g\}$ to denote the 3-windmill consisting of the triangles (a, b, c), (a, d, e) and (a, f, g).

A bowtie system is a Steiner triple system (STS) in which the triples or triangles have been paired together into bowties. Similarly, a 3-windmill system can be regarded as a STS in which the triples are arranged into 3-windmills. If the number of triples in a STS is *not* 0 (mod 3), we may consider a *near*-3-windmill system, in which all but one or two of the triples are arranged in 3-windmills.

Next we explain the concept of a metamorphosis of a design. Suppose there exists a G-design (X, B) of order n, and let G_1 be a proper subgraph of G. Let G_1' denote the complement of G_1 in G, so that $G = G_1 \cup G_1'$. For each copy, say β, of G in B, let $\beta = \beta_1 \cup \beta_1'$, where β_1 is isomorphic to G_1 and β_1' is isomorphic to G_1'. Let $B_1 = \{\beta_1 \mid \beta_1 \cup \beta_1' \in B\}$ and $B_1' = \{\beta_1' \mid \beta_1 \cup \beta_1' \in B\}$. Let B_1'' be a rearrangement of all the *edges* of the graphs in B_1' into copies of the graph G_1 (if such a rearrangement is possible). Then $(X, B_1 \cup B_1'')$ is a G_1-design. We shall say that $(X, B_1 \cup B_1'')$ is a *metamorphosis* of the G-design (X, B) into a G_1-design.

In [2], existence of metamorphoses of λ-fold 4-wheel systems into λ-fold 4-cycle systems was shown, for all except five possible cases when $\lambda = 2$.

In [1], existence of metamorphoses of λ-fold 4-wheel systems into λ-fold bowtie systems was shown, for all except three possible cases when $\lambda = 2$.

In [3], existence of metamorphoses of λ-fold 3-wheel systems into λ-fold triple systems was shown, with no exceptional cases.

In this paper we concentrate on 3-, 4- and 6-wheel systems, in the case $\lambda = 1$. We are interested in possible *simultaneous* metamorphisms of such systems. Let (X, B) be a G-design of order n, and suppose that $G = G_i \cup G'_i$, for $1 \le i \le r$. For each copy, say β, of G in B, let $\beta = \beta_i \cup \beta'_i$ where β_i is isomorphic to G_i (and so β'_i is isomorphic to G'_i). Then let $B_i = \{\beta_i \mid \beta_i \cup \beta'_i \in B\}$ and $B'_i = \{\beta'_i \mid \beta_i \cup \beta'_i \in B\}$. Let B''_i be a rearrangement of all the *edges* of the graphs in B'_i into copies of G_i (if such a rearrangement is possible). Then we say that $(X, B_i \cup B''_i)$ is a metamorphosis of the G-design (X, B) into a G_i-design.

Performing such a metamorphosis with the *same* set of G-blocks B, for subgraphs G_i of G, $1 \le i \le r$, yields a *simultaneous metamorphosis* of the G-design into G_i-designs for $1 \le i \le r$.

In this paper we exhibit:

- a simultaneous metamorphosis of a 3-wheel system into a triple system and a $K_{1,3}$-design;
- a simultaneous metamorphosis of a 4-wheel system into a 4-cycle system, a $K_{1,4}$-system and a bowtie system; and
- a simultaneous metamorphosis of a 6-wheel system into a 6-cycle system, a $K_{1,6}$-system and a 3-windmill system, or near-3-windmill system if the number of triples is not 0 (mod 3).

(See Figure 1.)

For further definitions such as that of a group divisible design (GDD), see for example [4]. We use the notation k-GDD of type $a^x b^y$ to denote a GDD with x groups of size a, y groups of size b, and block size k.

If a wheel system has a simultaneous metamorphosis as described above, we shall refer to it as a SM-wheel system (SM for "simultaneous metamorphosis").

THE CONSTRUCTION

Let the complete graph K_n have order $n = 1 + \ell s$, where generally ℓ will be 4, 8 or 12 in what follows. We use a k-GDD of order s, of type $s = a^t b^u$, and subsequently k will be 3 or 4. The vertex set for K_n is taken to be

$$\{\infty\} \cup \{(i, j) \mid 1 \le i \le s,\ \ 1 \le j \le \ell\}.$$

For each group $\{x_1, x_2, \ldots, x_g\}$ of the k-GDD, on the vertex set $\{\infty\} \cup \{(x_i, j) \mid 1 \le i \le g,\ 1 \le j \le \ell\}$, we place a SM-wheel design of order $1 + g\ell$.

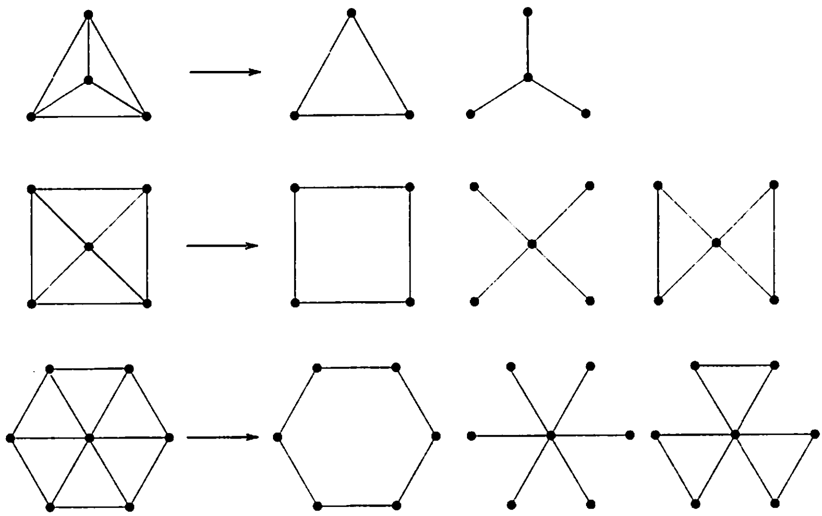

Figure 1 The simultaneous metamorphosis of small-wheel systems

For each block $\{x_1, x_2, \ldots, x_k\}$ of the k-GDD, on $\bigcup_{i=1}^{k}\{(x_i, j) \mid 1 \leq j \leq \ell\}$, we place a SM-wheel design of $K_{\ell,\ell,\ldots,\ell}$ (k-partite).

The result is a SM-wheel design of order $n = 1 + \ell s$.

2 The 3-wheel

Note that a 3-wheel is also a complete graph K_4. It is straightforward to check that the necessary condition for simultaneous metamorphosis of a 3-wheel system of order n into a triple system, and into a $K_{1,3}$-design, is that $n \equiv 1 \pmod{12}$.

Example 2.1 A SM-3-wheel system of order 13.

A metamorphosis of a 3-wheel system of order 13, simultaneously into a Steiner triple system and into a $K_{1,3}$-design (that is, a SM-3-wheel system of order 13), may be constructed as follows.

Let the vertex set be $\mathbb{Z}_{13}$. Take the 3-wheel (or K_4-) system given by developing the starter block $\{0, 1, 3, 9\}$ (mod 13). In wheel notation, call this 0–(1, 3, 9) (mod 13).

Then a metamorphosis into a STS is given by developing the starter blocks (1, 3, 9), (0, 1, 4) (mod 13).

And a (simultaneous) metamorphosis into a $K_{1,3}$-design is given by developing the starter blocks $[0:1,3,9]$, $[0:2,5,6]$ (mod 13).

Example 2.2 A SM-3-wheel system of order 25.

Let the vertex set be $\{i_j \mid 0 \le i \le 4,\ 1 \le j \le 5\}$. In each case, the required design is obtained by developing the given starter blocks mod $(5,-)$, so the base is cycled mod 5, and the subscript remains fixed.

The starter 3-wheels are:

0_3–$(0_4,1_4,0_5)$ 0_1–$(1_1,0_2,0_3)$ 0_1–$(2_1,3_4,1_5)$ 0_1–$(1_2,2_2,3_5)$
0_1–$(3_2,2_4,4_4)$ 0_1–$(1_3,2_3,0_4)$ 0_1–$(3_3,0_5,2_5)$ 0_2–$(2_2,3_3,0_4)$
0_2–$(2_3,4_3,0_5)$ 0_2–$(2_4,3_5,4_5)$.

Then rearranging the wheel spokes gives the following *extra* starter triples, taken together with the previous triangle wheel rims:

$(0_1,1_1,1_3)$ $(0_1,2_1,2_5)$ $(0_1,0_2,0_4)$ $(0_1,1_2,3_4)$ $(0_1,2_2,1_5)$
$(0_1,3_2,3_5)$ $(0_1,2_3,2_4)$ $(0_1,3_3,4_4)$ $(0_2,2_2,4_3)$ $(0_2,3_3,3_5)$.

Finally, we retain the wheel spokes (which are copies of $K_{1,3}$), and rearrange the wheel rims into the following further starter copies of $K_{1,3}$:

$[0_1:4_2,4_3,1_4]$ $[0_2:1_2,0_3,1_3]$ $[0_2:1_4,3_4,4_4]$ $[0_3:1_3,2_3,2_4]$
$[0_3:3_4,4_4,1_5]$ $[0_3:2_5,3_5,4_5]$ $[0_4:1_4,2_4,0_5]$ $[0_4:1_5,2_5,3_5]$
$[0_5:1_1,3_2,4_2]$ $[0_5:1_4,1_5,2_5]$.

Example 2.3 A SM-3-wheel system of $K_{4,4,4,4}$.
Let the vertex set be $\{0,1,2,3\}\cup\{4,5,6,7\}\cup\{8,9,10,11\}\cup\{12,13,14,15\}$.

The starter 3-wheels are:

0–(4,8,12)	0–(5,9,13)	0–(6,10,14)	0–(7,11,15)
1–(4,9,14)	1–(5,8,15)	1–(6,11,12)	1–(7,10,13)
4–(2,10,15)	4–(3,11,13)	5–(2,11,14)	5–(3,10,12)
12–(2,7,9)	13–(2,6,8)	14–(3,7,8)	15–(3,6,9).

Then rearranging the wheel spokes gives the following *extra* starter triples, taken together with the previous triangle wheel rims:

(0, 4, 10)	(0, 5, 11)	(0, 6, 13)	(0, 7, 12)
(0, 8, 14)	(0, 9, 15)	(1, 4, 11)	(1, 5, 10)
(1, 6, 15)	(1, 7, 14)	(1, 8, 13)	(1, 9, 12)
(2, 4, 13)	(2, 5, 12)	(3, 4, 15)	(3, 5, 14).

Finally, we retain the wheel spokes (which are copies of $K_{1,3}$), and rearrange the wheel rims into the following further starter copies of $K_{1,3}$:

[2 : 6, 7, 8]	[2 : 9, 10, 11]	[3 : 6, 7, 8]	[3 : 9, 10, 11]
[4 : 8, 9, 12]	[5 : 8, 9, 13]	[6 : 8, 9, 10]	[6 : 11, 12, 14]
[7 : 8, 9, 10]	[7 : 11, 13, 15]	[10 : 12, 14, 15]	[11 : 13, 14, 15]
[12 : 3, 8, 11]	[13 : 3, 9, 10]	[14 : 2, 4, 9]	[15 : 2, 5, 8].

THEOREM 2.1 *There exists a SM-3-wheel system of order n for all $n \equiv 1 \pmod{12}$.*

Proof We use the Construction described in the previous section, and let the order n be $n = 12x + 1$, where $\ell = 4$, $s = 3x$ and $k = 4$.

When $x \equiv 0$ or $1 \pmod 4$, $x \geq 4$, there is a 4-GDD of type 3^x (see [4]). Then Examples 2.1 and 2.3 above complete the construction for such x.

When $x \equiv 2$ or $3 \pmod 4$, $x \geq 7$, there is a 4-GDD of type $6^1 3^{x-2}$ (see [4]). Then we also use SM-3-wheel-designs given in Examples 2.1, 2.2 and 2.3 to complete the construction in this case.

Finally, when $x = 6$, a SM-3-wheel-design of order 73 is given in the Appendix. □

3 The 4-wheel

The necessary conditions for a 4-wheel design of order n to have simultaneous metamorphoses into C_4, $K_{1,4}$ and bowtie designs is that $n \equiv 1$ or $33 \pmod{48}$. We remark that in order for the number of removed edges to be a multiple of 3, when transforming the 4-wheel into a bowtie, the number of 4-wheels must be a multiple of 3, and then the number of removed edges is a multiple of 6, so that the number of extra triangles to form is always even. In other words, we are always able to obtain a bowtie system rather than a near-bowtie system, because the final number of triangles is always even.

Note that this time the graph is tripartite, so a 3-GDD may be used in the construction.

Example 3.1 A SM-4-wheel system of order 33.

Let the vertex set be $\{i_j \mid 0 \leq i \leq 10,\ 1 \leq j \leq 3\}$. In each case, the required design is obtained by developing the given starter blocks mod $(11, -)$, so the base is cycled mod 11, and the subscript remains fixed.

The starter 4-wheels are:

0_1–$(3_1, 10_2, 2_2, 4_3)$ 0_1–$(8_2, 3_3, 8_3, 6_3)$ 0_2–$(0_1, 2_1, 7_1, 1_2)$
0_2–$(5_2, 7_2, 0_3, 1_3)$ 0_3–$(0_1, 1_1, 8_3, 3_2)$ 0_3–$(2_1, 6_1, 1_2, 4_3)$.

The extra starter 4-cycles, formed by rearranging edges in the wheel spokes, are:

$(0_1, 3_1, 0_2, 4_2)$ $(0_1, 0_2, 1_1, 0_3)$ $(0_1, 2_2, 1_2, 9_3)$ $(0_1, 9_2, 3_2, 3_3)$
$(0_1, 4_3, 1_3, 5_3)$ $(0_1, 6_3, 7_2, 8_3)$

The extra starter copies of $K_{1,4}$, formed by rearranging edges in the wheel rims, are:

$[0_1 : 1_1, 2_1, 4_1, 5_1]$ $[0_1 : 1_2, 3_2, 5_2, 6_2]$ $[0_1 : 7_2, 1_3, 2_3, 7_3]$
$[0_2 : 2_2, 3_2, 2_3, 3_3]$ $[0_2 : 4_3, 5_3, 6_3, 7_3]$ $[0_3 : 2_2, 1_3, 2_3, 5_3]$.

The starter bowties contained in the wheels are:

$\{0_1, 3_1, 10_2; 0_1, 4_3, 2_2\}$ $\{0_1, 8_2, 3_3; 0_1, 6_3, 8_3\}$ $\{0_2, 0_1, 2_1; 0_2, 1_2, 7_1\}$
$\{0_2, 5_2, 7_2; 0_2, 1_3, 0_3\}$ $\{0_3, 0_1, 1_1; 0_3, 3_2, 8_3\}$ $\{0_3, 2_1, 6_1; 0_3, 4_3, 1_2\}$.

The extra starter bowties, formed by rearranging other (alternate) edges in the wheel rims, are given by:

$\{0_1, 5_1, 6_2; 0_1, 3_2, 1_3\}$ $\{0_3, 4_1, 6_3; 0_3, 4_2, 7_2\}$.

Example 3.2 A SM-4-wheel system of order 49.

Let the vertex set be $\mathbb{Z}_{49}$. In each case, the required design is obtained by developing the given starter blocks mod 49.

The starter 4-wheels are:

9–(41, 45, 48, 46) 0–(1, 7, 26, 35) 0–(8, 29, 11, 33).

The extra starter 4-cycles, formed by rearranging edges in the wheel spokes, are:

(0, 1, 8, 16) (0, 10, 22, 35) (0, 11, 37, 17).

The extra starter copies of $K_{1,4}$, formed by rearranging edges in the wheel rims, are:

[0 : 2, 3, 4, 5] [0 : 6, 9, 15, 18] [0 : 19, 21, 22, 24]

The starter bowties contained in the wheels are:

$\{9, 41, 45; 9, 46, 48\}$ $\{0, 1, 7; 0, 35, 26\}$ $\{0, 8, 29; 0, 33, 11\}$.

The extra starter bowtie, formed by rearranging other (alternate) edges in the wheel rims, is given by:

$\{0, 3, 18; 0, 5, 24\}$.

Example 3.3 A SM-4-wheel system of $K_{8,8,8}$.

Let the vertex set be $\{i_j \mid 0 \le i \le 3,\ j = 1,2\} \cup \{i_j \mid 0 \le i \le 3,\ j = 3,4\} \cup \{i_j \mid 0 \le i \le 3,\ j = 5,6\}$.
In each case, the required design is obtained by developing the given starter

blocks mod $(4,-)$, so the base is cycled mod 4, and the subscript remains fixed.

The starter 4-wheels are:

$0_1-(0_3,0_5,1_3,0_6)$ $0_2-(0_4,0_5,1_4,2_6)$ $0_3-(0_2,2_5,1_2,1_6)$
$0_4-(0_1,2_5,1_1,3_6)$ $0_5-(1_1,3_3,1_2,3_4)$ $0_6-(3_1,2_3,1_2,0_4)$

The extra starter 4-cycles, formed by rearranging edges in the wheel spokes, are:

$(0_1,0_3,3_1,3_4)$ $(0_1,0_5,1_1,1_6)$ $(0_2,0_3,1_2,1_4)$ $(0_2,0_5,1_2,3_6)$
$(0_3,1_5,3_3,1_6)$ $(0_4,1_5,3_4,3_6)$

The extra starter copies of $K_{1,4}$, formed by rearranging edges in the wheel rims, are:

$[0_1:2_3,3_3,1_4,2_4]$ $[0_1:1_5,2_5,2_6,3_6]$ $[0_2:1_3,2_3,2_4,3_4]$
$[0_2:1_5,2_5,0_6,1_6]$ $[0_3:0_5,3_5,0_6,3_6]$ $[0_4:0_5,3_5,1_6,2_6]$

The starter bowties contained in the wheels are:

$\{0_1,0_3,0_5;0_1,0_6,1_3\}$ $\{0_2,0_4,0_5;0_2,2_6,1_4\}$ $\{0_3,0_2,2_5;0_3,1_6,1_2\}$
$\{0_4,0_1,2_5;0_4,3_6,1_1\}$ $\{0_5,1_1,3_3;0_5,3_4,1_2\}$ $\{0_6,3_1,2_3;0_6,0_4,1_2\}$.

The extra starter bowties, formed by rearranging other (alternate) edges in the wheel rims, are given by:

$\{0_1,1_4,3_6;0_1,2_4,1_5\}$ $\{0_2,1_3,1_6;0_2,2_3,1_5\}$

THEOREM 3.1 *There exists a SM-4-wheel system of order n for all $n \equiv$ 1 or 33 (mod 48).*

Proof The Construction in Section 1 is used, with $n = 48x+1$ or $n = 48x+33$. In each case $\ell = 8$, so that $s = 6x$ or $s = 6x+4$. In the former case, a 3-GDD of type 6^x exists for all $x \geq 3$. In the latter case, a 3-GDD of type 6^x4^1 exists for all $x \geq 3$. For $x < 3$, the cases of orders 49, 97 and 33, 81, 129 remain. Orders 33 and 49 appear above (Examples 3.1 and 3.2) while orders 81, 97 and 129 appear in the Appendix. □

4 The 6-wheel

The necessary conditions for a 6-wheel design of order n to have simultaneous metamorphoses into C_6, $K_{1,6}$ and 3-windmill or near-3-windmill designs is that $n \equiv 1 \pmod{24}$. Unlike the 4-wheel case, when transforming into triangles or 3-windmills, the number of removed edges is three from each 6-wheel, (leaving behind a 3-windmill) and so the number of *new* triangles formed from these removed edges is not necessarily a multiple of 3. So in this case, according as the final number of triangles is 0, 1 or 2 (mod 3), we form a 3-windmill design, or a near-3-windmill design with 1 or else 2 left-over triangles.

As in the previous section, the graph is tripartite, so that a 3-GDD is used in the Construction.

Example 4.1 A SM-6-wheel system of order 25.

In order to obtain a near-3-windmill design in the metamorphosis, we are unable to use a cyclic 6-wheel design of order 25. So instead we take the vertex set $\{i_j \mid 0 \leq i \leq 4,\ 1 \leq j \leq 5\}$, and in most cases work modulo $(5, -)$, so the base is cycled mod 5 and the subscripts are fixed.

The starter 6-wheels are:

4_3–$(1_3, 1_4, 4_5, 3_4, 0_4, 0_5)$ 1_2–$(1_1, 0_4, 2_2, 2_3, 0_5, 3_4)$
1_2–$(3_1, 1_5, 3_2, 0_3, 2_5, 3_5)$ 0_1–$(3_1, 4_3, 0_3, 3_4, 2_5, 4_5)$
0_1–$(4_1, 1_2, 1_4, 0_4, 4_2, 2_3)$.

The extra starter 6-cycles, formed by rearranging edges in the wheel spokes, are:

$(0_1, 1_1, 3_1, 1_2, 2_1, 2_3)$ $(0_1, 0_2, 4_1, 3_3, 2_2, 1_4)$ $(0_1, 0_4, 2_1, 1_5, 0_2, 2_5)$
$(0_2, 1_2, 3_2, 2_3, 0_3, 2_4)$ $(0_2, 0_5, 0_3, 4_4, 3_3, 4_5)$.

The extra starter copies of $K_{1,6}$, formed by rearranging edges in the wheel rims, are:

$[0_4 : 2_4, 1_4, 1_1, 3_1, 4_5, 3_5]$ $[0_3 : 0_4, 3_4, 1_3, 0_2, 3_2, 2_2]$
$[0_1 : 0_5, 1_5, 3_5, 1_3, 2_2, 3_3]$ $[0_4 : 1_5, 0_5, 2_5, 2_2, 0_2, 4_2]$
$[0_5 : 2_2, 1_3, 2_3, 3_3, 1_5, 2_5]$.

The near-3-windmill design is obtained from the metamorphosis as follows. First, take the five starter 3-windmills from the 6-wheels:

$\{4_3, 1_3, 1_4; 4_3, 4_5, 3_4; 4_3, 0_4, 0_5\}$ $\{1_2, 1_1, 0_4; 1_2, 2_2, 2_3; 1_2, 0_5, 3_4\}$
$\{1_2, 3_1, 1_5; 1_2, 3_2, 0_3; 1_2, 2_5, 3_5\}$ $\{0_1, 3_1, 4_3; 0_1, 0_3, 3_4; 0_1, 2_5, 4_5\}$
$\{0_1, 4_1, 1_2; 0_1, 1_4, 0_4; 0_1, 4_2, 2_3\}$.

Then take the following 3-windmills (not cycled) and one remaining unused triangle formed by rearranging other (alternate) edges in the wheel rims:

$\{0_5, 3_3, 0_1; 0_5, 4_1, 1_4; 0_5, 1_3, 2_3\}$ $\{1_5, 4_3, 1_1; 1_5, 3_2, 3_4; 1_5, 2_3, 3_3\}$
$\{2_5, 0_3, 2_1; 2_5, 1_1, 3_4; 2_5, 3_3, 4_3\}$ $\{3_5, 1_3, 3_1; 3_5, 2_1, 4_4; 3_5, 4_3, 0_3\}$
$\{4_5, 2_3, 4_1; 4_5, 3_1, 0_4; 4_5, 0_3, 1_3\}$ $\{0_4, 0_2, 3_5; 0_4, 2_2, 3_4; 0_4, 4_2, 2_4\}$
$\{1_4, 1_2, 4_5; 1_4, 0_2, 3_4; 1_4, 3_2, 4_4\}$ $\{2_4, 0_1, 1_5; 2_4, 2_2, 0_5; 2_4, 1_2, 4_4\}$
$(4_2, 4_4, 2_5)$.

Example 4.2 A SM-6-wheel system of order 49.

Let the vertex set be $\mathbb{Z}_{49}$. In all but the 3-windmill design case, the required design is obtained by developing the given starter blocks mod 49.

The starter 6-wheels are:

$0\text{–}(1,3,7,12,18,33)$ $0\text{–}(8,35,11,40,10,36)$.

The extra starter 6-cycles, formed by rearranging edges in the wheel spokes, are:

$(0,1,4,11,2,10)$ $(0,11,24,8,26,12)$.

The extra starter copies of $K_{1,6}$, formed by rearranging edges in the wheel rims, are:

$[0:2,4,5,6,15,17]$ $[0:19,20,21,22,23,24]$.

The near-3-windmill design is obtained from the metamorphosis as follows. First, take the two starter 3-windmills from the 6-wheels:

$\{0,1,3;0,7,12;0,18,33\}$ $\{0,8,35;0,11,40;0,10,36\}$.

Then take the following 3-windmills (not cycled) and two remaining unused triangles, formed by rearranging other (alternate) edges in the wheel rims. (Note that these arise from the 98 triangles formed from $(0,4,21)$ and $(0,6,25)$ mod 49.)

$\{0,4,21;0,6,25;0,17,45\}$ $\{1,5,22;1,7,26;1,18,46\}$
$\{2,6,23;2,8,27;2,19,47\}$ $\{3,7,24;3,9,28;3,20,48\}$
$\{4,8,25;4,10,29;4,28,34\}$ $\{5,9,26;5,11,30;5,29,35\}$
$\{6,10,27;6,12,31;6,34,38\}$ $\{7,11,28;7,13,32;7,35,39\}$
$\{29,8,12;29,25,46;29,23,48\}$ $\{13,9,30;13,17,34;13,19,38\}$
$\{14,10,31;14,18,35;14,8,33\}$ $\{32,11,15;32,36,4;32,26,2\}$
$\{33,12,16;33,37,5;33,27,3\}$ $\{15,19,36;15,9,34;15,39,45\}$
$\{36,42,12;36,40,8;36,11,17\}$ $\{37,16,20;37,41,9;37,12,18\}$
$\{38,17,21;38,42,10;38,32,8\}$ $\{39,18,22;39,43,11;39,33,9\}$
$\{40,19,23;40,44,12;40,34,10\}$ $\{20,24,41;20,14,39;20,26,45\}$
$\{21,25,42;21,15,40;21,27,46\}$ $\{43,22,26;43,18,24;43,37,13\}$
$\{44,23,27;44,19,25;44,38,14\}$ $\{45,24,28;45,41,13;45,2,21\}$
$\{30,26,47;30,34,2;30,36,6\}$ $\{31,27,48;30,35,3;30,37,7\}$
$\{0,28,32;0,24,30;0,43,19\}$ $\{1,29,33;1,25,31;1,44,20\}$
$\{46,42,14;46,40,16;46,3,22\}$ $\{47,43,15;47,22,28;47,4,23\}$
$\{48,44,16;48,42,18;48,5,24\}$ $\{41,16,22;41,35,11;41,47,17\}$
$(10,16,35)$, $(17,23,42)$.

Example 4.3 A SM-6-wheel system of $K_{12,12,12}$.

Let the vertex set be $\{i_j \mid 0 \le i \le 5,\ j = 1,2\} \cup \{i_j \mid 0 \le i \le 5,\ j = 3,4\} \cup \{i_j \mid 0 \le i \le 5,\ j = 5,6\}$.
In each case, the required design is obtained by developing the given starter

blocks mod $(6,-)$, so the base is cycled mod 6, and the subscript remains fixed.

The starter 6-wheels are:

0_1–$(1_4, 4_5, 4_4, 1_6, 5_4, 5_6)$ 0_2–$(0_3, 0_5, 1_3, 2_5, 5_3, 1_6)$
0_3–$(1_1, 4_5, 3_1, 2_5, 4_2, 3_6)$ 0_4–$(4_1, 4_5, 1_2, 5_6, 5_2, 1_6)$
0_5–$(4_1, 1_4, 5_2, 5_4, 1_2, 4_4)$ 0_6–$(0_1, 1_3, 3_2, 0_3, 2_1, 2_3)$.

The extra starter 6-cycles, formed by rearranging edges in the wheel spokes, are:

$(0_1, 3_3, 4_1, 0_4, 1_1, 0_6)$ $(0_1, 1_4, 3_1, 1_5, 0_2, 2_5)$ $(0_1, 1_6, 0_2, 0_3, 1_2, 4_6)$
$(0_2, 1_3, 5_2, 0_4, 1_2, 0_5)$ $(0_3, 2_5, 4_3, 1_6, 1_3, 5_6)$ $(0_4, 1_5, 2_4, 0_5, 4_4, 5_6)$.

The extra starter copies of $K_{1,6}$, formed by rearranging edges in the wheel rims, are:

$[0_1 : 0_3, 1_3, 2_3, 4_3, 0_4, 3_4]$ $[0_1 : 0_5, 1_5, 3_5, 5_5, 2_6, 3_6]$
$[0_2 : 3_3, 4_3, 0_4, 2_4, 3_4, 4_4]$ $[0_2 : 3_5, 4_5, 0_6, 2_6, 4_6, 5_6]$
$[0_3 : 0_5, 1_5, 3_5, 5_5, 1_6, 2_6]$ $[0_4 : 0_5, 3_5, 0_6, 2_6, 3_6, 4_6]$.

The extra starter 3-windmills, formed by rearranging other (alternate) edges in the wheel rims, are given by:

$\{0_1, 2_3, 3_6; 0_1, 4_3, 1_5; 0_1, 0_4, 2_6\}$ $\{0_2, 4_3, 3_5; 0_2, 2_4, 0_6; 0_2, 4_4, 4_5\}$.

THEOREM 4.1 *There exists a SM-6-wheel system of order n for all $n \equiv 1 \pmod{24}$.*

Proof The Construction in Section 1 is used, with $n = 24x + 1$. Take $\ell = 12$; a 3-GDD of type 2^x exists for $x \equiv 0$ or $1 \pmod 3$, $x \geq 3$, and a 3-GDD of type $4^1 2^{x-2}$ exists for $x \equiv 2 \pmod 3$, $x \geq 5$ (see [4]).

When $x \equiv 0$ or $1 \pmod 3$, while taking SM-6-wheel designs of order 25, we ensure that the unused triangle in the near-3-windmill design is one containing the point ∞. The number of such SM-6-wheel designs of order 25 which we use is x, and since $x \equiv 0$ or $1 \pmod 3$, we can take these "unused" triangles three at a time with the common point ∞, with either 0 or 1 remaining.

When $x \equiv 2 \pmod 3$, we have one SM-6-wheel design of order 49 (which has two unused triangles in its near-3-windmill design) and then a further $x - 2$ SM-6-wheel designs of order 25. Again we ensure that each of the unused triangles in the near-3-windmill designs of order 25 contains the point ∞. Then since $x - 2 \equiv 0 \pmod 3$ here, the unused triangles from these near-3-windmill designs of order 25 may be grouped three at a time into 3-windmills, all containing the common point ∞. The two unused triangles from the design of order 49 remain. The result is a near-3-windmill design of order $24x+1$, since the SM-6-wheel system on $K_{12,12,12}$ forms an exact 3-windmill system.

The cases $x=1$ and 2 are dealt with in Examples 4.1 and 4.2 above. □

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Appendix

SM-3-wheel-design of order 73

Let the vertex set be $\mathbb{Z}_{73}$. In each case, the required design is obtained by developing the given starter blocks mod 73.

The starter 3-wheels are:

40–(30, 52, 65)   0–(1, 3, 7)   0–(5, 21, 45)   0–(8, 23, 42)
0–(9, 26, 46)   0–(11, 41, 55).

The extra starter triples, formed by rearranging edges in the wheel spokes, are:

(0, 1, 8)   (0, 3, 28)   (0, 5, 26)   (0, 9, 27)   (0, 10, 41)   (0, 11, 23).

The extra starter copies of $K_{1,3}$, formed by rearranging edges in the wheel rims, are:

[0 : 2, 4, 6]   [0 : 13, 14, 15]   [0 : 16, 17, 19]   [0 : 20, 22, 24]
[0 : 29, 30, 33]   [0 : 34, 35, 36].

SM-4-wheel-design of order 81

Let the vertex set be $\mathbb{Z}_{81}$. In each case, (unless otherwise stated) the required design is obtained by developing the given starter blocks mod 81.

The starter 4-wheels are:

0-(18, 24, 78, 76)   0-(41, 73, 45, 74)   0-(1, 10, 14, 31)
0-(12, 34, 13, 56)   0-(15, 26, 42, 61).

The extra starter 4-cycles, formed by rearranging edges in the wheel spokes, are:

(0, 1, 4, 12),   (0, 5, 12, 25),   (0, 10, 25, 45),
(0, 14, 55, 24),   (0, 18, 60, 26).

The extra starter copies of $K_{1,4}$, formed by rearranging edges in the wheel rims, are:

$[0:2,4,6,9]$ $[0:11,16,17,19]$ $[0:21,22,23,27]$
$[0:28,29,30,32]$ $[0:33,35,37,38]$.

The starter bowties contained in the wheels are:

$\{0,18,24;0,76,78\}$ $\{0,41,74;0,45,73\}$ $\{0,1,31;0,10,14\}$
$\{0,12,56;0,13,34\}$ $\{0,15,26;0,42,61\}$.

The extra starter bowties, formed by rearranging other (alternate) edges in the wheel rims, are given by:

$\{\{i,i+17,i+46;i,i+16,i+38\} \mid 0 \le i \le 80\}$;
$\{\{i,i+9,i+32;i,i+27,i+54\} \mid 0 \le i \le 26\}$;
$\{\{i+54,i+45,i+77;i+54,i+63,i+5\},$
$\{i+72,i+63,i+14;i+72,i,i+23\},$
$\{i+36,i+27,i+59;i+36,i+45,i+68\} \mid 0 \le i \le 8\}$.

SM-4-wheel-design of order 97

Let the vertex set be $\mathbb{Z}_{97}$. In each case the required design is obtained by developing the given starter blocks mod 97.

The starter 4-wheels are:

48–(71, 90, 76, 96) 23–(26, 38, 73, 79) 9–(35, 45, 85, 52)
0–(1, 5, 16, 46) 0–(2, 60, 68, 75) 0–(18, 27, 65, 31).

The extra starter 4-cycles, formed by rearranging edges in the wheel spokes, are:

(0, 1, 3, 18) (0, 3, 8, 29) (0, 16, 38, 61)
(0, 26, 58, 27) (0, 28, 79, 37) (0, 41, 90, 43).

The extra starter copies of $K_{1,4}$, formed by rearranging edges in the wheel rims, are:

$[0:4,6,7,8]$ $[0:9,10,11,12]$ $[0:13,14,17,19]$
$[0:20,24,25,30]$ $[0:33,34,35,38]$ $[0:39,40,44,45]$.

The starter bowties contained in the wheels are:

$\{48,71,90;48,76,96\}$ $\{23,26,38;23,73,79\}$ $\{9,35,45;9,85,52\}$
$\{0,1,5;0,16,46\}$ $\{0,2,60;0,68,75\}$ $\{0,18,27;0,65,31\}$.

The extra starter bowties, formed by rearranging other (alternate) edges in the wheel rims, are:

$\{0,8,25;0,11,35\}$ $\{0,13,53;0,14,52\}$.

SM-4-wheel-design of order 129

Let the vertex set be $\mathbb{Z}_{129}$. In each case, (unless otherwise stated), the required design is obtained by developing the given starter blocks mod 129.

The starter 4-wheels are:

$0\text{–}(104,122,113,126)$ $0\text{–}(6,37,67,82)$ $0\text{–}(33,68,63,77)$
$0\text{–}(1,11,19,21)$ $0\text{–}(4,27,39,73)$ $0\text{–}(17,49,100,57)$
$0\text{–}(24,79,41,83)$ $0\text{–}(26,84,36,101)$.

The extra starter 4-cycles, formed by rearranging edges in the wheel spokes, are:

$(0,1,4,11)$ $(0,4,10,26)$ $(0,17,36,57)$ $(0,24,49,77)$
$(0,27,56,92)$ $(0,33,95,39)$ $(0,41,91,45)$ $(0,47,110,49)$.

The extra starter copies of $K_{1,4}$, formed by rearranging edges in the wheel rims, are:

$[0:2,5,8,9]$ $[0:10,12,13,14]$ $[0:15,18,20,22]$
$[0:23,30,31,32]$ $[0:34,35,38,40]$ $[0:42,43,44,48]$
$[0:51,53,54,55]$ $[0:58,59,60,64]$.

The starter bowties contained in the wheels are:

$\{0,104,126;0,122,113\}$ $\{0,6,82;0,37,67\}$ $\{0,33,68;0,63,77\}$
$\{0,1,21;0,11,19\}$ $\{0,4,73;0,27,39\}$ $\{0,17,57;0,49,100\}$
$\{0,24,79;0,41,83\}$ $\{0,26,84;0,36,101\}$.

The extra starter bowties, formed by rearranging other (alternate) edges in the wheel rims, are given by:

$\{0,5,18;0,23,54\}$, $\{0,15,59;0,10,48\}$ (mod 129);

$\{\{i,i+43,i+86;i,i+2,i+34\} \mid 0 \le i \le 14 \text{ and } 16 \le i \le 42\}$;

$\{101,15,58;101,67,69\}$, $\{17,15,49;17,112,114\}$;

$\{\{4i+1,4i-1,4i+33;4i+1,4i+3,4i+35\},$
$\{4i+2,4i,4i+34;4i+2,4i+4,4i+36\} \mid 11 \le i \le 16\}$;

$\{\{4i-2,4i-4,4i+30;4i-2,4i,4i+32\},$
$\{4i-1,4i-3,4i+31;4i-1,4i+1,4i+33\} \mid 18 \le i \le 28\}$;

$\{\{4i-1,4i-3,4i+31;4i-1,4i+1,4i+33\},$
$\{4i,4i-2,4i+32;4i,4i+2,4i+34\} \mid 29 \le i \le 32\}$.

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