On the [24, 12, 8] Self-Dual Quaternary Codes *

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Abstract

We construct all self-dual [24, 12, 8] quaternary codes with a monomial automorphism of prime order $r > 3$ and obtain a unique code for $r = 23$ (which has automorphisms of orders 5, 7, and 11 too), two inequivalent codes for $r = 11$, 6 inequivalent codes for $r = 7$, and 12 inequivalent codes for $r = 5$. The obtained codes are with 12 different weight spectra.

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1 Introduction

In [2] and [9], all self-dual codes over the field of four elements, F_4, of length at most 16 are enumerated. It is reasonable for higher lengths n to investigate only those of the largest minimum weight $d = 2\lfloor n/6 \rfloor + 2$. These codes are called extremal. The extremal self-dual codes of lengths 18 and 20 are classified in [7]. All inequivalent extremal self-dual codes of lengths 22, 26, and 28 which have a nontrivial odd order automorphism are known [5, 6]. In [8], it is shown that there does not exist a [24, 12, 10] self-dual code over F_4. It is known that any [24, 12, 8] self-dual quaternary code has a weight enumerator of the form:

$$\begin{aligned}
&W(y) = 1 + A_8y^8 + (18216 - 8A_8)y^{10} + (156492 + 28A_8)y^{12} + \\
&(1147608 - 56A_8)y^{14} + (3736557 + 70A_8)y^{16} + \\
&(6248088 - 56A_8)y^{18} + (4399164 + 28A_8)y^{20} + \\
&(1038312 - 8A_8)y^{22} + (32778 + A_8)y^{24}
\end{aligned} \tag{1}$$

where A_8 is the number of weight 8 vectors.

In this paper, we examine [24, 12, 8] self-dual codes over F_4 possessing a monomial automorphism of prime order $r > 3$. We obtain all such codes up to equivalence. All these codes have weight enumerators (1) with A_8 = 2277, 1242, 1197, 1089, 837, 792, 756, 702, 657, 630, 522, or 513. We use a general method for constructing self-dual codes via an automorphism of odd prime order developed in [4, 5, 10, 11].

2 Description of the method and notations

We describe first the method for constructing quaternary self-dual codes, C, possessing a permutation automorphism of odd prime order $r \geq 5$. By Theorem 2 of [3], if C has a monomial automorphism of order $r \geq 5$, there is a code equivalent to C with a permutation automorphism σ of order r with the same cycle structure. Let C be an [n, k] code over F_4 with a permutation automorphism σ of odd prime order r which has c r-cycles and f fixed points. We can assume that

$$\sigma = (1, 2, \ldots, r)(r+1, r+2, \ldots, 2r) \ldots ((c-1)r+1, \ldots, cr). \quad (2)$$

Denote the r-cycles of σ by $\Omega_1, \Omega_2, \ldots, \Omega_c$ and the fixed points by $\Omega_{c+1}, \cdots, \Omega_{c+f}$. Consider the factor-ring $R = F_4[X]/\langle X^r + 1\rangle$, where X is indeterminate. Suppose $X^r + 1 = \prod_{j=0}^{g} m_j(X)$, where $m_j(X)$ is irreducible over F_4 with $m_0(X) = X + 1$. Denote by I_j the minimal ideal in R generated by $(X^r + 1)/(m_j(X))$ for $0 \leq j \leq g$. Each I_j is an extension field of F_4 and $R = I_0 \oplus I_1 \oplus \cdots \oplus I_g$.

If $\mathbf{x} \in F_4^n$, let $\mathbf{x}|\Omega_i$ be the restriction of $\mathbf{x}$ on Ω_i. For $1 \leq i \leq c$, the restriction $\mathbf{x}|\Omega_i$ can be viewed as an element $a_0 + a_1 X \cdots + a_{r-1} X^{r-1}$ from R. Let $C(\sigma) = \{\mathbf{x} \in C : \mathbf{x}\sigma = \mathbf{x}\}$. For $1 \leq j \leq c$, let $E_j(\sigma) = \{\mathbf{x} \in C : \mathbf{x}|\Omega_i \in I_j \text{ for } 1 \leq i \leq c$ and $\mathbf{x}|\Omega_i = 0$ for $c+1 \leq j \leq c+f\}$. It is known that $C = C(\sigma) \oplus E_1(\sigma) \oplus \cdots \oplus E_g(\sigma)$ [5]. Let $E_j(\sigma)^*$ be $E_j(\sigma)$ punctured on the fixed points. The codewords of $E_j(\sigma)^*$ are c-tuples from I_j^c. Define the map $\Phi : C(\sigma) \to F_4^{c+f}$, where $\mathbf{v}\Phi$ is the vector obtained from $\mathbf{v} \in C(\sigma)$, by choosing one coordinate from each cycle Ω_i , $1 \leq i \leq c+f$. This way, we obtain the code $C(\sigma)\Phi$ over F_4. The inner product $< \cdot, \cdot >$ in F_4^n has the form:

$$< \mathbf{u}, \mathbf{v} > = \sum_{i=1}^{n} u_i v_i^2, \quad (3)$$

where $\mathbf{u}, \mathbf{v} \in F_4^n$ and $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{v} = (v_1, \ldots, v_n)$.

We let $s = 0$ or 1 and t be a nonnegative integer. Choose an integer u such that $2^s 4^t u \equiv -1 (mod\ r)$. Define a form $(\cdot, \cdot)$ on R^c by

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{c} x_i y_i^{2^s 4^t}, \quad (4)$$

where $\mathbf{x}, \mathbf{y} \in R^c$, $\mathbf{x} = (x_1, \ldots, x_c)$, $\mathbf{y} = (y_1, \ldots, y_c)$.

Consider the ring automorphism $\tau_{2^a,u}(\sum_{i=0}^{r-1} a_i X^i) = \sum_{i=0}^{r-1} a_i^{2^a} X^{ui}$, where a is an integer $0 \leq a \leq 1$ and u is defined above. The map $\tau_{2^a,u}$ preserves I_0 and permutes the fields $I_1, \ldots, I_g$ [5]. Let us define the permutation λ on $1, \ldots, g$ with $\tau_{2^a,u}(I_i) = I_{\lambda(i)}$. Denote by $\Lambda_1, \ldots, \Lambda_l$ the orbits of λ.

The next theorem can be found in [5].

Theorem 1 *Let s, t be nonnegative integers, where $s \leq 1$. Choose an integer u such that $2^s 4^t u \equiv -1 (mod\, r)$. A quaternary [n,n/2,d] code C over F_4 with a permutation automorphism σ in the form (2) is self-dual under inner product (3) iff the following conditions hold:*

(i) $C(\sigma)\Phi$ is self-dual [c+f,(c+f)/2] under (3);

(ii) For $1 \leq i \leq g$ $E_{\lambda(i)}(\sigma)^*$ is dual of $\tau_{2^a,u} E_i(\sigma)^*$ under (4).

Some useful restrictions of the cycle structure of σ are given in the following theorem.

Theorem 2 *[2, 5] Let C be [n,n/2,d] code over F_4 with a permutation automorphism σ of prime order $r \geq 3$ with c r-cycles and f fixed points. We have:*

(i) if $f \leq d-1$, then $c \geq f$;

(ii) if $f \geq d$, then $c + f \geq 2d - 2$;

(iii) if $|\Lambda_j|$ is odd for some j, $1 \leq j \leq l$, then c is even.

Two linear quaternary codes C and C' of length n are said to be equivalent whenever $C' = CM\tau$, where M is a monomial $n \times n$ matrix over F_4 and $\tau \in Gal(F_4)$.

3 Results

Let C be a [24, 12, 8] self-dual quaternary code with a monomial automorphism M of odd prime order $r \geq 5$. Then the automorphism can be assumed to be a permutation automorphism σ of order r with the same cycle structure as M [3]. Let σ have c r-cycles, f fixed points and decomposition (2). Applying theorem 2 and using 8 as the minimum distance of C, we obtain the next theorem.

Theorem 3 *The only possibilities for r, c, f are as follows: 1) r=23, c=1, f=1, 2) r=11, c=2, f=2, 3) r=7, c=3, f=3, 4) r=5, c=4, f=4, 5) r=3, c=6, f=6, 6) r=3, c=8, f=0.*

We consider the cases for $r \neq 3$. In all of this cases $R = I_0 \oplus I_1 \oplus I_2$ and $C = C(\sigma) \oplus E_1(\sigma) \oplus E_2(\sigma)$. Generators of $I_j^* = I_j \setminus \{0\}$ for $j = 1, 2$ will be denoted α and β respectively. The elements of F_4 will be denoted 0, 1, ω and $\bar{\omega} = \omega^2$. It holds that $1 + \omega = \omega^2$.

We use the following transformations which produce equivalent codes possessing the automorphism σ given in (2):

a) permutation of the last f coordinates of C;

b) permutation of the r c-cycles of C;

c) multiplication of each cycle Ω_i, $1 \leq i \leq c$, and each fixed coordinate of C by a nonzero constant from F_4;

d) substitution $f_t : X \to X^t$ in each r-cycle of σ, where t is an integer $1 \leq t \leq r-1$;

e) cycle shifts of the entries of the r-cycles separately.

Denote the groups consisting of transformations of type a), b), c), d), and e) by S_f, S_c, D, F and W, respectively.

The next two theorems are particular cases of results provided in [5].

Theorem 4 *Let C and C' have the same monomial automorphism of order r= 23, 11, 7 or 5 (which we may assume to be a permutation σ). Then C and C' are equivalent if and only if $C' = CM\tau$, for some $M \in WS_fS_cDF$ and $\tau \in Gal(F_4)$. Denote the actions of $T \in WS_fS_cDFGal(F_4)$ on $C(\sigma)\Phi$ and $E_i(\sigma)^*$ by $\hat{T}$.*

Theorem 5 *Let C and C' have the same automorphism σ. Suppose $C = C(\sigma) \oplus E_1(\sigma) \oplus E_2(\sigma)$ and $C' = C'(\sigma) \oplus E_1'(\sigma) \oplus E_2'(\sigma)$.*

(i) If $CT = C'$ where $T \in WS_fS_cDFGal(F_4)$, then $C(\sigma)T = C'(\sigma)$ and $E_i(\sigma)T = E_j(\sigma)$, where $i, j \in \{1, 2\}$.

(ii) Suppose $C(\sigma) = C'(\sigma)$ and $CT = C'$, where $T \in S_f S_c D Gal(F_4)$. Then $\hat{T}$ is an automorphism of $C(\sigma)\Phi$.

We define the group $\hat{G} = \{\hat{T} | T \in S_f S_c D\} Gal(F_4) \cap Aut(C(\sigma)\Phi)$.

3.1 The case r=23, c=1, f=1.

Now $R = F_4[X]/\langle X^{23}+1\rangle = I_0 \oplus I_1 \oplus I_2$, where I_1 and I_2 are fields with 4^{11} elements, $\tau_{2^{1-s},u} = \tau_{1,-1}$ and $\tau_{1,-1}(\alpha) = \beta$. The code $C(\sigma)\Phi$ is a [2, 1] self-dual code over F_4. Therefore, $C(\sigma)\Phi = C_2$. The form (4) in R^1 is $(x,y) = xy^{2 \cdot 4^5}$ and $E_2(\sigma)^* = \tau_{1,-1}(E_1(\sigma)^*)$. Then $E_1(\sigma)^* \oplus E_2(\sigma)^*$ is an [1, 1] code over I_1 or I_2. Thus, up to equivalence, $E_1(\sigma)^* \oplus E_2(\sigma)^* = I_1$. We use the unit $e_1(X) = 1+X+X^2+X^3+X^4+X^6+X^8+X^9+X^{12}+X^{13}+X^{16}+X^{18}$ of I_1 to obtain a generator matrix of C of the form

$$G = \begin{pmatrix} 11\ldots1 & 1 \\ G_1 & 0 \end{pmatrix}.$$

The first row of G_1 is 11111010110011001010000 and it is an 11×23 circulant type matrix. The matrix G generates over $GF(2)$ the extended [24, 12, 8] Golay code. A computer check shows that over $GF(4)$, it generates a [24, 12, 8] code with 2277 vectors of weight 8. Denote the obtained code by $C^1_{(23)}$. Thus, we prove the following theorem:

Theorem 6 *There exists a unique self-dual [24, 12, 8] quaternary code with a monomial automorphism of order 23.*

3.2 The case r=7, c=3, f=3

In this case, $R = F_4[X]/\langle X^7+1\rangle = I_0 \oplus I_1 \oplus I_2$, where I_1 and I_2 are fields with 4^3 elements. $C(\sigma)\Phi$ is a self-dual [6, 3] code over F_4. Then, it is either $C_2 \oplus C_2 \oplus C_2$ or E_6 [2]. Now, $\tau_{2^{1-s},u} = \tau_{1,-1}$ and $\tau_{1,-1}(\alpha) = \beta$, where $\alpha = 1+\omega^2X+\omega^2X^2+\omega X^3+X^5+\omega X^6$ and $\beta = 1+\omega X+X^2+\omega X^4+\omega^2X^5+\omega^2X^6$. The form (4) in

R^3 now is given by $(x,y) = \sum_{i=1}^{3} x_i y_i^8$. As $E_2(\sigma)^*$ is the dual of $\tau_{1,-1}E_1(\sigma)^*$ under it, $dim_{I_1}E_1(\sigma)^* + dim_{I_2}E_2(\sigma)^* = 3$. The group $W = < f_3 >$ and $f_3^3(\alpha) = \beta$. Hence, f_3^3 interchanges $E_1(\sigma)^*$ and $E_2(\sigma)^*$. So we may assume that $1 \leq dim_{I_1}E_1(\sigma)^* \leq dim_{I_2}E_1(\sigma)^*$. Since the minimal distance of C is 8, we obtain that $E_1(\sigma)^*$ is a [3, 1, 3] code over I_1. As a consequence of the form of the inner product in R^3, we obtain the following lemma.

Lemma 1 *The code $E_1(\sigma)^*$ determines the whole $E_1(\sigma)^* \oplus E_2(\sigma)^*$. If the generator matrix of $E_1(\sigma)^*$ has the form $(\alpha^0, \alpha^i, \alpha^j)$, $0 \leq i \leq 4^3-1$, $0 \leq j \leq 4^3-1$, then $E_2(\sigma)^*$ is generated by*

$$\begin{pmatrix} \beta^{8i} & \beta^0 & 0 \\ \beta^{8j} & 0 & \beta^0 \end{pmatrix}.$$

Let $C(\sigma)\Phi$ be $C_2 \oplus C_2 \oplus C_2$. We can fix the generator matrix in the form

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

with cycle coordinates on the left. The group $\hat{G}$ contains the transformations S_3^2 (S_3 is the symmetric group of degree 3), τ, $diag < \omega, 1, 1, \omega, 1, 1 >$, $diag < 1, \omega, 1, 1, \omega, 1 >$, and $diag < 1, 1, \omega, 1, 1, \omega >$. Applying theorem 5, we obtain three classes of [24, 12, 8] self-dual codes with representatives for the generator matrix of $E_1(\sigma)^*$ in the form $(\alpha^0, \alpha^0, \alpha^0)$, $(\alpha^0, \alpha^0, \alpha^1)$, $(\alpha^0, \alpha^1, \alpha^2)$. Denote these codes by $C_{(7)}^1$, $C_{(7)}^2$ and $C_{(7)}^3$ respectively. A computer check shows that the number of weight 8 vectors, A_8, is 2277 for $C_{(7)}^1$ and 513 for $C_{(7)}^2$ and $C_{(7)}^3$. The matrices $(\alpha^0, \alpha^0, \alpha^1)$ and $(\alpha^0, \alpha^1, \alpha^2)$ generate over $GF(4)$ two [21, 3] codes with different spectra. Hence, from theorem *5(i)* we obtain that $C_{(7)}^2$ and $C_{(7)}^3$ are inequivalent.

Let $C(\sigma)\Phi$ be E_6. We fix a generator matrix in the form

$$G_6 = \begin{pmatrix} 1 & 0 & 0 & 1 & \omega & \omega \\ 0 & 1 & 0 & \omega & 1 & \omega \\ 0 & 0 & 1 & \omega & \omega & 1 \end{pmatrix},$$

with cycle coordinates on the left. The group $\hat{G}_6$ contains the transformations S_3^2, $diag < \omega, \ldots, \omega >$, $(4,5)diag < 1, 1, \omega, \bar{\omega}, \bar{\omega}, \omega > \tau$, $(4,6)diag < 1, \omega, 1, \bar{\omega}, \omega, \bar{\omega} > \tau$, and $(5,6)diag < \omega, 1, 1, \omega, \bar{\omega}, \bar{\omega} > \tau$, where τ is the generator of the Galois group $Gal(F_4)$. For example, the transformation $(4,5)diag < 1, 1, \omega, \bar{\omega}, \bar{\omega}, \omega > \tau$ results in transposing the fourth and fifth columns of the matrix G_6, then multiplying the third and the sixth columns by ω and the fourth and fifth columns by $\bar{\omega}$, and raising of all entries to the second power. This way, the matrix G_6 is transformed to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & \omega & \omega \\ 0 & 1 & 0 & \omega & 1 & \omega \\ 0 & 0 & \bar{\omega} & 1 & 1 & \bar{\omega} \end{pmatrix},$$

which generates the same code E_6.

Using the group $\hat{G}_6$, we obtain again three inequivalent classes, with the same representatives for the generator matrix of $E_1(\sigma)^*$. Denote these codes by $C^4_{(7)}, C^5_{(7)}$ and $C^6_{(7)}$. A computer check finds the number of weight 8 vectors in these three codes. Thus, we obtain the following table and theorem.

i for $C^i_{(7)}$	$C(\sigma)\Phi$	$genE_1(\sigma)^*$	A_8
1	$C_2 \oplus C_2 \oplus C_2$	$\alpha^0\,\alpha^0\,\alpha^0$	2277
2	$C_2 \oplus C_2 \oplus C_2$	$\alpha^0\,\alpha^0\,\alpha$	513
3	$C_2 \oplus C_2 \oplus C_2$	$\alpha^0\,\alpha^0\,\alpha^2$	513
4	E_6	$\alpha^0\,\alpha^0\,\alpha^0$	1197
5	E_6	$\alpha^0\,\alpha^0\,\alpha$	630
6	E_6	$\alpha^0\,\alpha^0\,\alpha^2$	756

Theorem 7 *The inequivalent codes $C^1_{(7)}, C^2_{(7)}, C^3_{(7)}, C^4_{(7)}, C^5_{(7)}$, and $C^6_{(7)}$ are up to equivalence the only [24, 12, 8] codes over F_4 with a monomial automorphism of order 7.*

The results for r=11 and 5 are obtained in a similar way.

3.3 The case r=11, c=2, f=2

Theorem 8 *There exist exactly two inequivalent [24, 12, 8] quaternary codes with a monomial automorphism of order 11.*

These codes are determined as follows. The code $C(\sigma)\Phi$ is $C_2 \oplus C_2$ with a generator matrix fixed in the form $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Generator matrices for $E_1(\sigma)^*$ and $E_2(\sigma)^*$ are given in next table. Denote these two [24, 12, 8] codes by $C^1_{(11)}$ and $C^2_{(11)}$. The number of weight 8 vectors, A_8, given in the table is determined using a computer. Here we have $\alpha = 1+\omega^2x+x^2+\omega x^5+\omega^2x^6+\omega x^7+\omega x^8+\omega^2x^9+x^{10}$ and $\beta = 1+\omega x+x^2+\omega^2x^5+\omega x^6+\omega^2x^7+\omega^2x^8+\omega x^9+x^{10}$.

i for $C^i_{(11)}$	$genE_1(\sigma)^*$	$genE_2(\sigma)^*$	A_8
1	$\alpha^0\ \alpha^0$	$\beta^0\ \beta^{31\cdot 11}$	2277
2	$\alpha^0\ \alpha^0$	$\beta^0\ \beta^{31\cdot 12}$	1089

3.4 The case r=5, c=4, f=4

Theorem 9 *There are exactly 12 inequivalent [24, 12, 8] quaternary codes with a monomial automorphism of order 5.*

The code $C(\sigma)\Phi$ is E_8, see [2], with a generator matrix fixed in the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

In this case, if $genE_1(\sigma)^* = (\alpha^0, \alpha^i, \alpha^j, \alpha^k)$, then

$$genE_2(\sigma)^* = \begin{pmatrix} \beta^{4i} & \beta^0 & 0 & 0 \\ \beta^{4j} & 0 & \beta^0 & 0 \\ \beta^{4k} & 0 & 0 & \beta^0 \end{pmatrix},$$

and if

$$genE_1(\sigma)^* = \begin{pmatrix} \alpha^0 & 0 & \alpha^i & \alpha^j \\ 0 & \alpha^0 & \alpha^k & \alpha^s \end{pmatrix},$$

then

$$genE_2(\sigma)^* = \begin{pmatrix} \beta^{4i} & \beta^{4k} & \beta^0 & 0 \\ \beta^{4j} & \beta^{4s} & 0 & \beta^0 \end{pmatrix}.$$

In the next table we give all the necessary information about the generator matrix of $E_1(\sigma)^*$ for the $[24, 12, 8]$ codes $C^i_{(5)}$, $i = 1, 2, \ldots, 12$, with a permutation automorphism of order 5, as well as the number of vectors of weight 8. Now, $\alpha = 1 + \omega x + \omega x^3 + x^4$ and $\beta = 1 + \omega^2 x + \omega^2 x^3 + x^4$.

i for $C^i_{(5)}$	$genE_1(\sigma)^*$	A_8
1	$\alpha^0\ \alpha^0\ \alpha^0\ \alpha^0$	657
2	$\alpha^0\ \alpha^0\ \alpha^0\ \alpha^5$	792
3	$\alpha^0\ \alpha^0\ \alpha^5\ \alpha^5$	837
4	$\alpha^0\ 0\ 0\ \alpha^0$ $0\ \alpha^0\ \alpha^0\ \alpha^0$	1242
5	$\alpha^0\ 0\ 0\ \alpha^5$ $0\ \alpha^0\ \alpha^5\ \alpha^5$	522
6	$\alpha^0\ 0\ \alpha^0\ \alpha^0$ $0\ \alpha^0\ \alpha^0\ \alpha^3$	657
7	$\alpha^0\ 0\ \alpha^0\ \alpha^0$ $0\ \alpha^0\ \alpha\ \alpha^2$	702
8	$\alpha^0\ 0\ \alpha^0\ \alpha$ $0\ \alpha^0\ \alpha\ \alpha^0$	657
9	$\alpha^0\ 0\ \alpha^0\ \alpha$ $0\ \alpha^0\ \alpha\ \alpha^3$	837
10	$\alpha^0\ 0\ \alpha^0\ \alpha$ $0\ \alpha^0\ \alpha\ \alpha^7$	837
11	$\alpha^0\ 0\ \alpha^0\ \alpha$ $0\ \alpha^0\ \alpha^2\ \alpha^8$	1197
12	$\alpha^0\ 0\ \alpha\ \alpha^2$ $0\ \alpha^0\ \alpha^2\ \alpha^{13}$	2277

Remark 1. The code $C^1_{(23)}$ has a binary generator matrix which generates over F_2 the extended Golay code [24, 12, 8]. It is known that this binary code has automorphisms of orders 5, 7, 11, and 23. Therefore, the quaternary code $C^1_{(23)}$ has such

automorphisms, too. Only the codes $C^{12}_{(5)}$, $C^{1}_{(7)}$, and $C^{1}_{(11)}$ have the same weight enumerators as $C^{1}_{(23)}$. Hence, these four codes are equivalent.

Remark 2. The elements α and β from the tables are taken from [5].

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